Thin Quad Lattices

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Consider several thin quad lattices, and their parameters. In particular, a FODO lattice, a doublet lattice (really a special case of the FODO lattice), and a triplet lattice.

I. GENERAL

A Symmetric Quad Lattices

Let's say that the transfer matrix for a set of (not necessarily thin) quadrupoles is given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{1}$$

This transfer matrix comes from the differential equations

$$\frac{dx}{ds} = p \qquad \frac{dp}{ds} = -K(s)x \tag{2}$$

Let's find the transfer matrix for a symmetric lattice consisting of the lattice described by the transfer matrix A followed by the same lattice but with the quadrupoles reversed and in the reverse order.

The differential equations describing the reversed lattice starting at L (the length of the original lattice) are

$$\frac{dx}{ds} = p \qquad \frac{dp}{ds} = -K(2L-s)x \tag{3}$$

Making a change of variables to

$$\bar{x} = x \qquad \bar{p} = -p \qquad \bar{s} = 2L - s \tag{4}$$

the differential equations in these new variables are

$$\frac{d\bar{x}}{d\bar{s}} = \bar{p} \qquad \frac{d\bar{p}}{d\bar{s}} = -K(\bar{s})\bar{x} \tag{5}$$

and thus the transfer map in these new coordinates is the same as the transfer map for the first half of the lattice in the original coordinates. It is easy to transform the transfer map back into the original coordinates (remembering that $\bar{s}$ and s have a sign difference); it is

$$\bar{A} = \begin{bmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{bmatrix} \tag{6}$$

Thus, the map from beginning to end is

$$\bar{A}A = \begin{bmatrix} a_{11}a_{22} + a_{12}a_{21} & 2a_{12}a_{22} \\ 2a_{21}a_{11} & a_{11}a_{22} + a_{12}a_{21} \end{bmatrix} \tag{7}$$

II. LATTICE DESIGNS

A FODO Lattice

Consider the lattice

- Focusing quadrupole, strength F
- Drift, length L
- Defocusing quadrupole, strength F
- Drift, length L.

The transfer map from the center of the focusing quadrupole is

$$\begin{bmatrix} 1 & 0 \\ -F/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ F & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -F/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{L^2F^2}{2} & 2L\left(1 + \frac{LF}{2}\right) \\ -\frac{LF^2}{2}\left(1 - \frac{LF}{2}\right) & 1 - \frac{L^2F^2}{2} \end{bmatrix}. \tag{8}$$

The beta function at this point is

$$\frac{2}{|F|}\sqrt{\frac{1 + LF/2}{1 - LF/2}}. \tag{9}$$

For a given value of L, the minimum value of the beta function occurs when $LF = \sqrt{5} - 1$, at which point the beta function is

$$\sqrt{\frac{11}{2} + \frac{5\sqrt{5}}{2}}L \approx 3.330L. \tag{10}$$

B Doublet Lattice

Consider the following lattice:

- Focusing quadrupole, strength F
- Drift, length x
- Defocusing quadrupole, strength F
- Drift, length L

The one-cell transfer matrix on the L side of the focusing quadrupole is

$$\begin{bmatrix}1 & 0\\ -F & 1\end{bmatrix}\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ F & 1\end{bmatrix}\begin{bmatrix}1 & L\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1+xF & L+x+xLF\\ -xF^2 & 1-xF-xLF^2\end{bmatrix}. \tag{11}$$

