

LEARNING OBJECTIVES:

- 1.01.01 Add, subtract, multiply, and divide fractions.
- 1.01.02 Add, subtract, multiply, and divide decimals.
- 1.01.03 Convert fractions to decimals and decimals to fractions.
- 1.01.04 Convert percent to decimal and decimal to percent.
- 1.01.05 Add, subtract, multiply, and divide signed numbers.
- 1.01.06 Add, subtract, multiply, and divide numbers with exponents.
- 1.01.07 Find the square roots of numbers that have rational square roots.
- 1.01.08 Convert between numbers expressed in standard form and in scientific notation.
- 1.01.09 Add, subtract, multiply, and divide numbers expressed in scientific notation.
- 1.01.1 Solve equations using the "Order of Mathematical Operations".
- 1.01.2 Perform algebraic functions.
- 1.01.3 Solve equations using common and/or natural logarithms.

INTRODUCTION

Radiological control operations frequently require the RCT to use arithmetic and algebra to perform various calculations. These include scientific notation, unit analysis and conversion, radioactive decay calculations, dose rate/distance calculations, shielding calculations, stay-time calculations. A good foundation in mathematics and algebra is important to ensure that the data obtained from calculations is accurate. Accurate data is crucial to the assignment of proper radiological controls.

SYMBOLS FOR BASIC OPERATIONS

The four basic mathematical operations are addition, subtraction, multiplication and division. Furthermore, it is often necessary to group numbers or operations using parenthesis or brackets. In writing problems in this course the following notation is used to denote the operation to be performed on the numbers. If a and b represent numbers or variables, the operations will be denoted as follows:

Table 1. Symbols for Basic Mathematical Operations

Operation	Notation			
Addition:	$a + b$			
Subtraction:	$a - b$			
Multiplication:	$a \times b$	$a \cdot b$	$a(b)$	ab
Division:	$a \div b$	a/b	$\frac{a}{b}$	$b \overline{)a}$
Grouping:	$(\quad) \quad [\quad]$			
Equality:	$=$			
Inequality:				
Less than:	$<$	Less than or equal to:	\leq	
Greater than:	$>$	Greater than or equal to:	\geq	

1.01.01 Add, subtract, multiply and divide fractions.

FRACTIONS

Whole numbers are the normal counting numbers and zero.

$$\{0, 1, 2, 3, 4, \dots\}$$

A *fraction* is part of a whole number. It is simply an expression of a division problem of two whole numbers. A fraction is written in the format:

$$\frac{a}{b} \quad \text{or} \quad a/b$$

The number above the bar a is called the *numerator* and the number below the bar b is called the denominator. A *proper fraction* is a fraction in which the number in the numerator is less than the number in the denominator. If the numerator is greater than the denominator then it is an *improper fraction*. For example, $\frac{1}{2}$ and $\frac{1}{4}$ are proper fractions, while $\frac{7}{5}$, $\frac{5}{5}$, $\frac{5}{1}$, or $\frac{61}{27}$ are improper fractions.

Any whole number can be written as a fraction by letting the whole number be the numerator and 1 be the denominator. For example,

$$5 = \frac{5}{1} \quad 2 = \frac{2}{1} \quad 0 = \frac{0}{1}$$

Five can be written as $\frac{10}{2}$, $\frac{15}{3}$, $\frac{20}{4}$, etc. Similarly, the fraction $\frac{1}{4}$ can be written as $\frac{2}{8}$, $\frac{3}{12}$, $\frac{4}{16}$, etc. These are called *equivalent fractions*. An equivalent fraction is **built up** by multiplying the numerator and the denominator by the same nonzero number. For example,

$$\frac{3}{4} = \frac{3 \cdot 2}{4 \cdot 2} = \frac{6}{8} \quad \frac{3}{4} = \frac{3 \cdot 5}{4 \cdot 5} = \frac{15}{20}$$

A fraction is **reduced** by dividing the numerator and the denominator by the same nonzero number. For example,

$$\frac{12}{18} = \frac{12 \div 2}{18 \div 2} = \frac{6}{9} \quad \frac{6}{9} = \frac{6 \div 3}{9 \div 3} = \frac{2}{3}$$

A fraction is reduced to lowest terms when 1 is the only number that divides both numerator and denominator evenly. This is done by finding the *greatest common multiple* between the numerator and denominator. In the previous example, two successive reductions were performed. For the fraction $12/18$, the greatest common multiple would be 6, or (2×3) , which results in a reduction down to a denominator of 3.

A whole number written with a fraction is called a *mixed number*. Examples of mixed numbers would be $1\frac{1}{2}$, $3\frac{1}{4}$, $5\frac{3}{4}$, etc. A mixed number can be simplified to a single improper fraction using the following steps:

1. Multiply the whole number by the denominator of the fraction
2. Add the numerator of the fraction to the product in step 1.
3. Place the sum in step 2 as the numerator over the denominator.

For example:

$$5\frac{3}{4} = \frac{(5 \cdot 4) + 3}{4} = \frac{23}{4}$$

Adding and Subtracting Fractions

In order to add and subtract fractions the denominators must be the same. If the denominators are not the same, then the fractions must be built up so that the denominators are equal.

Same Denominator

To add two fractions that have the same denominator:

1. Add the numerators
2. Place the sum of step 1 over the common denominator
3. Reduce fraction in step 2 to lowest terms (if necessary)

For example,

$$\frac{1}{5} + \frac{3}{5} = \frac{1+3}{5} = \frac{4}{5}$$

Subtraction of two fractions with the same denominator is accomplished in the same manner as addition. For example,

$$\frac{5}{8} - \frac{3}{8} = \frac{5-3}{8} = \frac{2}{8} = \frac{2 \div 2}{8 \div 2} = \frac{1}{4}$$

Different Denominator

To add two fractions with different denominators requires that the equations be built up so that they have the same denominator. This is done by finding the *lowest common denominator*. Once a common denominator is obtained, the rules given above for the same denominator apply.

For example, $1/3 + 2/5$. The fraction $1/3$ could be built up to $2/6$, $3/9$, $4/12$, $5/15$, $6/18$, $7/21$, etc. The fraction $2/5$ could be built up to $4/10$, $6/15$, $8/20$, $10/25$, etc. The lowest common denominator for the two fractions would be 15. The problem would be solved as follows:

$$\frac{1}{3} + \frac{2}{5} = \frac{1 \cdot 5}{3 \cdot 5} + \frac{2 \cdot 3}{5 \cdot 3} = \frac{5}{15} + \frac{6}{15} = \frac{5+6}{15} = \frac{11}{15}$$

Subtraction of fractions with different denominators is accomplished using the same steps as for addition. For example,

$$\frac{3}{4} - \frac{2}{3} = \frac{3 \cdot 3}{4 \cdot 3} - \frac{2 \cdot 4}{3 \cdot 4} = \frac{9}{12} - \frac{8}{12} = \frac{9-8}{12} = \frac{1}{12}$$

Multiplying and Dividing Fractions

Multiplication of fractions is much easier than addition and subtraction, especially if the numbers in the numerators and denominators are small. Fractions with larger numerators and/or denominators may require additional steps. In either case, the product of the multiplication will most likely need to be reduced in order to arrive at the final answer. To multiply fractions:

1. Multiply the numerators
2. Multiply the denominators.
3. Place product in step 1 over product in step 2.
4. Reduce fraction to lowest terms.

For example:

$$\frac{5}{6} \cdot \frac{3}{4} = \frac{15}{24} = \frac{15 \div 3}{24 \div 3} = \frac{5}{8}$$

A variation on the order of the steps to multiply fractions is to factor the numerators and denominators first, reduce and cancel, and then multiply. An example follows:

$$\frac{3}{8} \cdot \frac{20}{9} = \frac{3}{2 \cdot 2 \cdot 2} \cdot \frac{2 \cdot 2 \cdot 5}{3 \cdot 3} = \frac{\cancel{3} \cdot 2 \cdot 2 \cdot 5}{2 \cdot 2 \cdot 2 \cdot \cancel{3} \cdot 3} = \frac{5}{2 \cdot 3} = \frac{5}{6}$$

Reciprocals

Two numbers whose product is 1 are called *reciprocals*, or *multiplicative inverses*. For example:

$$5 \text{ and } \frac{1}{5} \text{ are reciprocals because } 5 \cdot \frac{1}{5} = 1.$$

$$\frac{4}{5} \text{ and } \frac{5}{4} \text{ are reciprocals because } \frac{4}{5} \cdot \frac{5}{4} = 1.$$

$$1 \text{ is its own reciprocal because } 1 \cdot 1 = 1.$$

0 has no reciprocal because 0 times *any* number is 0 *not* 1.

The symbol for the reciprocal, or multiplicative inverse, of a nonzero real number a is $\frac{1}{a}$.

Every real number except 0 has a reciprocal. We say, therefore, for every *nonzero* real number a , there is a unique real number $\frac{1}{a}$ such that

$$a \cdot \frac{1}{a} = 1$$

Now, look at the following product:

$$(ab)\left(\frac{1}{a} \cdot \frac{1}{b}\right) = \left(a \cdot \frac{1}{a}\right)\left(b \cdot \frac{1}{b}\right) = 1 \cdot 1 = 1$$

Relationship of multiplication to division

The operation of division is really just inverted multiplication (reciprocals). Notice from the first examples given on the previous page that the reciprocal of a fraction is merely "switching" the numerator and denominator. The number 5 is really $\frac{5}{1}$, and the reciprocal of 5 is $\frac{1}{5}$. Likewise, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, which can also be expressed as:

$$\frac{1}{\frac{2}{3}} = 1 \div \frac{2}{3} = 1 \cdot \frac{3}{2} = \frac{3}{2}$$

Fractions are a division by definition. **Division** of fractions is accomplished in two steps:

1. Invert the second fraction, i.e. change it to its reciprocal, and change the division to multiplication.
2. Multiply the two fractions using the steps stated above.

For example:

$$\frac{4}{7} \div \frac{2}{3} = \frac{4}{7} \cdot \frac{3}{2} = \frac{12}{14} = \frac{12 \div 2}{14 \div 2} = \frac{6}{7}$$

Practice Problems

Solve the following problems involving fractions. Answers should be reduced to lowest terms.

