ASYMPTOTICALLY OPTIMAL SPHERICAL CODES*

Jon Hamkins
hamkins@uiuc.edu

Kenneth Zeger
zeger@uiuc.edu

Coordinated Science Lab., University of Illinois, 1308 W. Main St., Urbana, IL 61801

Abstract

A technique analogous to laminated lattice construction is given for creating spherical codes. These "laminated spherical codes" outperform many other spherical codes, such as those derived from lattices, concatenated codes, and other lower bounds. Additionally, for fixed dimension k, the density of codepoints on the unit sphere approaches the best known sphere packing *density in $k-1$ dimensions for $k = 3-8$, as the minimum distance $d \to 0$. In particular, the three-dimensional laminated spherical code is asymptotically optimal, in the sense that its density approaches the well-known Fejes Tóth upper bound as $d \to 0$. Our codes are also highly structured, which simplifies decoding.*

1 Introduction

A *spherical code* is a set of codepoints on the surface of a multi-dimensional unit radius sphere. Let S_k be the surface of the unit radius k-dimensional sphere:

$$S_k \equiv \{(x_1, \ldots, x_k) : \sum_{i=1}^{k} x_i^2 = 1\}.$$

Denote the surface area of S_k by $A(S_k)$, and the volume by $V(S_k)$. For a k-dimensional spherical code $\mathcal{C} \subset S_k$, the *minimum distance* is defined as

$$d \equiv \min_{\substack{\mathbf{x}, \mathbf{y} \in \mathcal{C} \\ \mathbf{x} \neq \mathbf{y}}} ||\mathbf{x} - \mathbf{y}||,$$

where $||\mathbf{x} - \mathbf{y}||$ is the Euclidean distance between codepoints $\mathbf{x}$ and $\mathbf{y}$. $\mathcal{C}(k, d)$ denotes a k-dimensional spherical code with minimum distance d. The angular separation between two points $\mathbf{x}, \mathbf{y} \in S_k$ is $\cos^{-1}(\mathbf{x} \cdot \mathbf{y})$. The *minimum angular separation* (expressed in terms of d), is defined as

$$\theta \equiv 2 \sin^{-1}(d/2). \tag{1}$$

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The set of points on S_k whose angular separation from a fixed point $\mathbf{y}$ in S_k is at most ϕ is called a *spherical cap* centered at $\mathbf{y}$ with angular radius ϕ, denoted $c(k, \phi)$. That is,

$$c(k, \phi) \equiv \{\mathbf{x} \in S_k : \mathbf{x} \cdot \mathbf{y} \geq \cos \phi\}.$$

Note that if the codepoints of a spherical code with minimum distance d are the centers of spherical caps of angular radius $\theta/2$, then none of the caps overlap. Let $A(c(k, \phi))$ denote the surface area of $c(k, \phi)$.

The *density* $\Delta_{\mathcal{C}(k,d)}$ of spherical code $\mathcal{C}(k, d)$ is defined as the ratio of the surface area of $|\mathcal{C}(k, d)|$ nonoverlapping spherical caps with angular radius $\theta/2$ to the surface area of S_k:

$$\Delta_{\mathcal{C}(k,d)} \equiv \frac{|\mathcal{C}(k, d)| \cdot A(c(k, \theta/2))}{A(S_k)}.$$

Let

$$\Delta(k, d) \equiv \max_{\mathcal{C}(k,d)} \Delta_{\mathcal{C}(k,d)}, \tag{2}$$

and let $\Delta_k \equiv \limsup_{d \to 0} \Delta(k, d)$. Define Δ_k^{pack} to be the highest sphere packing density in k dimensions.

It is currently an open problem to determine $\Delta(k, d)$ for $k \geq 3$, although a few special cases have been solved and a number of bounds have been proposed [Ran55, Yag58, Fej59, Wyn65, Cox68, DGS77, GHSW87, EZ95]. For small d, [Fej59] and [Cox68] provide the tightest known upper bound and [GHSW87] provides the tightest known lower bound. There is a gap between these two bounds. We construct spherical codes which provide a new lower bound that asymptotically meets the upper bound of [Fej59] for dimension 3, as shown in Table 1.

