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ON SELF ORGANIZED CRITICALITY

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Self organized criticality refers to the tendency of highly dissipative systems to drive themselves to a critical state. This may explain why observed physics often displays a wide disparity of length and time scales. The phenomenon is observed in several simple cellular automaton models.

"Self organized criticality" describes the tendency of strongly dissipative systems to show relaxation behavior involving a wide range of length and time scales.[1] The phenomenon is expected to be quite universal; indeed, it has been looked for in such diverse areas as earthquake structure[2] and economics.[3] The idea provides an unifying concept for large scale behavior in systems with many degrees of freedom. It complements the concept of "chaos," wherein simple systems with a small number of degrees of freedom can display quite complex behavior.

I begin the discussion with the simple observation that real dissipative systems, unlike the ideals discussed in freshman physics class, rarely tend to go to their ground state. Consider, for example, a pendulum. The ideal motion is periodic and for small amplitudes is well approximated by a sine wave. To make the system more realistic, one can put in a drag term, giving rise to a damped oscillatory behavior, with motion theoretically continuing forever with a decreasing amplitude. However, in the real world the fulcrum for the pendulum will have some imperfections, perhaps in the form of grit. Thus once the amplitude gets small enough, the pendulum will suddenly stop, and this will generally occur at the end of a swing where the velocity is smallest. This is not the state of lowest energy, and indeed the probability is a minimum for stopping at exactly the bottom of the potential. In a sense the system is most likely to stop in a "minimally stable" state.

Generalizing to a multi-dimensional system of many coupled pendula, a new issue arises. A minimally stable state will be particularly sensitive to small perturbations which can "avalanche" through the system. This idea leads me to the prototypical example of self organized criticality, a sandpile. Adding sand slowly to an existing heap will result in the slopes increasing to a critical value, where an additional grain will give rise to an unpredictable behavior. If the slope were too steep, one would obtain a large avalanche and a collapse to a flatter and more stable configuration. On the other hand, if it were less steep the new sand will just accumulate to make the pile steeper. At the critical slope the distribution of avalanches has no overall scale, but rather shows a power law behavior. This has been extensively discussed by several authors. [4] [5] [6] [7] [8]

Self organized criticality is expected to be a quite universal phenomenon, applying whenever substantial dissipation is present. In a general sense, it explains why fractal structures, from $1/f$ noise to mountain ranges, are ubiquitous in our world. Self similar structures are also frequently associated with the idea of chaos. As usually discussed in non-linear science, chaos represents the extremely complex behavior often seen in the evolution of non-linear systems with only a few degrees of freedom. In contrast, self organized criticality emphasizes coherent features of the evolution of systems with many degrees of freedom.

Ref. [1] presented a simple mathematical system to demonstrate this behavior. This is a cellular automaton model formulated on a regular lattice, and uses an integer variable z_i on each site i to represent the local sandpile slope. For this discussion, consider a two dimensional lattice with open boundaries. When the slope exceeds a critical value, here taken to be 3, an avalanche ensues. In one time step, each site with slope z_i exceeding 3 has z_i drop by 4, with the "sand" spilling onto neighboring sites, increasing their local slopes by one. The updating is done concurrently, with all sites updated simultaneously. Studies of this model gave evidence of a self organized critical state where the ultimate distribution of avalanches was indeed a power law. In addition, interesting non-trivial geometric structures appear from the relaxation of uniform initial states. [9]

To explore self organized criticality in this model, one can randomly add sand and have the system relax. After an initial relaxation period, the results of such an addition become unpredictable, with one only being able to find the outcome of an addition by actually simulating the resulting avalanche.

In Fig. (1a) I show a typical state of the sandpile after a large amount of sand has been dropped pseudo-randomly. The lattice here is 178 by 190 and the boundaries are open. Note the absence of any notable structure. It is easy to understand some features, such as the fact that no cells of height 0 are next to each other. This follows from the fact that in tumbling a site to height 0, a grain of sand is dumped onto each neighbor. In Fig. (1b) I show an intermediate stage of an avalanche obtained by adding sand to make one site of the previous figure unstable. To trace the progress of the avalanche, I color sites which have tumbled a muddy red. Sites which are still active are shades of yellow. Fig. (1c) shows the system after the avalanche has completed.