- | | | | |
|-----|--------------------|-----|-------------------|
| 1. | $1/3 + 2/3$ | 2. | $5/7 - 3/7$ |
| 3. | $5/9 + 2/3$ | 4. | $6/7 - 1/2$ |
| 5. | $2 - 1/3$ | 6. | $3/8 + 15/16$ |
| 7. | $25/32 - 3/4$ | 8. | $15/21 - 4/7$ |
| 9. | $13/20 - 2/5$ | 10. | $7/18 + 5/9$ |
| 11. | $2/3 \times 1/5$ | 12. | $4/7 \times 3/4$ |
| 13. | $1/2 \times 2$ | 14. | $3/5 \times 4$ |
| 15. | $4/9 \div 2/3$ | 16. | $8/13 \times 2/3$ |
| 17. | $12/15 \times 3/5$ | 18. | $20/25 \div 4/5$ |
| 19. | $7/8 \times 2/5$ | 20. | $14/21 \div 2/7$ |

1.01.02 *Add, subtract, multiply and divide decimals.*

DECIMALS

A *decimal* is another way of expressing a fraction or mixed number. It is simply the numerical result of division (and fractions are division). Recall that our number system is based on 10 ("deci" means ten) and is a **place-value system**; that is, each *digit* {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} in a numeral has a particular value determined by its location or place in the number. For a number in decimal notation, the numerals to the left of the *decimal point* is the whole number, and the numerals to the right are the decimal fraction, the denominator being a power of ten.



Figure 1. Decimal Place Value System

For example, the numeral 125.378 (decimal notation) represents the expanded numeral

$$100 + 20 + 5 + \frac{3}{10} + \frac{7}{100} + \frac{8}{1000}$$

If this numeral were written as a mixed number, we would write it as:

$$125\frac{378}{1000}$$

Addition and Subtraction of Decimals

In order to add or subtract decimals use the following steps:

1. Arrange the numbers in a column so that the decimal points are aligned.
2. Add or subtract in columns from right to left. (Additional zeros may need to be added to the right of the subtrahand.)
3. Place the decimal point in the answer in line with the other decimal points.

For example:

$$\begin{array}{r} 21.3 \\ + 4.2 \\ \hline 25.5 \end{array}$$

$$\begin{array}{r} 654.200 \\ - 26.888 \\ \hline 627.312 \end{array}$$

Multiplying Decimals

To multiply decimal numbers, do the following:

1. Multiply the numbers as if there were no decimal points.
2. Count the number of decimal places in each number and add them together.
3. Place the decimal point in the product so that it has the same number of decimal places as the sum in step 2.

For example:

$$\begin{array}{r} 5.28 \\ \times 3.7 \\ \hline 3696 \\ 1584 \\ \hline 19.536 \end{array}$$

$$\begin{array}{r} 0.04 \\ \times 0.957 \\ \hline 0.03828 \end{array}$$

Division of Decimals

The steps for division of decimals are as follows:

1. Move the decimal point of the divisor to the right until it becomes a whole number.
2. Move the decimal point of the dividend to the right the number of places moved in step 1.
3. Divide the numbers as if they were whole numbers.

For example:

$$\begin{array}{r} 28 \\ 0.25 \overline{)7.00} \\ \underline{50} \\ 200 \\ \underline{200} \\ 0 \end{array}$$

Decimal Forms

As we have learned, decimals are the result of division (or a fraction). When the remainder of the division is zero, the resulting decimal is called a *terminating* or *finite decimal*. For example, fractions like $2/5$, $3/8$ and $5/6$ will all result in finite decimals. These fractions and the resulting decimals are known as *rational numbers*.

On the other hand, fractions like $1/3$, $2/7$, $5/11$ and $7/13$ result in a *nonterminating* or *infinite decimal*. For example, $2/7$ results in the decimal $0.285714286 \dots$, the dots meaning that the decimal continues without end. These numbers are known as *irrational numbers*. Note that even though irrational numbers are nonterminating, some, such as $1/3$ and $5/11$ are *repeating* or *periodic* decimals because the same digit or block of digits repeats unendingly. For example:

$$\frac{1}{3} = 0.3333\dots$$

$$\frac{5}{11} = 0.454545\dots$$

A bar is often used to indicate the block of digits that repeat, as shown below:

$$\frac{1}{3} = 0.3$$

$$\frac{5}{11} = 0.\overline{45}$$

Practice Problems

Solve the following problems involving decimals.

1. $0.23 + 3.4$

2. $5.75 - 2.05$

3. $6.1 - 1.6$

4. $0.018 + 0.045$

5. $468.75 - 192.5$

6. 2.3×3.2

7. 0.007×2.18

8. $5.2 \div 1.4$

9. $12.26 \div 0.04$

10. 4.0×0.25

1.01.03 *Convert fractions to decimals and decimals to fractions.*

FRACTION TO DECIMAL CONVERSION

To convert a fraction to a decimal we simply perform the operation of division that the fraction represents. For example, the fraction $\frac{3}{4}$ represents "3 divided by 4", and would be converted as follows:

$$\begin{array}{r} 0.75 \\ 4 \overline{)3.00} \\ \underline{28} \\ 20 \\ \underline{20} \\ 0 \end{array}$$

Practice Problems

Convert the following fractions to decimals.

- | | | | |
|----|-----------------|----|-----------------|
| 1. | $\frac{1}{2}$ | 2. | $\frac{2}{5}$ |
| 3. | $\frac{5}{8}$ | 4. | $\frac{12}{25}$ |
| 5. | $\frac{13}{39}$ | 6. | $\frac{7}{16}$ |

Convert the following decimals to fractions. Reduce answers to lowest terms.

- | | | | |
|----|-------|-----|--------|
| 7. | 0.125 | 8. | 0.6666 |
| 9. | 4.25 | 10. | 0.2 |

1.01.04 Convert percent to decimal and decimal to percent.

PERCENT

Percentage is a familiar concept. Most people know the meaning of terms such as 100 percent and 50 percent. Percentage is such a convenient concept for expressing common and decimal fractions that it is widely used. The word *percent* actually means "out of a hundred." (Consider that there are 100 "cents" in a dollar, and that a "century" is 100 years.) A percent is simply a fraction whose denominator is 100. Thus, 50 percent means $\frac{50}{100}$ or 0.50, while 25 percent means $\frac{25}{100}$ or 0.25.

Percent is abbreviated by the symbol %. So, 75 percent is written 75%.

Converting Decimal to Percent

A decimal fraction is changed to a percent by moving the decimal point two places to the right and adding a percent sign. For example, $\frac{1}{8}$ equals 0.125. Therefore:

$$0.\underline{1}25 = 12.5\%$$

A percent is changed to a common fraction by omitting the percent sign, placing the number over 100, and reducing the resulting fraction if possible. For example, 32% equals $\frac{32}{100}$ which reduces to $\frac{8}{25}$. When the percent consists of a mixed decimal number with a percent sign, the resulting fraction will contain a mixed decimal numerator. This can be

changed to a whole number by multiplying the numerator and the denominator by 10, 100, 1,000, etc.

Percentage is most frequently used to indicate a fractional part. Thus 20% of the total power output for 75% of the employees refer to fractional parts of some total number. To perform arithmetic operations with a percent, it is normally changed to a common or decimal fraction. Usually, a decimal fraction is more convenient.

Converting Percent to Decimal

A percent is changed to a decimal fraction by omitting the percent sign and moving the decimal point two places to the left. For example:

$$48\% = 0.48$$

Thus, 92% equals 0.92, 8% equals 0.08, and so on.

*
*
*

Practice Problems

Convert the following decimals to percent.

- | | | | |
|----|-------|----|--------|
| 1. | 0.5 | 2. | 0.782 |
| 3. | 1.1 | 4. | 0.06 |
| 5. | 0.049 | 6. | 0.0055 |

Convert the following percentages to decimals.

- | | | | |
|----|------|-----|-------|
| 7. | 65% | 8. | 0.25% |
| 9. | 300% | 10. | 0.09% |

1.01.05 Add, subtract, multiply and divide signed numbers.

SIGNED NUMBERS

The numbers that are used to describe the number of objects in a group, the counting numbers, are always *positive* numbers; that is, they are always greater than zero. However, there are many occasions when *negative* numbers, numbers less than zero, must be used. These numbers arise when we try to describe measurement in a direction opposite to the positive numbers. For example, if we assign a value of +3 to a point which is 3 feet above the ground, what number should be assigned to a point which is 3 feet below the ground? Perhaps the most familiar example of the use of negative numbers is the measurement of temperature, where temperatures below an arbitrary reference level are assigned negative values.

Every number has a *sign* associated with it. The plus (+) sign indicates a positive number; the minus (-) sign indicates a negative number. When no sign is given, a plus sign is implied. The fact that the plus and minus signs are also used for the arithmetic operations of addition and subtraction should not be a cause for confusion, for we shall see that they have equivalent meanings.

Every number has an *absolute value*, regardless of its sign. The absolute value of a number is indicated by a pair of vertical lines enclosing the number: $|5|$. The absolute value indicates the distance from zero, without regard to direction. The number 5 is 5 units from zero, in the positive direction. The number -5 is also 5 units from zero, but in the negative direction. The absolute value of each of these numbers is 5.

Operations with Signed Numbers

The arithmetic operations of addition, subtraction, multiplication, and division of signed numbers can be more easily visualized if the numbers are placed on a *number line* (see *Figure 2*). The positive numbers are greater than zero, and they lie to the right of zero on the number line. The negative numbers are less than zero, and lie to the left of zero on the number line.

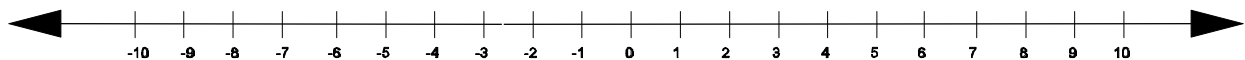


FIGURE 2. Number Line

The number line extends an infinite distance in each direction and therefore includes all numbers. The process of addition can be considered as counting in one direction or the other from a starting point on the number line. For example, let us add $1 + 2$. We locate $+1$ on the number line and then count 2 units to the right, since we are adding $+2$. The result will be $+3$. To illustrate further, let us add $+2$ and -4 . We first locate $+2$ on the number line and then count 4 units to the left. We end up at -2 .

