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SMALL VOLUME PHYSICS WITH TWISTED BOUNDARY CONDITIONS

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We use perturbation theory to compute the low-lying energy levels in $SU(2)$ gauge theory with twisted boundary conditions. Finite lattice spacing corrections, which can be large, are taken into account. A comparison with recent numerical simulations is made. Nonperturbative effects are discussed.

1. INTRODUCTION

We will consider nonabelian gauge theory in a 3-dimensional box of side L. If the box is small ($L \ll \Lambda_{\rm QCD}^{-1}$), asymptotic freedom implies that "physics" can be calculated in perturbation theory. This is not the physics ultimately relevant for the real world of large volumes ($L \gg \Lambda_{\rm QCD}^{-1}$), but it may have some bearing on our understanding and interpretation of numerical simulations of lattice gauge theories which, for practical reasons, are typically performed in intermediate volumes ($L \sim \Lambda_{\rm QCD}^{-1}$).

Physics in a small volume is strongly dependent on the boundary conditions imposed. Ideally we wish to have boundary conditions which lead to a fast (exponential) approach to the infinite volume limit for L sufficiently large. Periodic boundary conditions are known to have this character[1], however in small volumes they induce large finite size effects which qualitatively alter the spectrum. These effects, which have their origin in the existence of zero momentum modes for the gauge field, persist into intermediate volumes. This can be seen in both numerical simulations and in analytic calculations, for both of which we refer to a fairly recent review by Micheal[2].

An alternative to periodic boundary conditions are the twisted boundary conditions of 't Hooft[3] where we have periodicity modulo a gauge transformation. These respect the exponential approach to the infinite volume limit, but eliminate the zero modes of the gauge fields and the associated small volume behaviour. It has been argued that twisted boundary conditions "seed" the vacuum for the formation of strings, and that as a result the transition to the large volume "stringy" physics should be more rapid, with smaller finite size effects.

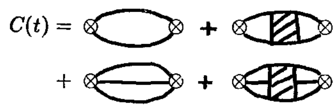
2. GLUEBALL MASSES

(a) *Perturbation theory.* To calculate the glueball masses we examine the (Euclidean) time behaviour of the connected correlation function between Wilson loops at zero momentum:

$$C(t) = \langle W(t)W(0)\rangle - \langle W(t)\rangle\langle W(0)\rangle, \\ = \sum_n A_n \exp(-E_n t).$$

To isolate the lowest lying states, which we identify with the glueballs, t is taken to be large. The masses of different glueballs are found by considering correlations of Wilson loop combinations in various representations of the cubic group.

In a small volume we can use perturbation theory and we expect $C(t)$ to have the resolution



$+ 4$ gluon $+ \ldots,$

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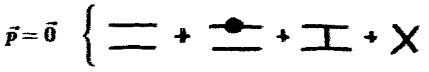
where —— represents a gluon propagator, and ⊗ represents a source. Taking t large then has the effect of putting all legs emanating from the source on shell. Furthermore, as a consequence of the Ward identities, only physical polarizations propagate and contribute to the energy. This can be verified explicitly, and is a very useful (indeed essential) check on the complicated lattice Feynman diagrams which arise in the calculation.

At $O(g^2)$ the lowest energies are given by



$$= 2|\vec{p}_{\rm min}| + cg^2$$

in the 2-gluon sector, and by

$$= 3|\vec{p}_{\rm min}| + dg^2$$

in the 3-gluon sector, and so on. Here we have used the notation $\vec{p}_{\rm min}$ to denote a (discrete) momentum of the minimal length allowed by the boundary conditions. For periodic boundary conditions we have $\vec{p}_{\rm min} = \vec{0}$, and this results in infrared problems which make the perturbative strategy outlined above impossible. Instead a less straightforward approach must be adopted[4].