We refer to a class of codes $\{\mathcal{C}(k, d)\}$ as *asymptotically optimal* if $\Delta_{\mathcal{C}(k,d)}/\Delta(k, d) \to 1$ as $d \to 0$. Although $\Delta(k, d)$ is not known, it is not hard to prove that $\Delta_k \leq \Delta_{k-1}^{pack}$ for all k, which can be used to show that laminated spherical codes developed in this paper are asymptotically optimal for $k = 3$. For $4 \leq k \leq 8$, the laminated spherical codes are asymptotically optimal if and only if the the best known sphere packing

Reference	Asymptotic density achieved	k						
		2	3	4	5	6	7	8
	$\Delta_k^L \equiv \frac{\Delta_{k-1}^L V(S_{k-1})}{2\sqrt{1-c_k^2}V(S_{k-2})} =$	1	.9069	.7405	.6169	.4653	.3729	.2953
[GHSW87]	$\frac{\Delta_{k-1} V(S_{k-1})}{2V(S_{k-2})} =$	1	.7854	.6046	$\leq .4585$	$\leq .3455$	$\leq .2581$	$\leq .1916$
[Yag58]	$\frac{\Delta_{k-1}^{pack} V(S_{k-1})}{A(S_k)} =$	1	.2500	.1925	$\leq .1460$	$\leq .1100$	$\leq .08214$	$\leq .06100$
[Wyn65]	$2^{1-k} =$	.5000	.2500	.1250	.06250	.03125	.01563	.007813
	Parameters used							
[Lee71]	$\Delta_{k-1}^{pack} \geq$ [best packing density]	1	$\frac{\pi}{2\sqrt{3}}$ $= .9069$	$\frac{\pi}{3\sqrt{2}}$ $= .7405$	$\frac{\pi^2}{16}$ $= .6169$	$\frac{\pi^2}{15\sqrt{2}}$ $= .4653$	$\frac{\pi^3}{48\sqrt{3}}$ $= .3729$	$\frac{\pi^3}{105}$ $= .2953$
[Rog58]	$\Delta_{k-1}^{pack} \leq$	1	$\pi/(2\sqrt{3})$	.7784	.6478	.5257	.4192	.3298
[Sch85]	$V(S_k) \equiv$ Volume of S_k	π	$4\pi/3$	$\pi^2/2$	$8\pi^2/15$	$\pi^3/6$	$16\pi^3/105$	$\pi^4/24$
[Sch85]	$A(S_k) \equiv$ Area of S_k	2π	4π	$2\pi^2$	$8\pi^2/3$	π^3	$16\pi^3/15$	$\pi^4/3$
	c_k^2		$1/4$	$1/3$	$1/2$	$1/2$	$5/8$	$2/3$

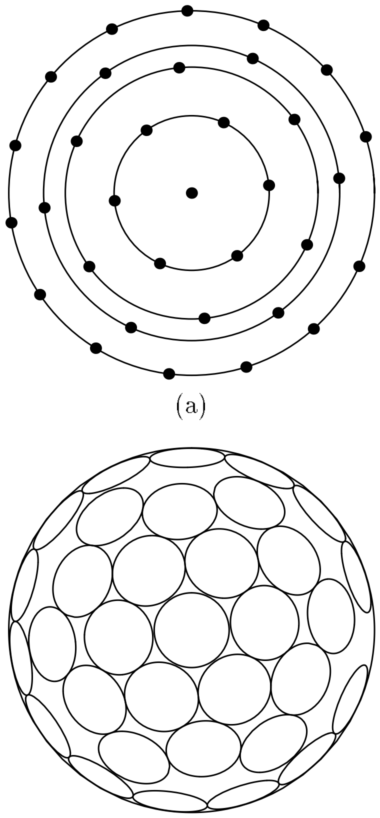
Table 1: Summary of asymptotic performance of various spherical codes, as $d \to 0$. The asymptotic performance may be expressed precisely in terms of the parameters of the bottom rows.

in the previous dimension is optimal. A corollary of this is that the apple-peeling lower bound [GHSW87] is *not* asymptotically optimal for $3 \leq k \leq 8$.

A summary of the densities of the best spherical codes and sphere packings is given in Table 1. In Section 2, our laminating technique for creating spherical codes is given and shown to be asymptotically optimal for $k = 3$. A plot compares the codes constructed here to many other codes. In Section 3 the structure of the code is shown to allow easy decoding of a received point.