The number line is useful for illustrating the principles of addition, but it clearly would be inconvenient to use in the case of large numbers. Consequently, the following rules were developed to govern the addition process:

Adding and Subtracting Signed Numbers.

To **add** two numbers with the **same signs**, add their absolute values and attach the common sign. For example:

$$(-3) + (-2) = -5$$

To **add** two numbers with **opposite signs**, find the difference of their absolute values, then attach the sign of the original number which had the greater absolute value. For example:

$$(-2) + 3 = 1$$

Notice that -3 and $+3$ are the same distance but in *opposite* directions from 0 on the number line. What happens when you add two numbers like 3 and -3 ?

$$3 + (-3) = 0$$

$$-7 + 7 = 0$$

If the sum of two signed numbers is 0, the numbers are called *additive inverses* or *opposites*.

For example: $7 - 3 = 4$ is the same as: $7 + (-3) = 4$
 $8 - 2 = 6$ is the same as: $8 + (-2) = 6$

So, it can be seen that subtracting a number is equivalent to adding its additive inverse or opposite.

To **subtract** a signed number, **add** its **opposite** or additive inverse. In other words, change the subtraction symbol to addition and change the sign of the second signed number. For example:

$$5 - (-8) = 5 + (+8) \quad \Leftarrow \text{add } +8$$

$$6 - 11 = 6 + (-11) \quad \Leftarrow \text{add } -11$$

$$-4 - (-7) = -4 + (+7) \quad \Leftarrow \text{add } +7$$

Multiplying and Dividing Signed Numbers.

The **product** of two numbers with **like signs** is a **positive** number. The product of two numbers with **unlike signs** is a **negative** number. In symbols:

$$\begin{array}{ll} (+) \times (+) = (+) & (+) \times (-) = (-) \\ (-) \times (-) = (+) & (-) \times (+) = (-) \end{array}$$

For example:

$$\begin{array}{l} (-4) \times (-3) = (+12) \\ (-4) \times (+3) = (-12) \end{array}$$

The **division** of numbers with **like signs** gives a **positive** quotient. The division of numbers with **unlike signs** gives a **negative** quotient. In symbols:

$$\begin{array}{ll} (+)/(+) = (+) & +/(-) = (-) \\ (-)/(-) = (+) & (-)/(+) = (-) \end{array}$$

For example:

$$\begin{array}{l} (-24)/(-6) = (+4) \\ (-24)/(+6) = (-4) \end{array}$$

Remember that multiplication is really a short form of addition. When we say $+4 \times (-3)$, we are adding the number -3 four times, that is, $(-3) + (-3) + (-3) + (-3) = -12$. Also, since division is a short form of subtraction, when we say $-24 \div (-6)$, we subtract the number -6 from the number -24 four times in order to reach 0, i.e., $-24 - (-6) - (-6) - (-6) - (-6) = 0$. Although we could repeat the process for the multiplication and division of any signed numbers, usage of the two rules will produce equivalent results.

Practice Problems

Solve the following problems involving signed numbers.

- | | | | |
|-----|-----------------|-----|---------------|
| 1. | $(-28) + (-51)$ | 2. | $(-2) + (-5)$ |
| 3. | $40 + (-21)$ | 4. | $(-87) + 50$ |
| 5. | $48 + (-27)$ | 6. | $56 - (-5)$ |
| 7. | $81 - 4$ | 8. | $-48 - (-2)$ |
| 9. | $(-4)(5)$ | 10. | $6/(-3)$ |
| 11. | $4(-5)$ | 12. | $(-6)/(-3)$ |
| 13. | $(-6)/3$ | 14. | $(-8)(-5)$ |
| 15. | $(-7)(6)$ | | |

1.01.06 *Add, subtract, multiply and divide numbers with exponents.*

EXPONENTS

An *exponent* is a small number placed at the right and a little above another number, called the *base*, to show how many times the base is to be multiplied by itself. Thus, 3^4 (read "three to the fourth *power*") means 3 used as a factor four times or $3 \times 3 \times 3 \times 3$. 4 is the exponent; 3 is the base.

In general, if a is any real number and n is any positive integer, the ***n*th power of a** is written a^n , where a is the base and n is the exponent, is read as "a to the *n*th power," and tells you that a is used as a factor n times. *

Thus, 5^2 is called "5 raised to the second power" (or 5 "*squared*"), and 2^3 is called "2 raised to the third power" (or 2 "*cubed*"). When no exponent is shown for a number or no power is indicated, the exponent or power is understood to be 1. Thus, 7 is the same as 7^1 . Any number raised to the power of zero equals one; that is, $7^0 = 1$. Normally, exponents of zero and one are not left as the final value, but are changed to the simpler form of the base.

Exponents can be expressed as integers as in the examples above or as fractions or decimals such as $9^{1/2}$ or $10^{3.2}$. They may also be positive or negative.

Exponents and powers are particularly useful in mathematics not only because they shorten the writing of mathematical expressions but also because they simplify many mathematical operations. However, there are several special rules which govern mathematical operations involving numbers with exponents.

Addition and Subtraction

The addition or subtraction of numbers with exponents can be performed only if both the bases and the exponents of the numbers are the same. When the bases of the exponents are different, the multiplication indicated by the exponent must be performed and the numbers then added or subtracted. Thus, 2^5 and 2^4 cannot be added directly because their exponents are different. They can be added only by carrying out the indicated multiplication first. Thus, 2^5 equals $2 \times 2 \times 2 \times 2 \times 2$ which equals 32, and 2^4 equals $2 \times 2 \times 2 \times 2$ which equals 16. Therefore, $2^5 + 2^4$ equals $32 + 16$ which equal 48.

When the bases and the exponents are the same, the numbers can be added or subtracted directly. Thus,

$$3^5 + 3^5 = 2(3^5) = 2(243) = 486$$

Multiplication

The multiplication of numbers with exponents of the same base is performed by adding the exponents. The general form is as follows:

$$(a^m)(a^n) = a^{m+n}$$

It is important to remember that the **bases of the numbers must be the same** before they can be multiplied by adding their exponents. The base of the product is the same as the base of the two factors. Thus,

$$3^2 \times 3^3 = 3^5 = 243$$

Division

The division of numbers with exponents of the same base is performed by subtracting the exponent of the divisor (numerator) from the exponent of the dividend (denominator). The general form is:

$$\frac{a^m}{a^n} = a^{m-n}$$

Again, it is important to remember that the bases of the numbers must be the same before they can be divided by subtracting their exponents. The base of the quotient is the same as the base of the number divided. Thus,

$$\frac{2^5}{2^2} = 2^{5-2} = 2^3 = 8$$

Division of numbers with exponents can be used to show why any number raised to the power of zero equals one. We know that any fraction in which the numerator equals the denominator can be reduced to 1; that is, $2/2 = 1$. Similarly:

$$\frac{2^3}{2^3} = 2^{3-3} = 2^0 = 1$$

Exponent Raised to a Power

Raising a number with an exponent to a power is performed by multiplying the exponent by the power. The general form is:

$$(a^m)^n = a^{mn}$$

The base of the answer is the same as the base of the number raised to the power. Thus:

$$(5^2)^2 = 5^{2 \times 2} = 5^4 = 625$$

Product Raised to a Power

Raising a product of several numbers to a power is performed by raising each number to the power. The general form is as follows:

$$(ab)^n = a^n b^n$$

Thus:

$$[(2)(3)(4)]^2 = (2^2)(3^2)(4^2) = (4)(9)(16) = 576$$

This same result can also be obtained like this:

$$[(2)(3)(4)]^2 = 24^2 = 576$$

Mixed Product and Exponents Raised to a Power

The same rule can be used to raise a product of several numbers with exponents to a power. The general form looks like this:

$$(A^a B^b C^c)^n = A^{a \times n} B^{b \times n} C^{c \times n}$$

Thus:

$$[(2)^4 (3)^3 (4)^2]^2 = (2^{4 \times 2})(3^{3 \times 2})(4^{2 \times 2}) = (2^8)(3^6)(4^4) = (256)(729)(256)$$

Fraction Raised to a Power

Raising a fraction to a power is performed by raising both numerator and denominator to the power. It should be remembered that with a proper fraction (numerator is less than the denominator) the resulting number must be less than one. Also, the resulting number will be less than the value of the original fraction. Thus, $(2/3)^3$ equals $2^3/3^3$ which equals $8/27$, which is also less than one and less than the original fraction, $2/3$.

Negative Exponents and Powers

A negative exponent or power has a special meaning. Any number, except 0, with a negative exponent equals the *reciprocal* of the same number with the same positive exponent. For example:

$$6^{-2} = \frac{1}{6^2} = \frac{1}{36}$$

The same rules for addition, subtraction, multiplication, division and raising to a power apply to negative exponents that apply to positive exponents. However, in adding, subtracting or multiplying the exponents, the rules for signed numbers must also be observed.

Fractional Exponents

Fractional exponents are used to represent roots (see next section). The general form of a fractional exponent is $a^{\frac{m}{n}}$, which reads "the n th root of a^m ." For example, $a^{1/2}$, means the square root of a^1 , or a . In other words, $a^{1/2} = \sqrt{a}$.

Calculator Method

To raise a number to a power using a scientific calculator, use the following steps:

- 1) Enter the number. (Fractions must first be converted to decimal form.)
- 2) Press the y^x key
- 3) Enter the power. (Fractions must first be converted to decimal form.)
- 4) Press the $=$ key. The number displayed will be the number entered in step 1 raised to the power entered in step 3.

Practice Problems

1. $(3^2)(3^3)$
2. $7^5/7^3$
3. $(10^5)(10^6)$
4. $(10^6)(10^{-4})$
5. $6^{-4}/6^{-3}$
6. $6^{-8}/6^3$

1.01.07 Find the square roots of numbers that have rational square roots.

SQUARE ROOTS

To *square* a number means to multiply the number by itself, i.e. raise it to the second power. (Consider that a square is two-dimensional.) For example, 2 squared is 4, since $2 \times 2 = 4$. The square of 3 is 9, 4 squared is 16, and so on.

Just as subtraction "undoes" addition and division "undoes" multiplication, squaring a number can be "undone" by finding the *square root*. The general definition is as follows:

If $a^2 = b$, then a is a **square root** of b .

Be careful not to confuse the terms *square* and *square root*. For example, if $5^2 = 25$, this indicates that 25 is the **square** of 5, and 5 is the **square root** of 25. To be explicit, we say that it is a *perfect square* because 5 times itself is 25.