This particular model was recently shown to have some rather remarkable mathematical properties.[10] [11] In particular, the critical states of the system are fully characterized in terms of an Abelian group. If two grains of sand are added to the system in arbitrary locations, the resulting final state is independent of the orders of addition and the intermediate relaxation steps. This result enables an exact calculation of the number of states important in the self organized critical state and determination of the average number of tumblings occurring at a site i given a grain of sand has been added at site j. In this picture the group elements are in one to one correspondence with the critical states. In particular there is a unique identity state which has the property that it will relax back to itself if all sand heights are doubled. The construction of this state was the subject of my talk at last year's lattice conference.[12]

One rather surpising exact result is that in the critical state all avalanches in the sandpile model are simply connected. That is, regardless of how or where a set of sand grains are dumped on the system to start an avalanche, the resulting disturbed region contains no closed paths that cannot be shrunk to a point. Note that the avalanche region in Fig. (1c) satisfies this property. This is not true for avalanches begun on an arbitrary state, only for those in the critical set, which

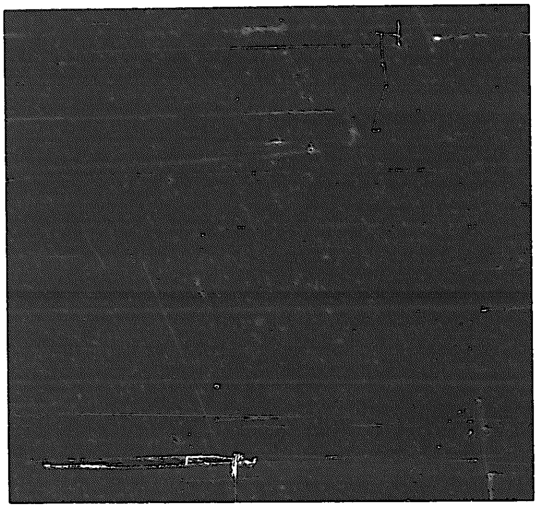

Fig. (1a) A critical state of the sandpile model on a 178 by 190 lattice with open boundaries. Sand heights of 0-3 are denoted by black, red, blue, and green, respectively.

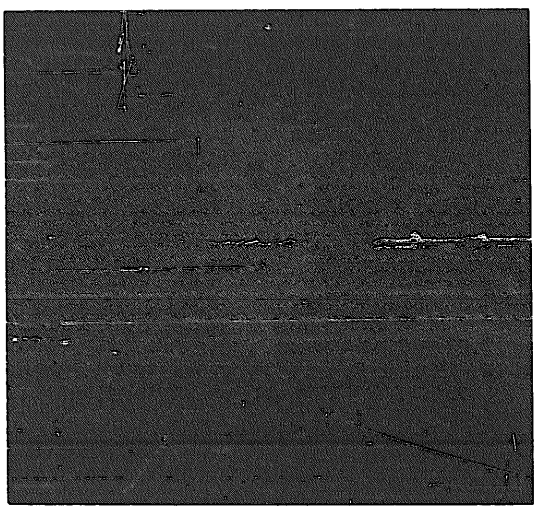

Fig. (1b) An avalanche was started by increasing a single site in Fig. (1a) to height 4. Here the avalanche is in progress, and previously tumbled cells are colored a muddy red. The still active sites are various shades of yellow.

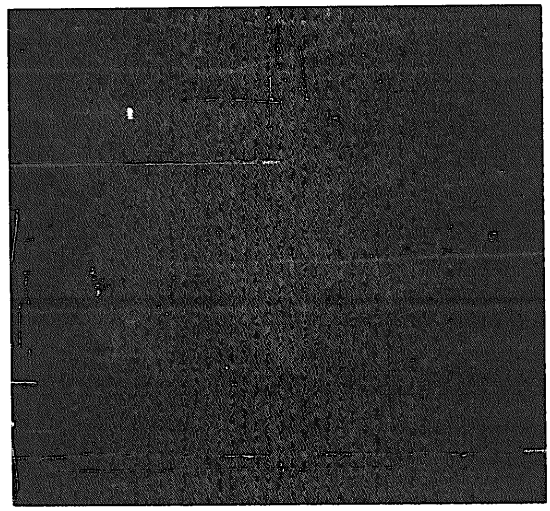

Fig. (1c) The final avalanche region. Note that it is simply connected.