The phase advance per cell μ is given by

$$\sin\mu = \sqrt{xLF^2 - \frac{x^2L^2F^4}{4}} \tag{12}$$

and the maximum beta function in the cell is

$$\beta_{\max} = \frac{L+x+xLF}{\sqrt{xLF^2 - \frac{x^2L^2F^4}{4}}} \tag{13}$$

Fixing x and L and minimizing with respect to x, the minimum will occur when $u = \sqrt{xL}F$ is a solution of

$$\frac{\sqrt{xL}}{4(L+x)}u^3 + \frac{u^2}{2} - 1 = 0. \tag{14}$$

We can write the solution to this as a power series in $a = \sqrt{xL}/(L+x)$:

$$u = \sqrt{2} - \frac{a}{2} + \frac{5\sqrt{2}a^2}{16} + O(a^3) \tag{15}$$

and the maximum β at this value is

$$x + L + \sqrt{2xL} - \frac{1}{4}\frac{xL}{x+L} + (x+L)O\left(\left(\frac{\sqrt{xL}}{L+x}\right)^3\right). \tag{16}$$

The maximum value a can be is 0.5, and at that value of a, $u = \sqrt{5} - 1$ and

$$\beta_{\max} = \frac{1}{2}\sqrt{\frac{11}{2} + \frac{5\sqrt{5}}{2}}(L+x) \approx 1.665(L+x). \tag{17}$$

Thus, the maximum value for the double lattice can be made to be less than half of what the FODO lattice would give for an equivalent spacing between magnets. The cost is that the doublet requires a pair of magnets for each single magnet in the FODO lattice, and those magnets must be substantially stronger than the magnets in the FODO lattice were.

C Triplet

Consider the following lattice:

- Defocusing quadrupole, strength D
- Drift, length x
- Focusing quadrupole, strength F
- Drift, length x
- Defocusing quadrupole, strength D
- Drift, length L.

The map from the center of the focusing quadrupole to the center of the drift is

$$\begin{bmatrix}1 & L/2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ D & 1\end{bmatrix}\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -F/2 & 1\end{bmatrix} = \begin{bmatrix}1 + \frac{LD}{2} - \frac{xF}{2} - \frac{LxDF}{4} & x + \frac{L}{2} + \frac{LDx}{2}\\ D - \frac{F}{2} - \frac{xDF}{2} & 1 + xD\end{bmatrix} \tag{18}$$

TABLE I: Minimum momenta for beam to fit in beam pipe.

f MHz	cells	k	$p_{\min}$ MeV/c
200	1	4	160
400	2	4	639
800	4	4	2554

III. APPLICATION TO NEUTRINO FACTORY LINACS

Let's start out with a scaled TESLA cavity design. The inside beam pipe radius for the cavity at 1300 MHz is 3.5 cm. Call the cavity radius a, and set this equal to $k\sigma$ for some k, where σ is the RMS beam sigma at the maximum of the β-function.

A Beta Function Constraints

The beam will fit into the beam pipe as long as

$$p \geqslant mc\frac{k^2\epsilon_\perp\beta_{\max}}{a^2}. \tag{19}$$

Assuming a 50% filling factor, Tab. I gives the minimum momenta for various scenarios, assuming a doublet lattice.

B Magnet Lengths

The maximum value that the focusing strength will have is

$$F_{\max} = \sqrt{\frac{2}{xL}}. \tag{20}$$

The focusing strength is

$$F = \frac{qB\ell}{pa} \tag{21}$$

where ℓ is the magnet length, B is the pole tip field, q is the particle charge, and p is the particle's momentum. The distance between the quad centers x must be at least equal to ℓ; define $\alpha = x/\ell$. Then, setting the expressions for F equal to one another, and calling $D = x + L$ and keeping D fixed, we have an expression for $v = x/D$:

$$v^3(1 - v) = 2\left(\frac{pa\alpha}{qBD^2}\right)^2 \tag{22}$$

First of all, note that the maximum value of the left hand side occurs when $v = 3/4$, and is $27/256$. Second, note that an approxite expression for v in the solution of $v^3(1 - v) = y$, good to a relative accuracy better than 2.3%, is

$$y^{1/3}\left(1 - \frac{256}{27}y\right) + \frac{256}{27}y\left(\frac{3}{4} - \sqrt{\frac{8}{9}\left(\frac{27}{256} - y\right)}\right) \tag{23}$$

REFERENCES

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