* All perfect squares other than 0 have two square roots, one positive and one negative. For example, because $7^2 = 49$ and $(-7)^2 = 49$, both 7 and -7 are square roots of 49. The symbol $\sqrt{\quad}$, referred to as the *radical*, is used to write the *principal*, or positive, square root of a positive number.

$\sqrt{49} = 7$ is read "The *positive* square root of 49 equals 7."

A negative square root is designated by the symbol $-\sqrt{\quad}$.

$-\sqrt{49} = -7$ is read "The *negative* square root of 49 equals -7 ."

It is often convenient to use *plus-or-minus* notation:

$\pm\sqrt{49}$ means the positive **or** negative square root of 49.

Therefore, the rule is that every positive real number a has two square roots: \sqrt{a} and $-\sqrt{a}$. It follows from the definition of square root that $(\sqrt{a})^2 = a$, and that $\sqrt{a^2} = a$. Because the square of every real number is either positive or zero, negative numbers do not have square roots in the set of real numbers.

Notice that $\sqrt{4 \cdot 25} = \sqrt{100} = 10$ and $\sqrt{4} \cdot \sqrt{25} = 2 \cdot 5 = 10$. Therefore, $\sqrt{4 \cdot 25} = \sqrt{4} \cdot \sqrt{25}$. Therefore, in general, we can say:

For any nonnegative real numbers a and b : $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$

It also follows that $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Calculator Method

To calculate the square root of any number using a scientific calculator, follow these steps:

- 1) Enter the number. (Fractions must first be converted to decimal form.)
- 2) Press the \sqrt{x} key. The number displayed will be the square root of the number entered in step 1. An alternate method is to press the y^x key and then type 0.5. This raises the number in step 1 to the power of 0.5, or $\frac{1}{2}$.

Other roots

For informational purposes only, we mention the fact that other roots may be found for a number. One of these is the cube root. To *cube* a number means to multiply the number by itself three times, i.e. raise it to the third power. (Consider that a cube is three-dimensional.) For example, 2 cubed is 8, since $2 \times 2 \times 2 = 8$. The *cube root* (or third root) of a number, then, is the number that, when raised to the third power (cubed), equals the first number. The notation for a cube root is $\sqrt[3]{a}$.

Note that any root may be taken from a number to "undo" an exponent, such as the fourth or fifth root. The general definition for a root is:

If $a^n = b$, then a is the n th root of b .

The notation for the n th root is $\sqrt[n]{a}$. These roots follow the same general rules as the square root.

Practice Exercises

Find the indicated square roots.

1. $\sqrt{16}$

3. $\sqrt{144}$

5. $(\sqrt{6})^2$

7. $\sqrt{\frac{1}{64}}$

9. $\sqrt{\frac{81}{36}}$

2. $-\sqrt{81}$

4. $\sqrt{15^2}$

6. $-\sqrt{400}$

8. $\sqrt{9 \cdot 49}$

10. $-\sqrt{625}$

1.01.08 *Convert between numbers expressed in standard form and in scientific*

SCIENTIFIC NOTATION

The difficulty in writing very large or very small numbers in the usual manner is that a large number of zeros are required to write these numbers. This difficulty is overcome by using *scientific notation*, in which integral powers of ten are used instead of a large number of zeros to indicate the position of the decimal point. In addition to simplifying the writing of very large or very small numbers, scientific notation identifies clearly the number of significant digits in a number and simplifies arithmetic calculations involving multiplication, division or raising to a power. For these reasons, it is good practice to write numbers in scientific notation when these operations are involved.

The following demonstrates how the number 1 million can be represented by various factors of 10:

$$\begin{array}{r}
 0.00001 \times 10^{11} \\
 0.0001 \times 10^{10} \\
 0.001 \times 10^9 \\
 0.01 \times 10^8 \\
 0.1 \times 10^7 \\
 1,000,000 = 1 \times 10^6 \\
 10 \times 10^5 \\
 100 \times 10^4 \\
 1000 \times 10^3 \\
 10,000 \times 10^2 \\
 100,000 \times 10^1 \\
 1,000,000 \times 10^0 \\
 10,000,000 \times 10^{-1} \\
 100,000,000 \times 10^{-2}
 \end{array}$$

Converting From Standard Form To Scientific Notation

There are two steps involved in writing a number in scientific notation.

- 1) Move the decimal point just to the right of the first significant digit. The first significant digit is the first nonzero digit counting from the left.
- 2) Indicate multiplication of the resulting number by a power of ten that makes its value equal to the original value. The power of ten is found by counting the number of places the decimal point was moved from its original position. If counted to the left, the power is positive; if counted to the right, it is negative. For example:

Suppose you want to express a number such as 700 in scientific notation.

$$700 = 7 \times 10^2$$

Suppose you want to express 0.0014 in scientific notation.

$$0.0014 = 1.4 \times 10^{-3}$$

Converting from Scientific Notation to Standard Form

To transform from scientific notation to standard form, follow the opposite procedure.

$$1.96 \times 10^5 = 196000$$

$$2.27 \times 10^{-2} = 0.0227$$

There are two parts of a number written in scientific notation, the *significant digits* and the power of ten. Thus, in the number 3.21×10^6 , 3, 2, and 1 are the significant digits and 10^6 is the power of ten.

The ability to clearly see the number of significant digits can be helpful in performing arithmetic calculations. For example, the number of significant digits which should be reported in the product of two numbers can be readily determined if the two numbers are first written in scientific notation.

When numbers are expressed in scientific notation, calculations can be more easily visualized. This is because they involve only numbers between 1 and 10 and positive and negative integral powers of ten which can be treated separately in the calculations using the rules for numbers with exponents.

1.01.09 *Add, subtract, multiply, and divide numbers expressed in scientific*

Addition and Subtraction Using Scientific Notation

Addition and subtraction cannot normally be performed directly using scientific notation because they require adding or subtracting digits of equal place value. Thus, when numbers expressed in scientific notation are to be added or subtracted, they must first be converted to forms having equal place value. This is commonly done by expressing them as numbers which are multiplied by the same integral power of ten. The sum or difference of these significant digits, multiplied by their common power of ten, is the sum or difference of the original numbers.

For example: $(3.54 \times 10^5) + (2.51 \times 10^4)$

3.54×10^5 is first changed to 35.4×10^4

$$\begin{array}{r} 35.4 \times 10^4 \\ + 2.51 \times 10^4 \\ \hline 37.91 \times 10^4 = 3.79 \times 10^5 \end{array}$$

Multiplication and Division Using Scientific Notation

Multiplication or division of numbers using scientific notation is performed by multiplying or dividing the significant digits and the powers of ten separately. The significant digits are multiplied or divided in the same manner as other mixed decimals. The powers of ten are multiplied or divided by adding or subtracting their exponents using the rules for multiplication and division of numbers with exponents. Thus,

$$(2.7 \times 10^2)(3.1 \times 10^{-3}) = (2.7)(3.1) \times (10^2)(10^{-3}) = 8.37 \times 10^{-1}$$

which should be rounded off to 8.4×10^{-1} .

One of the most useful applications of scientific notation is in arithmetic calculations which involve a series of multiplications and divisions. The use of scientific notation permits accurate location of the decimal point in the final answer. For example:

Perform the following calculation using scientific notation.

$$\frac{(219)(0.00204)}{(21.2)(0.0312)}$$

- 1) Write each term in scientific notation.

$$\frac{(2.19 \times 10^2)(2.04 \times 10^{-3})}{(2.12 \times 10^1)(3.12 \times 10^{-2})}$$

- 2) Multiply and divide the significant digits.

$$\frac{(2.19)(2.04)}{(2.12)(3.12)} = \frac{4.46}{6.61} = 0.675$$

- 3) Multiply and divide the powers of ten by adding and subtracting exponents.

$$\frac{(10^2)(10^{-3})}{(10^1)(10^{-2})} = \frac{10^{-1}}{10^{-1}} = 10^{-1+1} = 10^0$$

- 4) Combine the results.

$$0.675 \times 10^0 = 6.75 \times 10^{-1}$$

"E" Notation

An alternate method for annotating scientific notation is often used by pocket calculators, computers and some references. The method uses an **E** in place of the "**× 10**," and the number written after the E is the exponent of 10. The standard and alternate methods for scientific notation are equivalent and can be converted from one form to another without a change in value. The expressions below use both methods in equivalent expressions.

$$3.79 \times 10^5 = 3.79E5$$

$$4.02 \times 10^{-6} = 4.02E-6$$

$$5.89 \times 10^0 = 5.89E0$$

Using "E" Notation with a Calculator

Numbers in scientific notation are entered into a scientific calculator as follows:

- 1) Enter the significant digits.
- 2) Press the **E** or **EXP** key. (Actual key label may vary.)
- 3) Enter the power of 10. If the power is negative press the **+/-** key in conjunction with entering the power.

Practice Problems

1. $(2 \times 10^{-2})(3 \times 10^{-2})$
2. $(6 \times 10^{-8})/(3 \times 10^3)$
3. $(9 \times 10^4)(-1 \times 10^{-2})$
4. $(3 \times 10^{-7})(9 \times 10^2)$
5. $(7E2)(6E4)$
6. $(5E-3)/(5E-2)$

1.01.10 Solve equations using the "Order of Mathematical Operations".

ORDER OF MATHEMATICAL OPERATIONS

In solving any equation it is necessary to perform the operations in the equation in accordance with a certain hierarchy or *order of operations*. Equations are solved by simplifying operations of higher order first, according to group, left to right. The order for solving equations is as follows:

- 1) Simplify expressions within grouping symbols, beginning with the innermost set if more than one set is used.
- 2) Simplify all powers.
- 3) Perform all multiplications and divisions in order from left to right.
- 4) Perform all additions and subtractions in order from left to right.