No such obstacle arises with twisted boundary conditions. As an example we take $SU(2)$ gauge theory with a cubically invariant twist (characterized by the abelian magnetic flux $\vec{m} = (1,1,1)$). Explicitly the boundary conditions we impose are

$$A_\mu(x + \vec{e}_k L) = i\sigma_k A_\mu(x)(i\sigma_k)^\dagger,$$

where $\vec{e}_k$ is a unit 3-vector and σ_k are the Pauli matrices. Performing a Fourier decomposition, we find $\vec{p}_{\rm min} = 2\pi/2L\,(1,1,0)$ (or a cubic rotation of this). The perturbative calculation then proceeds with the results for the 2-gluon sector given in figure 1 [5]. Here we show the dependence of the energies on the lattice spacing (or more accurately on the lattice size). For $4^3 \times \infty$ lattices, for example, the lattice spacing corrections are seen to be sizable, particularly for the negative parity states A_1^- and E^-.

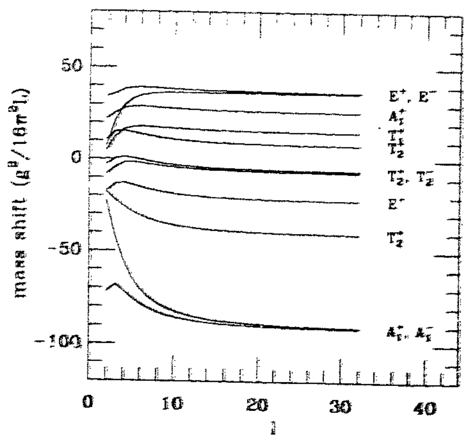


Figure 1. The $O(g^2)$ energy shifts of the two gluon states as a function of $l = L/a$ [5].

(b) *Comparison with simulation.* We now have analytic results for the "glueball" mass spectrum in small volumes for $SU(2)$ gauge theory with cubically invariant twisted boundary conditions. But how small is small? To answer this question we must turn to numerical simulations. This theory has recently been simulated by Stephenson[6] at a variety of values of β and $l = 4, 6, 8$, and 12 where $l = L/a$. We include some of his results in figure 2. The horizontal scale here may be regarded as $Z_\kappa/3$, where $Z_\kappa = \sqrt{\kappa}L$ is the finite size scaling variable associated to the infinite volume string tension, although for large β it is in fact calculated indirectly by assuming asymptotic scaling. It measures the physical size of the box.

A few remarks about the L dependence of the mass gap (the mass of the A_1^+ state) are in order. In figure 2, the zeroth order perturbative result for the mass gap would be a constant ($mL =$ constant), whereas for large L, mL must grow linearly (unless $m = 0$). However, the first order perturbative results drive mL *down*. This is similar to the Coulomb binding energy in a bound state like the Hydrogen atom. It is evident that nonperturbative effects are instrumental in pushing the mass gap up once again as we move to larger L. From figure 2 we can see

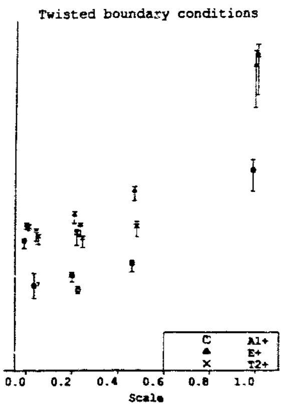


Figure 2. Glueball masses as function of physical volume[6].

that such effects come in to play at $Z_\kappa \sim 1$.

When it comes to comparing our perturbative results to the simulation, we have to decide on what our coupling g means. We are at liberty to make two choices. First, we can take $g = g_0$, the bare coupling. Then, for $\beta = 4/g_0^2$ sufficiently large, our results should agree quantitatively with simulation. It turns out that we need to go as high as $\beta = 4.7$ before this is achieved within errors[6], which are unfortunately rather large. Second, we can take $g = g_r$, the renormalized coupling at a scale $\mu = L^{-1}$. Since we do not know g_r, we have to fit our results to the numerical data. If we do this then there is reasonable agreement between perturbation theory and simulation down to $\beta = 3.0$. After this there are distinct differences. In making these comparisons, we ignore the A_1^+ mass found in the simulation, which lies consistently below the perturbative prediction, but which has unfortunately large systematic errors[6].

Reference[6] also includes a comparison between twisted and periodic boundary conditions. In the latter case, the small volume behaviour is for mL to go to zero[4]. For $Z_\kappa \sim 1$ (intermediate volumes), the A_1^+ and E^+ states remain very close together, a feature which is absent from the case with twist. For $Z_\kappa \sim 3$, the glueball spectra with twisted and periodic boundary conditions are found to coincide, indicating that we have reached the large volume region.