2 Laminated spherical codes

By placing codepoints on concentric $(k-1)$-dimensional spheres and projecting the points onto S_k, a k-dimensional spherical code may be obtained. By nesting the concentric spheres closely, and placing codepoints of one sphere at the radial extension of the deep-holes of codepoints of the previous sphere, the method of constructing laminated lattices comes to mind (see, e.g., [CS88]), and thus we denote our codes by $\mathcal{C}^L$. The procedure is illustrated with $k = 3$ in Figure 1. The method is similar to the Yaglom [Yag58] and apple-peeling [GHSW87] lower bounds in that a projection from $k-1$ dimensions to k dimensions is used; the difference lies in the placement of points before the projection. The technique is practical for creating codes of any size and thus provides a lower bound on achievable minimum distance as a function of code size. For $k = 3$, this lower bound is tighter than the Yaglom and apple-peeling lower bounds, and even tight enough to prove its asymptotic optimality.



(a)

(b)

Figure 1: (a) 5 scaled 2-dimensional codes. (b) A 3-dimensional code derived from the 2-dimensional codes by projecting codepoints out of the page. Codepoints are the centers of the caps.

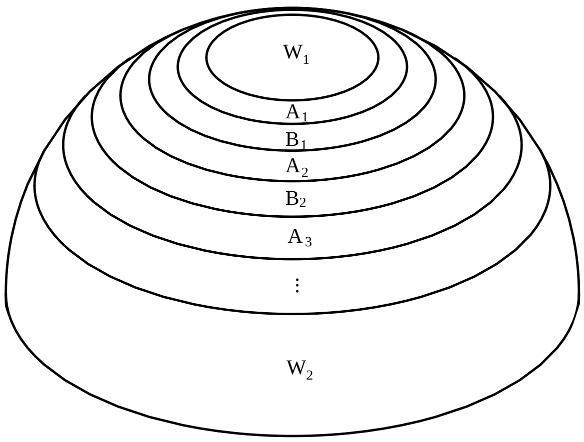


Figure 2: The sphere consists of annuli, buffer zones, and wasted regions. In general, W_1 and W_2 may encompass one or more annuli. Shown is the case $k=3$.

2.1 The laminated spherical code construction

Fix the dimension k and minimum distance $d \in (0,1]$. Given a finite increasing sequence $r_0, \ldots, r_N$ of real numbers in $[0,1]$, we define a collection of *shells,* where the ith shell consists of those points in $\mathbb{R}_{k-1}$ whose distance to the origin is in the interval $(r_{i-1}, r_i]$. Define $g\colon \mathbb{Z} \to \mathbb{Z}$ by $g(i) = \min\{j\colon r_j \geq i\sqrt{d}\}$ and $s\colon \mathbb{Z} \to \mathbb{Z}$ by $s(i) = g(\max\{j\colon r_{g(j)} \leq r_i\})$. We refer to the *ith annulus* as the set of points in $\mathbb{R}_{k-1}$ whose distance to the origin is in the interval $(r_{g(i)}, r_{g(i+1)-1}]$. That is, an annulus is a collection of consecutive shells. Note that $s(i)$ is the smallest integer for which the set of points whose distance from the origin is in the interval $(r_{s(i)}, r_i]$ is a subset of one annulus. Between the $(i-1)$th and ith annuli is the *ith buffer zone.* We call shells, annuli, and buffer zones which are projected onto the unit k-dimensional sphere S_k again shells, annuli, and buffer zones, respectively. These quantities are formally defined by:

$$
\begin{aligned}
T_i &= \left\{(x_1,\ldots,x_k) \in S_k\colon \sqrt{\sum_{i=1}^{k-1} x_i^2} \in (r_{i-1}, r_i]\right\} \\
A_i &= \bigcup_{j=g(i)+1}^{g(i+1)-1} T_j \\
B_i &= \left\{(x_1,\ldots,x_k) \in S_k\colon \sqrt{\sum_{i=1}^{k-1} x_i^2} \in (r_{g(i)-1}, r_{g(i)}]\right\} \\
W_1 &= \left\{(x_1,\ldots,x_k) \in S_k\colon \sqrt{\sum_{i=1}^{k-1} x_i^2} < d^{1/4}\right\} \\
W_2 &= \left\{(x_1,\ldots,x_k) \in S_k\colon \sqrt{\sum_{i=1}^{k-1} x_i^2} > 1 - d^{1/4}\right\} \\
W &= W_1 \cup W_2 \\
T &= \cup_i T_i = \cup_i A_i \\
B &= \cup_i B_i .
\end{aligned}
$$