For example:

$$(3 + 1)^2 \times 3 - 14 \div 2$$

- | | |
|---|---|
| 1) Simplify parentheses. | $\underbrace{(3 + 1)}^2 \times 3 - 14 \div 2$ |
| 2) Simplify powers. | $\underbrace{(4)^2} \times 3 - 14 \div 2$ |
| 3) Perform multiplication and division left to right. | $\underbrace{16 \times 3} - \underbrace{14 \div 2}$ |
| 4) Perform subtraction. | $\underbrace{48 - 7}$ |
| 5) Final answer: | 41 |

Practice Problems

- | | |
|---------------------------------|--|
| 1. $5 + (-3) \cdot (-2) - 6$ | 2. $18 - [52 \div (7 + 6)]$ |
| 3. $19 - 7 + 12 \cdot 2 \div 8$ | 4. $2^3 - 20 \div 4 + 4 \cdot 3$ |
| 5. $10 + 6(0.5)^3$ | 6. $3^3 + 10 \div 5$ |
| 7. $(8 + 4 \cdot 3)^2$ | 8. $7(4^2 - 10) \div (12 \cdot \frac{3}{4})$ |
| 9. $7(6 - 2^2)$ | 10. $(57 - 2^5)^{1/2}$ |

1.01.11 *Perform algebraic functions.***ALGEBRA**

Algebra is the branch of mathematics which deals with the manipulation of words and letters, generically called *symbols*, which represent numbers. Two factors contribute to the widespread use of algebra in scientific calculations. First, by using words and letters to represent the values of physical quantities, physical relationships can be expressed in clear, concise, and completely generalized form. Second, by using words and letters in place of numbers, combinations of physical relationships may be simplified to yield results applicable to any set of numbers.

For example, the area of a rectangle equals the product of the length of the rectangle multiplied by its width. In generalized terms, this statement can be written as:

$$\text{Area} = \text{Length} \times \text{Width}$$

This expression is a simple rule which tells the relationship between the area and the length and width of a rectangle. It does not mean that words are multiplied together but rather that numbers are inserted for the length and the width to obtain the area. For example, if the length is 4 feet and the width is 2 feet, the area is 2×4 feet or 8 square feet. This expression can be further simplified by using symbols or letters instead of words. For example, if area is designated by the letter A , length designated by the letter l , and width designated by the letter w , the following expression results:

$$A = l \times w \text{ or } A = lw$$

In algebraic expressions, when two or more letters representing numbers are written next to each other without a symbol between them, multiplication is indicated.

Variables vs. Numbers

When words or letters are used to represent numbers, they are called *variables*. Thus, when x , y , z , f or k are used to represent the values of physical quantities, they are called variables because their value varies with the actual numbers they may be chosen to represent. In the area calculation above, A , l , and w are variables used to represent the numerical values of area, length and width, respectively.

Properties of Variables

Recall that every number has a sign and an exponent associated with it. Recall also that any number can be written as a fraction by putting that number as the numerator and 1 as the denominator, e.g. $5 = \frac{5}{1}$. These properties also apply to any symbols that we might use to represent numbers.

Additionally, a symbol by itself stands for one of whatever the variable represents. That is to say, the symbol a by itself means "one of the variable represented by the letter a ", or $1a$. Combining this with the other "invisible" properties mentioned, the symbol a is understood to represent "positive one of the variable represented by the letter a to the power of one, over 1", which would be expressed as:

$$\frac{+1a^1}{1}$$

An expression that is either a numeral, a variable or the product of a numeral and one or more variables is called a *monomial*. A combination or sum of monomials is called a *polynomial*.

Examples of each are:

Monomials: 12 z $\frac{2}{3}r$ $-4x^3$

Polynomials: $3x + 9$ $6a^2 - 15$

Equations

An *equation* is a statement that two quantities or expressions are equal. The two quantities are written with an equal sign (=) between them. For example,

$$\begin{array}{ll} 1 + 1 = 2 & 10 = 6 - (-4) \\ 5 \times 3 = 15 & 18 \div 2 = 9 \end{array}$$

are all equations because in each the quantity on the left side is equal to the quantity on the right side.

In algebra we use variables to represent numbers in equations. In this lesson we will *manipulate* and *solve* equations involving more than one variable, but we will find the *solution*, i.e. the final answer, to equations having only one variable.

*
*
*

Algebraic Manipulation

The basic principle or *axiom* used in solving any equation is: **whatever operation is performed on one side of an equation)be it addition, subtraction, multiplication, division, raising to an exponent, taking a root)it must also be performed on the other side if the equation is to remain true.** This principle **MUST** be adhered to in solving all types of equations.

This axiom can be thought of by visualizing the balancing of a scale. If the scale is initially balanced, it will remain balanced if the same weight is added to both sides, if the same weight is removed from both sides, if the weights on both sides are increased by the same factor, or if the weights on both sides are decreased by the same factor.

Here are the general forms for algebraic manipulation of equations. For the real numbers a , b , c and n :

Table 2. Rules for Algebraic Operations

Addition:	If $a = b$, then $a + c = b + c$
Subtraction:	If $a = b$, then $a - c = b - c$
Multiplication:	If $a = b$, then $a \cdot c = b \cdot c$
Division:	If $a = b$, then $a \div c = b \div c$
Involution:	If $a = b$, then $a^n = b^n$
Evolution:	If $a = b$, then $\sqrt[n]{a} = \sqrt[n]{b}$

Manipulating and Solving Linear Equations

The addition or subtraction of the same quantity from both sides of an equation may be accomplished by *transposing* a quantity from one side of the equation to the other. Transposing is a shortened way of applying the addition or subtraction axioms. Any term may be transposed or transferred from one side of an equation to the other if its sign is changed. Thus, in the equation below the +4 can be transposed to the other side of the equation by changing its sign:

$$\begin{aligned} 5x + 4 &= 14 \\ 5x &= 14 - 4 \\ 5x &= 10 \end{aligned}$$

Transposing also works with multiplication and division. Remembering that any number can be expressed as a fraction we can rewrite the last line of the equation above. We can then move the 5 in the numerator of the left side to the denominator of the right side:

$$\begin{aligned} \frac{5x}{1} &= \frac{10}{1} \\ \frac{x}{1} &= \frac{10}{5} \\ x &= \frac{10}{5} \\ x &= 2 \end{aligned}$$

Some linear equations may contain multiple terms (monomials) involving the same variable. In order to simplify the equation like terms must be combined. Don't forget those "invisible properties of variables." Here's an example:

$$\begin{aligned} 3x - 5 &= x + 3 && \Leftarrow \text{To solve for } x: \\ 3x - 5 - x &= x + 3 - x && \Leftarrow \text{Subtract } x \text{ from both sides.} \\ 2x - 5 &= 3 && \Leftarrow \text{Combine like terms and cancel.} \\ 2x - 5 + 5 &= 3 + 5 && \Leftarrow \text{Add 5 to both sides.} \\ 2x &= 8 && \Leftarrow -5 \text{ and } +5 \text{ cancel.} \\ \frac{2x}{2} &= \frac{8}{2} && \Leftarrow \text{Divide both sides by 2.} \\ x &= 4 && \Leftarrow 2 \text{ over } 2 \text{ cancels. Reduce.} \end{aligned}$$

Quadratic Equations

In manipulating an equation involving multiple variables, the "variable of interest" must be moved to one side of the equal sign and all other variables must be moved to the other side. In order to accomplish this, operations must be performed on both sides of the equation that will result in a variable, or group of variables, to be canceled out from one side. This cancellation can only occur if the "opposite function" is performed on a function that already exists on that side of the equation. This means that a variable that is being multiplied can be canceled by dividing by the same variable. Addition can be canceled with subtraction, multiplication with division, etc.

For example: $a + b = c$ \Leftarrow To solve for a :

$$\begin{aligned} a + b - b &= c - b && \Leftarrow \text{Subtract } b \text{ from both sides} \\ a &= c - b && \Leftarrow +b \text{ and } -b \text{ cancel} \end{aligned}$$

Another example: $ab = c$ \Leftarrow To solve for a :

$$\begin{aligned} \frac{ab}{b} &= \frac{c}{b} && \Leftarrow \text{Divide both sides by } b \\ a &= \frac{c}{b} && \Leftarrow b \text{ over } b \text{ cancels} \end{aligned}$$

Do not forget that the order of operations must be observed when manipulating equations. Otherwise a completely different solution may result. The key is to **do the opposite function in reverse order**. Here is an example which shows how this is done.

$$\begin{aligned} \frac{a + b}{c} &= d && \Leftarrow \text{To solve for } a: \\ c \cdot \frac{(a + b)}{c} &= d \cdot c && \Leftarrow \text{Multiply both sides by } c. \\ a + b &= dc && \Leftarrow c \text{ over } c \text{ cancels.} \\ a + b - b &= dc - b && \Leftarrow \text{Subtract } b \text{ from both sides.} \\ a &= dc - b && \Leftarrow b - b \text{ cancels} \end{aligned}$$

Once the order of the arithmetic functions has been established, manipulation of the formula can begin. In the example above, if the values for a , b and c were known, the first step would be to add b to a . The second step would be to divide by c . Therefore, in order to solve it, we do the opposite functions in reverse order. So, we first multiply by c . Then, we would subtract b . It is a good idea to rewrite the equation each time so that the operations can be reevaluated before the next step.

One final example:	$ab^2 = cd^2$	\Leftarrow To solve for d :
	$\frac{ab^2}{c} = \frac{cd^2}{c}$	\Leftarrow Divide both sides by c .
	$\frac{ab^2}{c} = d^2$	$\Leftarrow c$ over c cancels.
	$\sqrt{\frac{ab^2}{c}} = \sqrt{d^2}$	\Leftarrow Take the square root of both sides.
	$\sqrt{\frac{ab^2}{c}} = d$	\Leftarrow Square root of square cancels.

The above example involves the equation known as the *inverse square law* and will be seen again in Lesson 1.11.

Substitution

Linear equations are solved by combining like terms and reducing to find the solution. Quadratic equations are solved by substituting given values into the equation for all but one of the variables, thus making it a linear equation. The best approach is to first solve for the variable of interest by algebraic manipulation. Then find the solution by substituting the given values into the equation for the respective variables. The single, unknown variable, or the *variable of interest*, will be left on one side, being set equal to the solution. For example:

Given the equation $2x - y^2 = 3a - b$, that $x = 5$, $y = (-4)$ and $a = 3$, solve for b :

$2x - y^2 = 3a - b$	\Leftarrow To solve for b :
$2x - y^2 + b = 3a - b + b$	\Leftarrow Add b to both sides, and cancel.
$2x - y^2 + b = 3a$	$\Leftarrow -b + b$ cancels.
$b = 3a - 2x + y^2$	\Leftarrow Transpose $2x - y^2$ to right side.
$b = 3(3) - 2(5) + (-4)^2$	\Leftarrow Substitute known values.
$b = 9 - 10 + 16$	\Leftarrow Perform operations.
$b = 15$	\Leftarrow Simplify.