3. NONPERTURBATIVE ASPECTS

(a) *Vacuum structure.* In doing perturbation theory we assumed the existence of a unique vacuum. For $SU(2)$ gauge theory with periodic boundary conditions there are in fact 2^3 perturbative vacua, characterized by abelian electric fluxes, $\vec{e}$, whose components are integers modulo 2. These vacua are mapped into one another by the action of spatial Polyakov loops, $P(\vec{e})$, of given flux, $\vec{e}$, which is the vector normal to the loop.

In the presence of a twist $\vec{m}$, a state of flux $\vec{e}$ acquires a "Poynting" momentum, $\pi/L(\vec{e} \times \vec{m} \bmod 2)$. For our case where $\vec{m} = (1,1,1)$, this means that only the $\vec{e} = (0,0,0)$ and $\vec{e} = (1,1,1)$ vacua survive. Similarly $P(001)$ and $P(011)$ pick up momentum, but $P(111)$ does not. In perturbation theory the surviving vacua remain degenerate, but this is lifted by tunneling between them.

It may be helpful to have a concrete picture of these ground states on a lattice of size $L^3 \times L_t$. If L is odd, then the boundary conditions can be transformed to the case where every spatial plaquette is twisted, that is each plaquette is multiplied by an extra factor of -1. Then the potential energy is just $\sum \mathrm{tr}(1+U_{\rm pl})$, where the sum is over all spacelike plaquettes. This is minimized by one of two gauge inequivalent configurations: $\vec{U} = (\pm i\sigma_1, \pm i\sigma_2, \pm i\sigma_3)$. These we identify with the perturbative vacua.

(b) *Instantons.* It is known from rigorous arguments[7] that smooth tunneling solutions exist in the presence of twisted boundary conditions. Their shape and other properties have been explored on the lattice using cooling methods in reference[8]. The salient results of this study are:

- the solutions are (anti)self-dual to a very good approximation, and the action is close to $4\pi^2$.
- the action integrated over 3-space has the peaked profile indicated in figure 3.

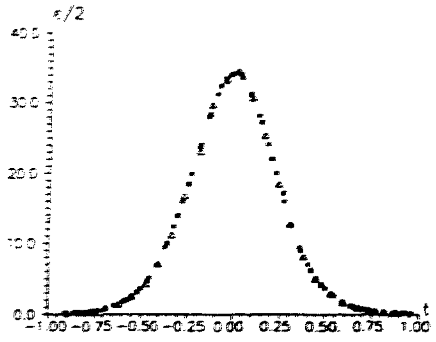


Figure 3. The instanton shape for $\vec{m} = (1,1,1)$[8].

- the width of the peak does not depend on L_t, only on L, but there is L_t dependence in the tail.
- the tail of the peak extrapolated to $L_t = \infty$ is of the form $\exp(-2|\vec{p}_{\min}|t)$.

(c) *String energies.* These are obtained by measuring the decay in time of spatial Polyakov loops. For large L we are in the string regime where these should behave like the $|\vec{e}|\kappa L$ (κ is the infinite volume string tension). This is confirmed within errors by the results of simulations at $Z_\kappa \approx 3$[6]. For small Z_κ we can compute $E(001)$ and $E(011)$ in perturbation theory with the result

$$E(\vec{e}) - E(\vec{0}) = |\vec{p}_{\min}| + g^2 S(\vec{e}),$$

where $S(\vec{e})$ is the projection of the gluon self energy onto the polarization parallel to $\vec{e}$. $E(111)$ remains is 0 in this approximation, however, since $P(111)$ relates the degenerate vacua and so must have zero energy. However, it does pick up an exponentially small contribution from instanton effects[8].

4. CONCLUSIONS

The glueball spectra of $SU(2)$ gauge theory in periodic and twisted boxes coincide within the errors of current simulations at $Z_\kappa \approx 3$. Also at this value, Polyakov loop energies become stringy, a nice confirmation of the soundness of the numerical simulations. Physically the perturbative analytic results show the effect of Coulombic forces on the energy splittings of the 2-gluon states. These effects drive the mass of the A_1^+ down, as in a bound state. At $Z_\kappa \approx 1$ the level starts to go up again due to a nonperturbative mechanism yet to be explained.

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