W_1 and W_2 are "wasted" regions, in which codepoints are not necessarily as tightly packed as the rest of the sphere. Each of the sets above is a function of the sequence of radii $\{r_i\}$ or d. For any $\{r_i\}$ and d, $S_k = W \cup T \cup B$, as shown in Figure 2.

For each $i \in \{0,\ldots,N\}$ let

$$\theta_i = 2\sin^{-1}\left(\frac{d}{2r_{s(i)}}\right). \tag{3}$$

We define the k-dimensional *laminated spherical code* with minimum distance d by

$$
\begin{aligned}
\mathcal{C}(k,d) \equiv \Bigg\{ & \left(x_1,\ldots,x_{k-1}, \pm\sqrt{1-\sum_{i=0}^{k-1} x_i^2}\right) : \\
& (x_1,\ldots,x_{k-1}) \in \bigcup_{i=0}^{N} \mathcal{C}_i(k-1, r_i d/r_{s(i)}) \Bigg\},
\end{aligned}
$$

where $\mathcal{C}_i(k-1, r_i d/r_{s(i)})$ is a special type of $(k-1)$-dimensional code with minimum distance $d/r_{s(i)}$ that has been scaled to lie on a sphere of radius r_i (and hence to have minimum distance $r_i d/r_{s(i)}$). We describe the construction of these $(k-1)$-dimensional codes next. The innermost code of an annulus, i.e., one of the form $\mathcal{C}_{g(j)}(k-1,d)$, is defined to be merely a $(k-1)$-dimensional spherical code determined by this method. The remaining codes of the annulus are built up in layers, using the deep-holes of the previous layer. In particular, the number of codepoints in each layer of an annulus will be the same. The codepoints of $\mathcal{C}_{g(j+2m)}(k-1, r_{g(j+2m)} d/r_{g(j)})$, or $\mathcal{C}_{g(j+2m)}$ for short, will lie at radial extensions of codepoints of $\mathcal{C}_{g(j)}$, while the codepoints of $\mathcal{C}_{g(j)+2m}$ will be radial extensions of deep-holes of codepoints of $\mathcal{C}_{g(j)}$. The determination of the deep-holes is the same as the determination of the deep-holes for the laminated lattices—they are the centroids of certain codepoints.

When $k=3$, the layers of 2-dimensional codes is easily defined explicitly. For each $i \in \{0,\ldots,N\}$, the set defined by

$$
\mathcal{C}_i(2,d) \equiv \begin{cases} \{(r_i\cos j\theta_i, r_i \sin j\theta_i)\} & i \text{ odd} \\ \{(r_i\cos(j+\frac{1}{2})\theta_i, r_i\sin(j+\frac{1}{2})\theta_i)\} & i \text{ even,} \end{cases}
$$

$j \in \{0,\ldots,\lfloor 2\pi/\theta_i\rfloor - 1\}$, is a 2-dimensional spherical code on a circle of radius r_i. Note that every code in a given annulus has the same number of points.

It remains to determine a formula for the smallest allowable radius r_i for the next concentric sphere, given $r_0, \ldots, r_{i-1}$. The quantity r_i is chosen as small as possible and yet large enough so that the codepoints at radius r_i are at least distance d from the codepoints at radius r_{i-1} after projecting them onto the k-dimensional sphere. The solution is obtained by:

$$\begin{aligned} r_i \;=\; & \left(r_{i-1}\left(1-\frac{d^2}{2}\right)\sqrt{1-(\frac{c_k d}{r_{s(i)}})^2} \right. \\ & \left. + \, d\sqrt{(1-r_{i-1}^2)(1-\frac{d^2}{4}-(\frac{c_k r_{i-1}}{r_{s(i)}})^2)} \right) \Bigg/ \\ & \left(1-\left(\frac{r_{i-1}c_k d}{r_{s(i)}}\right)^2\right). \end{aligned} \tag{4}$$