Practice Problems

Solve for the unknown variable:

1. $x + 3 = 10$

2. $5 + z = 8$

3. $\frac{y}{4} = 2$

4. $\frac{r}{2} = 6$

5. $5(j + 5) = 45$

6. $5d = 25$

7. $9n^2 = 81$

8. $a + 6 - 6a = -14$

Solve for a :

9. $a^2b = c^2d$

10. $\frac{b}{a} = \frac{c}{d}$

Substitute the values given and simplify the expression.

11. $3x + 14y - xy$

where $x = 5, y = 2$

12. $\frac{4s + 1}{6t + s}$

where $s = -3, t = 10$

13. $\frac{16a + 5b}{1 - 4a}$

where $a = -1, b = 5$

14. $3i + \frac{j}{1 + k} = 0$

where $k = 2, j = -12$, solve for i

15. $7(2m - \frac{3}{4}n) + l = 10$ where $m = \frac{1}{2}, n = 8$, solve for l

1.01.12 Solve equations using common and/or natural logarithms.

LOGARITHMS

In many cases, arithmetic operations can be performed much more quickly if the numbers involved are numbers with exponents to the same base. For example, the multiplication or division of numbers with exponents to the same base can be performed by merely adding or subtracting the exponents. Raising to a power or taking a root can be performed by merely multiplying or dividing the exponents by the power or root. It is this feature of numbers with exponents which led to the development of *logarithms*. If all numbers could be readily written as numbers with exponents to the same base, multiplication, division, raising to powers and taking roots could be performed much more quickly.

Any number can be expressed as a power of any other number. For example, 64 equals 2^6 , 4^3 , or 8^2 . 64 also equals $7^{2.137}$ or $10^{1.806}$. The use of logarithms involves expressing numbers as powers of a common number, such as 10, so that arithmetic operations with these numbers can be performed more quickly.

Simply put, a logarithm is an exponent. More explicitly, the logarithm of a number n is the exponent x of a given base a that is required to produce that number. The symbol \log_a is used to denote taking a logarithm. The base is usually indicated by a small number written to the right and slightly below the symbol log. The general relationship and form are as follows:

$$\text{If } n = a^x, \text{ where } a > 0 \text{ and } a \neq 1, \text{ then: } \log_a n = x$$

For example:

$$1000 = 10^3 \Rightarrow \log_{10} 1000 = 3$$

This says that the base ten logarithm of 1000 is 3, which means that the base number, 10, must be raised to the power of 3 to equal 1000.

Here are some additional examples:

$$\begin{aligned} 2^3 = 8 &\Rightarrow \log_2 8 = 3 \\ 4^{3/2} = 8 &\Rightarrow \log_4 8 = 3/2 \end{aligned}$$

Before the development of the scientific calculator, the use of logarithms saved considerable computation time. For example, the evaluation of the following expression by hand would take a very long time.

$$\frac{(865)^3(0.0247)^{1/2}(93,800)}{(0.00186)(4.53)^2}$$

However, using logarithms the above expression could be evaluated in a matter of minutes. Thus, logarithms, or *logs*, became one of the most useful tools in mathematics. In addition to simplifying arithmetic calculations and shortening computation time, logs are also important in engineering applications. The relationship between a number and its logarithm is used frequently to assist in measuring physical quantities when they vary over a wide range. For example, logarithmic scales are used to measure the neutron flux in nuclear reactors. Logarithms are also used for scales on charts and meters.

Properties of Logarithms

Since logarithms are exponents, the basic rules of exponents can be used to develop several useful properties of logarithms. Suppose that a , x and y are numbers, and a is a suitable base for a logarithm ($a > 0$, $a \neq 1$). The product rule for exponents says:

$$a^x \cdot a^y = a^{x+y}$$

Let us say that: $u = a^x$ and $v = a^y$

If we write each of these in logarithmic form we would have:

$$x = \log_a u \quad \text{and} \quad y = \log_a v$$

Then: $u \cdot v = a^x \cdot a^y = a^{x+y}$

If we write this in logarithmic form it would be:

$$\log_a (u \cdot v) = x + y$$

If we substitute the values for x and y from above we have:

$$\log_a (u \cdot v) = \log_a u + \log_a v$$

This results in one of the rules for logarithms, the **product rule**. Using similar methods, we could also prove the other two rules that have been developed for logarithms.

Table 3. Rules for Logarithms

*	Product Rule: $\log_a uv = \log_a u + \log_a v$ (The log of a product is the sum of the logs.)
*	
*	
*	Quotient Rule: $\log_a \frac{u}{v} = \log_a u - \log_a v$ (The log of a quotient is the difference of the logs.)
*	
*	
*	Power Rule: $\log_a u^n = n \log_a u$
*	

A common logarithm is most often a mixed number consisting of a whole number part and a decimal fraction part. The whole number part is called the *characteristic* of the logarithm. The decimal fraction part is called the *mantissa*. For example, in the logarithm of 527.3, which equals 2.7221, the characteristic is 2 and the mantissa is 0.7221. The mantissas of most logarithms are rounded off to a specified number of significant digits, typically four.

Table 5. Four-place Logarithms

N											Proportional Parts								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
1.1	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
1.2	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
1.3	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
1.4	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
1.5	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
1.6	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
1.7	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
1.8	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
1.9	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
2.0	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
2.1	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
2.2	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
2.3	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
2.4	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
2.5	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
2.6	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
2.7	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
2.8	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
2.9	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
3.0	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
3.1	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
3.2	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
3.3	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
3.4	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
3.5	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
3.6	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
3.7	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
3.8	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
3.9	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
4.0	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
4.1	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9

Table 5. Four-place Logarithms

N											Proportional Parts								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
4.2	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
4.3	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
4.4	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
4.5	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
4.6	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
4.7	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
4.8	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	5	5	6	7	8
4.9	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	5	5	6	7	8
5.0	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
5.1	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
5.2	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
5.3	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
5.4	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Table 5. Four-place Logarithms

N											Proportional Parts								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
5.5	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
5.6	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	1	2	3	4	5	5	6	7
5.7	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
5.8	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
5.9	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
6.0	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
6.1	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
6.2	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
6.3	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
6.4	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
6.5	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
6.6	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
6.7	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
6.8	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
6.9	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
7.0	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
7.1	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
7.2	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
7.3	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
7.4	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
7.5	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
7.6	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
7.7	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
7.8	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
7.9	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
8.0	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
8.1	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
8.2	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
8.3	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
8.4	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
8.5	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
8.6	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5

Table 5. Four-place Logarithms

N											Proportional Parts								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
8.7	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
8.8	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
8.9	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
9.0	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
9.1	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
9.2	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
9.3	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
9.4	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
9.5	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
9.6	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
9.7	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
9.8	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
9.9	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Log-Table Method

To find the common logarithm of a number using four-place log tables, use the following steps:

- 1) Write the number in scientific notation with up to four significant digits.
- 2) Using the product rule for logarithms, write the log of the product as the sum of the logs of the factors.
- 3) Determine the mantissa as follows:
 - a) Find the row in the table corresponding to the first two digits of the number, then move over to the column corresponding to the third digit of the number.
 - b) Find the number under the proportional parts section corresponding to the fourth digit of the number, and add it to the last digit of the decimal obtained in step 3.a.
- 4) Determine the characteristic by using the power of 10 written in step 1 and write the logs in the form of a sum.

NOTE: If the power of 10 is negative, the log may be left in this form.

- 5) Using the product rule, write the sum of the logs as the log of a product. *

NOTE: If the power of 10 is negative, the mantissa will be changed because of the addition of a whole number. *

Example:

- | | | |
|---|------------------------------|---|
| | * $\log 45,830$ | * |
| 1) Write the number in scientific notation. | * $\log (4.583 \times 10^4)$ | * |
| 2) Use the product rule to write the product as the sum of the logs. | * $\log 4.583 + \log 10^4$ | * |
| | * $\log 4.583 + 4$ | * |
| 3) a) Find the 4.5 row of the log table. Move over to the 8 column to find the mantissa. | * 0.6609 | * |
| b) Find the 3 column under the proportional parts section of the log table. Add this to the last digit of the mantissa in step 3.a. | * $\frac{+ 0.0003}{0.6612}$ | * |
| 4) Write the mantissa and characteristic as a sum. | * $0.6612 + 4$ | * |
| 5) Add the characteristic to the mantissa. | * 4.6612 | * |

Example:

- | | | |
|--|---------------------------------|---|
| | * $\log 0.004583$ | * |
| 1) First, we write the number in scientific notation. | * $\log (4.583 \times 10^{-3})$ | * |
| 2) Next, we use the product rule to write the product as the sum of the logs. | * $\log 4.583 + \log 10^{-3}$ | * |
| | * $\log 4.583 + (-3)$ | * |
| 3) The mantissa will be the same as in the previous example because the significant digits are the same. | * 0.6612 | * |
| 4) Now, we write the mantissa and characteristic as a sum. Since the characteristic is negative we could leave the logarithm in this form. | * $0.6612 + (-3)$ | * |
| | * $0.6612 - 3$ | * |

- * 5) Add the characteristic to the mantissa. Note that the * -2.3388
* mantissa has changed. Also note that this is not the
* same as -3.6612.

As you may have observed, the mantissa of the base ten logarithm of a number depends only on the succession of significant digits in the number. The position of the decimal point in the number does not affect the mantissa. Of course, the characteristics are different for each of these numbers.

Table 6. Mantissa for Successive Significant Digits

Number	Scientific Notation	Log Long form	Log Short form
0.000524	5.24×10^{-4}	0.7193 - 4	-3.2807
0.00524	5.24×10^{-3}	0.7193 - 3	-2.2807
0.0524	5.24×10^{-2}	0.7193 - 2	-1.2807
0.524	5.24×10^{-1}	0.7193 - 1	-0.2807
5.24	5.24×10^0	0.7193 + 0	0.7193
52.4	5.24×10^1	0.7193 + 1	1.7193
524	5.24×10^2	0.7193 + 2	2.7193
5240	5.24×10^3	0.7193 + 3	3.7193
52,400	5.24×10^4	0.7193 + 4	4.7193

Note that any time the logarithm of a number is rounded off it would be considered an approximate answer since each digit is necessary to exactly duplicate the number when the base is raised to that exponent. Since each significant digit of the logarithm affects the actual value of the number, a standard of four significant digits should be maintained to ensure appropriate accuracy in the answers.