The solution above is used only for shells within the same annulus. Between annuli, the radius r_i is defined such that every point on the ith shell is at least a distance d from every point on the $(i-1)$th shell. This gives:

$$r_i = r_{i-1}(1-\frac{d^2}{2}) + d\sqrt{(1-r_{i-1}^2)(1-\frac{d^2}{4})}. \tag{5}$$

Summarizing, the r_i's may be determined by the following algorithm:

$r_s := r_0 := i := 0$;
while $r_i \le \sqrt{1-d^2/4}$ {
  $i := i+1$;
  $r :=$ (RHS of (4));
  if $(r \in \mathbb{R})$ **and** $(r > r_i)$ **and** $(r - r_s < \sqrt{d})$
    then $r_i := r$; /* regular solution */
  else { /* begin new annulus */
    $r_s := r_i :=$ (RHS of (5));
  }
}
$N := i - 1$;

The apple-peeling spherical code and $\mathcal{C}^L$ are compared in Figure 3 when $k=3$ and $d=0.05$.

2.2 Laminated spherical code density

We are now ready to compute the density of a laminated spherical code. Let $\Delta^L(k,d)$ be the density of the k-dimensional laminated spherical code with minimum distance d, and let $\Delta_k^L = \lim_{d\to 0}\Delta^L(k,d)$.

Theorem 1 *The density of a laminated spherical code in k dimensions satisfies* $\Delta^L(k,d) = \frac{\Delta_{k-1}^L V(S_{k-1})}{2\sqrt{1-c_k^2}V(S_{k-2})} - O(d^{1/4})$.

Sketch of proof: Within an annulus, layers of shells are stacked much the same as layers of lattices are stacked in the construction of laminated lattices. Thus, we expect that the density of $\mathcal{C}^L(k,d)$ within an annulus should be nearly the best packing density in $k-1$ dimensions. After much algebra and asymptotic analysis, the density of codepoints within region T, denoted δ_T can be bounded as:

$$\delta_T \ge \frac{\Delta_{k-1}^L V(S_{k-1})}{2\sqrt{1-c_k^2}V(S_{k-2})} - O(d^{1/4}). \tag{6}$$

Consider now the buffer zones. Since $r_{g(j)} - r_{g(j)-1} < d$, the area of the jth buffer zone B_j is bounded above by $A(S_{k-1})d$. By construction, the total number of annuli in $\mathcal{C}^L$is $2\lceil d^{-1/2}\rceil$, where we include annuli with both positive and negative kth coordinates. Hence, the total area of $B = \cup_i B_i$ is

$$A(B) < 2A(S_{k-1})d\lceil d^{-1/2}\rceil = O(d^{1/2}),$$

and $A(W) = O(d^{1/4})$. Thus,

$$\begin{aligned} \Delta^L(k,d) \;&\ge\; \frac{\delta_T A(T)}{A(S_k)} = \delta_T(1-O(d^{1/4})) \\ &=\; \frac{\Delta_{k-1}^L V(S_{k-1})}{2\sqrt{1-c_k^2}V(S_{k-2})} - O(d^{1/4}).\blacksquare \end{aligned}$$

Corollary 1 *As $d \to 0$, the density of the laminated spherical code $\mathcal{C}^L(k,d)$ approaches the density of the best known $(k-1)$-dimensional sphere packing. In particular, $\mathcal{C}^L(3,d)$ is asymptotically optimal, and the Fejes Tóth upper bound is asymptotically tight.*

Shown in Figure 4 is a plot of the proposed lower bound $\Delta^L(3,d)$ versus d. The quantity $\Delta^L(3,d)$ is derived from actual code constructions, and a logarithmic scale is used for d. All code sizes are normalized by the Fejes Tóth upper bound, i.e., $\Delta^L(3,d)/\Delta_F(3,d)$ is plotted versus d, where $\Delta_F(3,d)$ is the upper bound on density. For small d, the code performance is better than any previous codes, and the convergence to the upper bound is evident. Included in the plot are other codes obtained by the authors using a simulating annealing approach which improves upon [GHSW87]. This method produces good codes, but due to time constraints is limited in the code size that can be constructed. Another unstructured code approach is used in a program by [Slo94], which has produced many of the best known codes for small code sizes. Spherical codes can also be generated from shells of lattices (see e.g., [CS88]). Figure 4 shows codes generated from the first 1000 shells of the face-centered cubic and Z_3 lattices, whose exact minimum distances were obtained by a computer search. Figure 4 also shows spherical codes formed from concatenations of MPSK and 2-AM codes. As can be seen, for $d < 0.1$ all of these codes are outperformed by the proposed laminated codes.

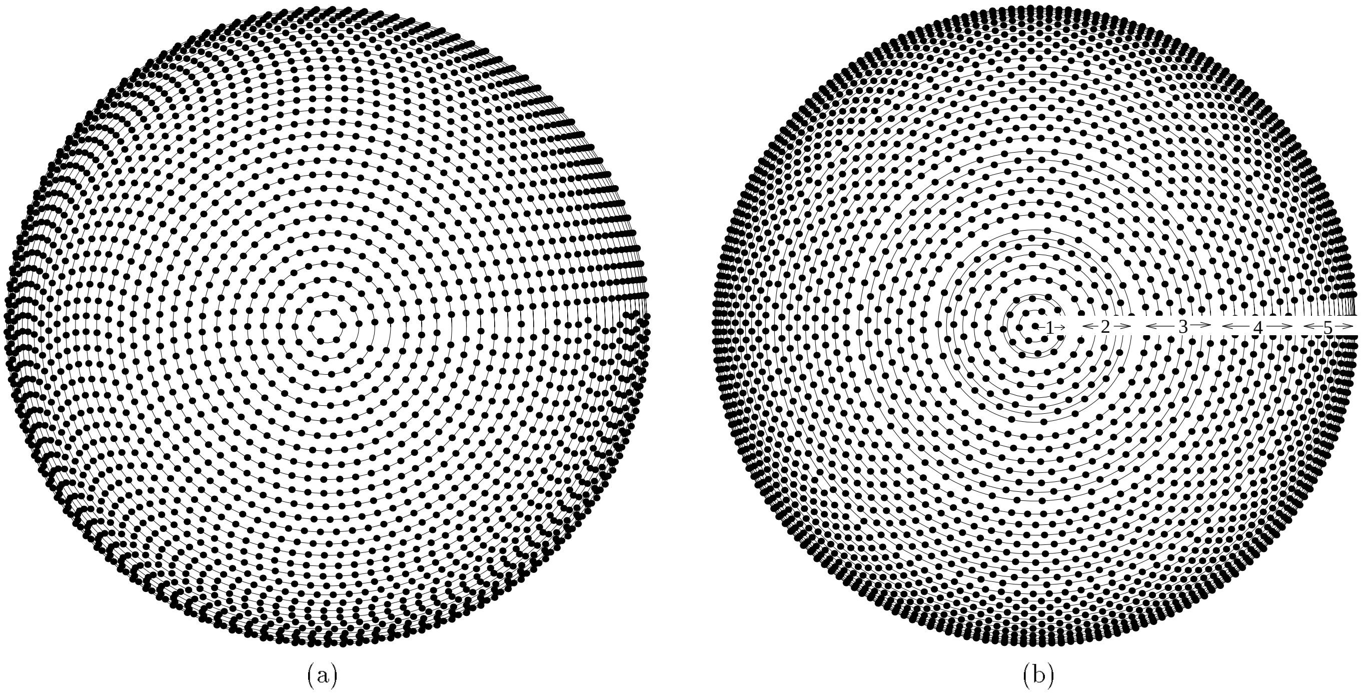


Figure 3: $k = 3$, $d = 0.05$. (a) The apple-peeling code has 4764 codepoints. Any shell of this code may be rotated without affecting the minimum distance. (b) $\mathcal{C}^L$ has 5230 codepoints. Codepoints in adjacent shells are interlaced. Any annulus may be rotated without affecting the minimum distance, but individual shells may not be rotated. The 5 annuli are numbered.