Calculator Method

Since hand-held scientific calculators are readily available today, it is impractical to use log tables. To find the logarithm of a number with a calculator:

1. Enter the number.
2. Press the $\boxed{\log}$ key. The number displayed is the logarithm of the number entered in step 1.

Natural Logarithms

A logarithm can be written to any base. For most practical computations, base ten logarithms are used because of their relationship to the place values in the decimal system. However, in many scientific and engineering problems, it is convenient to use another base, symbolized by the letter e . e is an irrational number whose value is 2.71828 . . . The actual value of e is the limiting value of $(1 + 1/n)^n$ as n gets larger and larger.

Although it is an irrational number, it can still be used as the base for logarithms in the same way as 10 is used for base ten logarithms. e is the basis for many laws of nature, such as the laws of growth and decay of physical quantities, including the decay of radioactive substances and the growth and decay of neutron population in a nuclear reactor. Because of the relationship of e to natural

phenomena, logarithms to the base e are called *natural logarithms*. Logs to base e are also known as *Napierian logarithms*, named for the Scottish mathematician John Napier (1550-1617).

The natural logarithm of a number is the exponent to which e must be raised in order to get that number. The symbol **ln** is used to denote a natural logarithm which is the same as saying \log_e . The relationship is expressed as follows:

$$\text{If } e^x = n \quad \text{then } \ln n = x$$

For example:

$$\begin{array}{ll} \ln 2 = 0.693147 \dots & \text{which means that } e^{0.693147\dots} = 2. \\ \ln 10 = 2.302585 \dots & \text{which means that } e^{2.302585\dots} = 10. \\ \ln e = 1 & \text{which means that } e^1 = e \end{array}$$

Natural logarithms are not often used for computations. However, they appear frequently in decay and shielding calculations problems because of the relationship of e to natural phenomena. As a result, it is important to know how to determine the natural logarithms of numbers.

Tables of natural logarithms are available in several standard handbooks. However, there are several important differences between natural logarithms and base ten logarithms which must be understood to use natural logarithms. A natural logarithm is not separated into a characteristic and a mantissa. This is because the whole number part of a natural logarithm does not relate to the position of the decimal point. Therefore, tables of natural logarithms give the entire logarithm, not just the decimal fraction part. Moreover, if a natural logarithm is negative, the entire logarithm is negative and is shown as such in a table of natural logarithms. Further, there is no part of the natural logarithm of a number which is not affected by the position of the decimal point. For all these reasons, tables of natural logarithms cannot be made concise.

To find the natural log of a number using a hand-held calculator:

1. Enter the number.
2. Press the ln key. The number displayed is the natural logarithm of the number entered in step 1.

Antilogarithms

An *antilogarithm*, usually shortened to "antilog," is the opposite of a logarithm and is much easier to do. The antilog of a given number is the value obtained by raising the base to that number. Finding antilogs is an important part in the overall use of logarithms in computations. If numbers are converted to logarithms to perform calculations, the answer must be converted back from logarithms once the calculations have been performed. The symbol \log^{-1} is used in calculations to indicate the antilog is going to be taken. The base of 10 is assumed unless otherwise noted. The general form is:

$$\log^{-1} x = n \quad \text{which means } 10^x = n$$

For example:

$$\log^{-1} 3 \quad \text{which means } 10^3 = 1000$$

To find the antilog of a number using log tables:	*	$\log^{-1} 2.7832$	*
1) Write the number in log long form.	*	$\log^{-1} (0.7832 + 2)$	*
2) Find the mantissa in the table. Take the value at the head of the row and attach to it the value at the head of the column.	*		6.0 *
	*		7 *
	*		6.07 *
3) Write this value in scientific notation, putting the characteristic of the original number as the power of ten.	*	6.02×10^2	*
	*		
4) Write this number in standard form.	*	602	*
			*

On a scientific calculator the antilog of a number is obtained by raising the base (10) to that number.

- 1) Enter the number.
- 2) Press the \log^{-1} or 10^x key. The number displayed is the antilog of the number of entered in step 1. In other words, 10 raised to that power.

The symbol \ln^{-1} is used to denote the inverse natural log, i.e. the antilog of base e .

$$\ln^{-1} x = n \quad \text{which means } e^x = n$$

For example:

$$\ln^{-1} 0.693 \dots \text{ which means } e^{0.693\dots} = 2$$

On a scientific calculator the inverse natural log of a number is obtained as follows:

- 1) Enter the number.
- 2) Press the $\boxed{\ln^{-1}}$ or $\boxed{e^x}$ key. The number displayed is the inverse natural log of the number of entered in step 1. In other words, e raised to that power.

Table 7. Summary of Common and Natural Log Definitions

Expression	Meaning	Example	Key-in
<u>10 is the base of the common logarithm.</u>			
$\log n$	Asks: "10 raised to what power will equal n ?"	$\log 100 = 2$	$\boxed{\log}$
$\log^{-1} n$	Asks: "10 raised to the n power equals what?"	$\log^{-1} 2 = 100$	$\boxed{10^x}$
<u>e is the base of the natural logarithm.</u>			
$\ln n$	Asks: " e raised to what power will equal n ?"	$\ln 2 = 0.693\dots$	$\boxed{\ln}$
$\ln^{-1} n$	Asks: " e raised to the n power equals what?"	$\ln^{-1} 0.693\dots = 2$	$\boxed{e^x}$

Solving for Variables as Exponents

One of the useful applications for logarithms is to solve algebraic equations with unknown exponents. In the following example, for instance, if the exponent is not known, it would be difficult to determine the correct value of x in order to make the statement (or equation) true.

$$2356 = 3^x$$

With the use of logarithms, however, this type of problem can be easily solved. The steps for solving an equation of this type are:

- 1) Make sure the base raised to the unknown exponent is isolated on one side of the equation (this may involve some manipulation of the formula in more complicated equations).
- 2) Take the log of both sides of the equation:

$$\log 2356 = \log 3^x$$

- 3) The right side of the equation can be rewritten using the power rule:

$$\log 2356 = x (\log 3)$$

- 4) Divide both sides by $\log 3$ which moves it to the right side of the equation.

$$\frac{\log 2356}{\log 3} = \frac{x(\log 3)}{\log 3}$$

- 5) Cancel terms and rewrite the equation:

$$\frac{\log 2356}{\log 3} = x$$

- 6) Perform the operations and solve:

$$\frac{3.372}{0.477} = x$$

$$7.068 = x$$

This answer can now be checked by substituting it back into the original equation to see if it makes the statement true,

$$2356 = 3^{7.068}$$

Some problems may involve the base of the natural logarithm e raised to an unknown power. This exponent can be determined by isolating e on one side of the equation and then taking the natural log of both sides. This is done because the natural log of e is 1. For example:

$$125 = 1000e^n$$

$$\frac{125}{1000} = e^n$$

$$\ln \frac{125}{1000} = \ln e^n$$

$$\ln 0.125 = (\ln e)(n)$$

$$\ln 0.125 = (1)(n)$$

$$-2.0794 = n$$

Practice Problems

Perform the following operations:

- | | | | |
|----|------------------|----|--------------------|
| 1. | $\log 6.40$ | 2. | $\log 0.5$ |
| 3. | $\log^{-1} 16$ | 4. | $\log^{-1} 3.7846$ |
| 5. | $\ln 86$ | 6. | $\ln 0.5$ |
| 7. | $\ln^{-1} 0.695$ | 8. | $\ln^{-1} 1$ |

Use logarithms to solve for the unknown variable:

- | | | | |
|-----|---------------|-----|----------------------------|
| 9. | $5 = 4^x$ | 10. | $23 = 6^t$ |
| 11. | $9 = 22(3)^y$ | 12. | $50 = 2000(\frac{1}{2})^n$ |

SUMMARY

A good foundation in mathematics is essential for the RCT. Calculations of various types are performed routinely in radiological control operations. The skills learned in this lesson will be applied in many of the lessons that follow as well as in the workplace.

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ANSWERS TO PRACTICE PROBLEMS:**Fractions**

1. $1/3 + 2/3 = 3/3 = 1$
2. $5/7 - 3/7 = 2/7$
3. $5/9 + 2/3 = 5/9 + 6/9 = 11/9 = 1 \frac{2}{9}$
4. $6/7 - 1/2 = 12/14 - 7/14 = 5/14$
5. $2 - 1/3 = 6/3 - 1/3 = 5/3 = 1 \frac{2}{3}$
6. $3/8 + 15/16 = 6/16 + 15/16 = 21/16 = 1 \frac{5}{16}$
7. $25/32 - 3/4 = 25/32 - 24/32 = 1/32$
8. $15/21 - 4/7 = 5/7 - 4/7 = 1/7$
9. $13/20 - 2/5 = 13/20 - 8/20 = 5/20 = 1/4$
10. $7/18 + 5/9 = 7/18 + 10/18 = 17/18$
11. $2/3 \times 1/5 = 2/15$
12. $4/7 \times 3/4 = 3/7$
13. $1/2 \times 2 = 1$
14. $3/5 \times 4 = 12/5 = 2 \frac{2}{5}$
15. $4/9 \div 2/3 = 4/9 \times 3/2 = 2/3$
16. $8/13 \times 2/3 = 16/39$
17. $12/15 \times 3/5 = 36/75$
18. $20/25 \div 4/5 = 20/25 \times 5/4 = 4/5 \times 5/4 = 1$
19. $7/8 \times 2/5 = 7/4 \times 1/5 = 7/20$
20. $14/21 \div 2/7 = 14/21 \times 7/2 = 2/3 \times 7/2 = 7/3 = 2 \frac{1}{3}$

*

Decimals

1. $0.23 + 3.4 = 3.63$
2. $5.75 - 2.05 = 3.7$
3. $6.1 - 1.6 = 4.5$
4. $0.018 + 0.045 = 0.063$
5. $468.75 - 192.5 = 276.25$
6. $2.3 \times 3.2 = 7.36$
7. $0.007 \times 2.18 = 0.01526$
8. $5.2 \div 1.4 = 3.7143$
9. $12.26 \div 0.04 = 306.5$
10. $4.0 \times 0.25 = 1$

Fraction/Decimal Conversion

1. $1/2 = 0.5$
2. $2/5 = 0.4$
3. $5/8 = 0.625$
4. $12/25 = 0.48$
5. $13/39 = 0.3$
6. $7/16 = 0.4375$

7. $0.125 = 125/1000 = 1/8$
8. $0.6666 = 2/3$
9. $4.25 = 4 \frac{25}{100} = 4\frac{1}{4}$
10. $0.2 = 2/10 = 1/5$

Percent

1. $0.5 = 50\%$
2. $0.782 = 78.2\%$
3. $1.1 = 110\%$
4. $0.06 = 6\%$
5. $0.049 = 4.9\%$
6. $0.0055 = 0.55\%$

7. $65\% = 0.65$
8. $0.25\% = 0.0025$
9. $300\% = 3$
10. $0.09\% = 0.0009$

Signed Numbers.