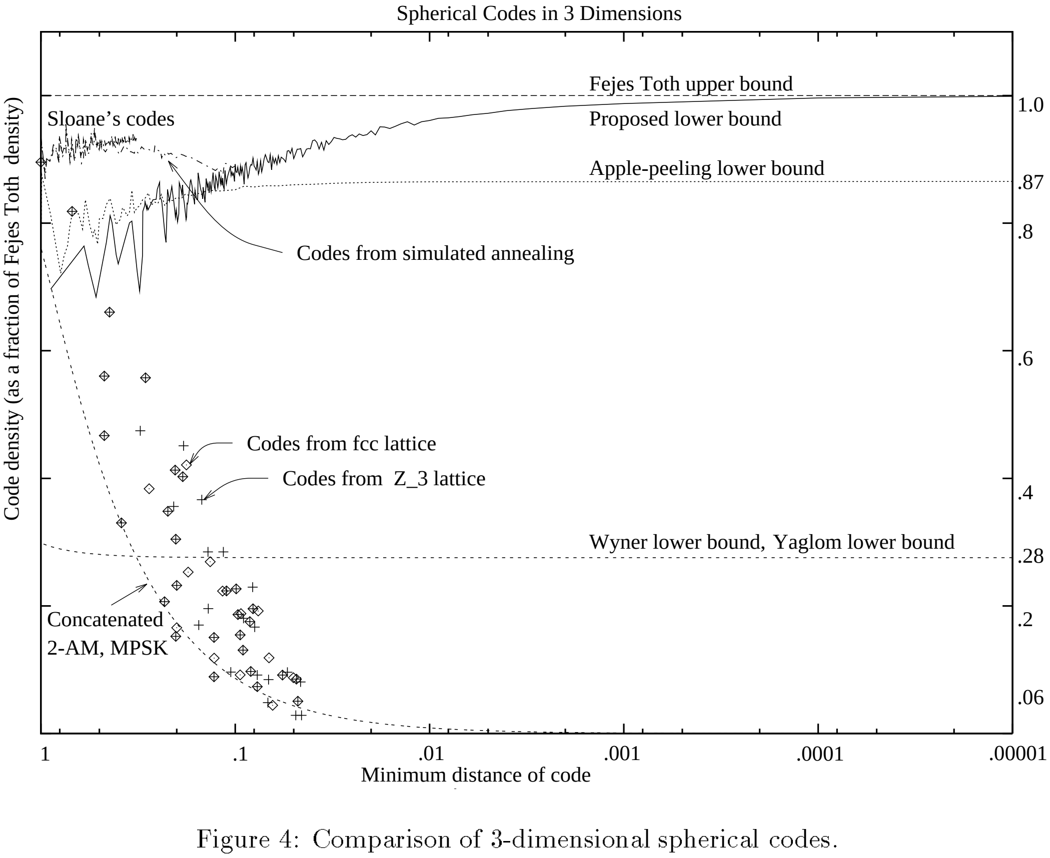


Figure 4: Comparison of 3-dimensional spherical codes.

3 Decoding

Let $\mathbf{r} \in S_k$. An important question in channel coding and quantization theory is how to find the nearest codepoint to $\mathbf{r}$ in an efficient manner [GRH88, MN91]. An advantage of having a structured code in this case is that the codepoints themselves need not be stored. We consider two approaches to decoding, reflecting a tradeoff in the time and space complexity of the decoding: namely, (I) the decoder stores the sequence of radii $\{r_i\}$, or (II) the decoder stores only the minimum distance d.

We can show that if the spherical code $\mathcal{C}^L(k,d)$ has size M, then there is an optimal decoder using $O(\sqrt{M})$ space and $O(\log M)$ time, or an optimal decoder using $O(1)$ space and $O(\sqrt{M})$ time. This may be proved as follows. Given the received vector $\mathbf{r} = (x_1, \ldots, x_k)$, the decoder may perform a binary search of the sequence $\{r_i\}$ in $O(\log M)$ time to determine the index i such that $r_{i-1} \leq \sum_{j=1}^{k-1} x_j^2 \leq r_i$. Under (I), the search may be performed directly. Under (II), the radii are generated. It is not hard to show that the length of the sequence $\{r_i\}$ is $O(\sqrt{M})$. Only codepoints arising from a constant number of $\mathcal{C}_i$'s are candidates for being a nearest neighbor, and the nearest neighbor to $\mathbf{r}$ from a shell $\mathcal{C}_i$ may be determined in constant time.

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