1. $(-28) + (-51) = -79$
2. $(-2) + (-5) = -7$
3. $40 + (-21) = 19$
4. $-87 + 50 = -37$
5. $48 + (-27) = 21$
6. $56 - (-5) = 61$
7. $81 - 4 = 77$
8. $-48 - (-2) = -46$
9. $-4(5) = -20$
10. $6/(-3) = -2$
11. $4(-5) = -20$
12. $(-6)/(-3) = 2$
13. $(-6)/3 = -2$
14. $(-8)(-5) = 40$
15. $(-7)(6) = -42$

Exponents

1. $(3^2)(3^3) = 3^5$
2. $7^5/7^3 = 7^2$
3. $(10^5)(10^6) = 10^{11}$
4. $(10^6)(10^{-4}) = 10^2$
5. $6^{-4}/6^{-3} = 6^{-1}$
6. $6^{-8}/6^3 = 6^{-11}$

Square Roots

1. $\sqrt{16} = 4$
2. $-\sqrt{81} = -9$
3. $\sqrt{144} = 12$
4. $\sqrt{15^2} = 15$
5. $(\sqrt{6})^2 = 6$
6. $-\sqrt{400} = -20$
7. $\sqrt{\frac{1}{64}} = \frac{1}{8}$
8. $\sqrt{9 \cdot 49} = \sqrt{3^2 \cdot 7^2} = \sqrt{3^2} \cdot \sqrt{7^2} = 3 \cdot 7 = 21$
9. $\sqrt{\frac{81}{36}} = \frac{\sqrt{81}}{\sqrt{36}} = \frac{\sqrt{9^2}}{\sqrt{6^2}} = \frac{9}{6} = \frac{3}{2}$
10. $-\sqrt{625} = -\sqrt{25^2} = -25$

Scientific Notation

1. $(2 \times 10^{-2})(3 \times 10^{-2}) = 6 \times 10^{-4}$
2. $(6 \times 10^{-8})/(3 \times 10^3) = 2 \times 10^{-11}$
3. $(9 \times 10^4)(-1 \times 10^{-2}) = -9 \times 10^2$
4. $(3 \times 10^{-7})(9 \times 10^2) = 2.7 \times 10^{-4}$
5. $(7E2)(6E4) = 4.2E7$
6. $(5E-3)/(5E-2) = 1E-1$

Order of Operations

1. $5 + (-3) \cdot (-2) - 6 = 5 + 6 - 6 = 5$
2. $18 - [52 \div (7 + 6)] = 18 - (52 \div 13) = 18 - 4 = 14$
3. $19 - 7 + 12 \cdot 2 \div 8 = 12 + 24 \div 8 = 12 + 3 + 15$
4. $2^3 - 20 \div 4 + 4 \cdot 3 = 8 - 5 + 12 = 3 + 12 = 15$
5. $10 + 6(0.5)^3 = 10 + 6(0.125) = 10 + 0.75 = 10.75$
6. $3^3 + 10 \div 5 = 27 + 2 = 29$
7. $(8 + 4 \cdot 3)^2 = (8 + 12)^2 = 20^2 = 400$
8. $7(4^2 - 10) \div (12 \cdot \frac{3}{4}) = 7(16 - 10) \div 9 = 7(6) \div 9 = 42 \div 9 = 4.667$
9. $7(6 - 2^2) = 7(6 - 4) = 7(2) = 14$
10. $(57 - 2^5)^{\frac{1}{2}} = (57 - 32)^{\frac{1}{2}} = 25^{\frac{1}{2}} = 5$

Algebra

$$\begin{aligned}
 1. \quad x + 3 &= 10 \\
 x + 3 - 3 &= 10 - 3 \\
 x &= 7
 \end{aligned}$$

$$\begin{aligned}
 2. \quad 5 + z &= 8 \\
 5 + z - 5 &= 8 - 5 \\
 z &= 3
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \frac{y}{4} &= 2 \\
 4 \cdot \frac{y}{4} &= 2 \cdot 4 \\
 y &= 8
 \end{aligned}$$

$$\begin{aligned}
 \frac{r}{2} &= 6 & 4. \\
 2 \cdot \frac{r}{2} &= 6 \cdot 2 \\
 r &= 12
 \end{aligned}$$

$$\begin{aligned}
 5. \quad 5(j + 5) &= 45 \\
 \frac{5(j + 5)}{5} &= \frac{45}{5} \\
 j + 5 &= 9 \\
 j + 5 - 5 &= 9 - 5 \\
 j &= 4
 \end{aligned}$$

$$\begin{aligned}
 6. \quad 5d &= 25 \\
 \frac{5d}{5} &= \frac{25}{5} \\
 d &= 5
 \end{aligned}$$

$$\begin{aligned}
 7. \quad 9n^2 &= 81 \\
 \frac{9n^2}{9} &= \frac{81}{9} \\
 n^2 &= 9 \\
 \sqrt{n^2} &= \sqrt{9} \\
 n &= 3
 \end{aligned}$$

$$\begin{aligned}
 8. \quad a + 6 - 6a &= -14 \\
 6 - 5a &= -14 \\
 6 - 5a - 6 &= -14 - 6 \\
 -5a &= -20 \\
 \frac{-5a}{-5} &= \frac{-20}{-5} \\
 a &= 4
 \end{aligned}$$

$$\begin{aligned}
 9. \quad a^2b &= c^2d \\
 \frac{a^2b}{b} &= \frac{c^2d}{b} \\
 a^2 &= \frac{c^2d}{b} \\
 \sqrt{a^2} &= \sqrt{\frac{c^2d}{b}} \\
 a &= \sqrt{\frac{c^2d}{b}}
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \frac{b}{a} &= \frac{c}{d} \\
 a \cdot \frac{b}{a} &= \frac{c}{d} \cdot a \\
 b &= \frac{ca}{d} \\
 d \cdot b &= \frac{ca}{d} \cdot d \\
 db &= ca \\
 \frac{db}{c} &= \frac{ca}{c} \\
 \frac{db}{c} &= a
 \end{aligned}$$

$$\begin{aligned}
 11. \quad &3x + \\
 &3(5) \\
 &15 + \\
 &43 - \\
 &33
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \frac{4s + 1}{6t + s} \quad &\text{where } s = -3, t = 10 \\
 \frac{4(-3) + 1}{6(10) + (-3)} & \\
 \frac{-12 + 1}{60 + (-3)} & \\
 \frac{-11}{57} &
 \end{aligned}$$

$$\begin{aligned}
 13. \quad \frac{16a + 5b}{1 - 4a} \quad &\text{where } a = -1, b = 5 \\
 \frac{16(-1) + 5(5)}{1 - 4(-1)} & \\
 \frac{-16 + 25}{1 + 4} & \\
 \frac{9}{5} & \\
 1\frac{4}{5} &
 \end{aligned}$$

14. $3i + \frac{j}{1+k} = 0$ where $k = 2, j = -12$, solve for i *
- $$3i + \frac{-12}{1+2} = 0$$
- $$3i - 4 = 0$$
- $$3i = 4$$
- $$i = \frac{4}{3}$$
- $$i = 1\frac{1}{3}$$
-
15. $7(2m - \frac{3}{4}n) + l = 10$ where $m = \frac{1}{2} n = 8$, solve for l *
- $$7[2(\frac{1}{2}) - \frac{3}{4}(8)] + l = 10$$
- $$7(1 - 6) + l = 10$$
- $$7(-5) + l = 10$$
- $$-35 + l = 10$$
- $$l = 45$$

Logarithms

1. $\log 6.40 = \mathbf{0.8062}$
2. $\log 0.5 = \mathbf{-0.301}$ *
3. $\log^{-1} 16 = \mathbf{1 \times 10^{16}}$
4. $\log^{-1} 3.7846 = \mathbf{6089.76}$
5. $\ln 86 = \mathbf{4.45}$
6. $\ln 0.5 = \mathbf{-0.693}$
7. $\ln^{-1} 0.695 = \mathbf{2.004}$
8. $\ln^{-1} 1 = e = \mathbf{2.7182 \dots}$

$5 = 4^x$	$23 = 6^t$	9.	10.
$\log 5 = \log [4^x]$	$\log 23 = \log [6^t]$		
$\log 5 = (\log 4)(x)$	$\log 23 = (\log 6)(t)$		
$\frac{\log 5}{\log 4} = x$	$\frac{\log 23}{\log 6} = t$		
$\frac{0.69897}{0.60206} = x$	$\frac{1.36173}{0.77815} = t$		
$1.161 = x$	$1.74995 = x$		

$9 = 22(3^y)$	$50 = 2000\left(\frac{1}{2}\right)^n$	11.	12.
$\frac{9}{22} = 3^y$	$\frac{50}{2000} = \left(\frac{1}{2}\right)^n$		
$\log \frac{9}{22} = \log [3^y]$	$0.025 = (0.5)^n$		
$\log \frac{9}{22} = (\log 3)(y)$	$\log 0.025 = \log [0.5^n]$		
$\frac{\log \frac{9}{22}}{\log 3} = y$	$\frac{\log 0.025}{\log 0.5} = n$		
$\frac{-0.38818}{0.47712} = y$	$\frac{-1.60206}{-0.30103} = n$		
$-0.8136 = y$	$5.3219 = n$		