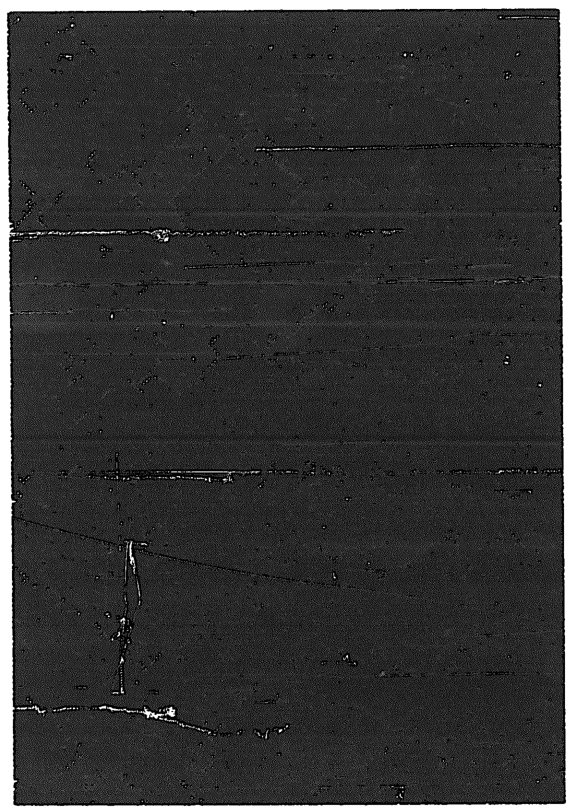

Fig. (2) A state in the evolution of the forest fire model mentioned in the text. On a black background, trees are green and fires yellow.

has been precisely defined in the above mentioned papers. This connectivity property is a consequence of a simple algorithm presented by Dhar[10] for determining if a state is in the critical set. While these exact results have not yet enabled determination of all critical properties of

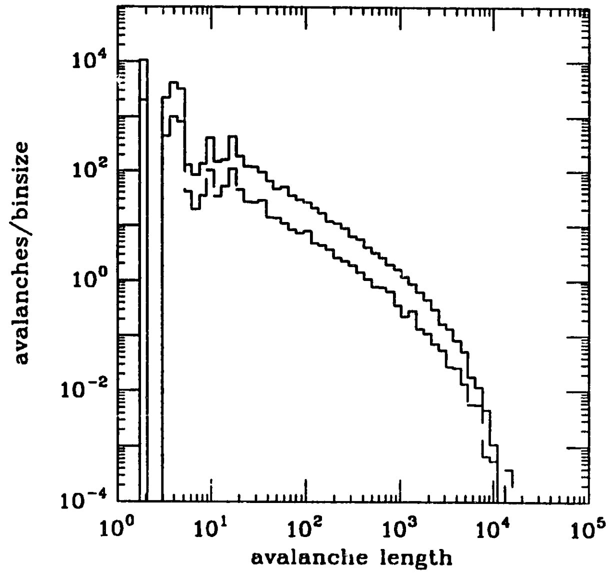

Fig. (3) The distribution of avalanche lengths for life on 512 by 512 (solid line) and 1024 by 1024 (dashed line) periodic lattices.

interest, they suggest further exact results may be found for this special model.

The sandpile model has a conservation law. The total amount of sand is unchanged in a toppling at any site in the lattice interior. Sand is only lost at the system edges. One might wonder whether this is essential. Recent tests on non-conservative variations of the model where the variables are continuous also found critical behavior.[13] However, the exponents appear not to be universal.

Fig. (2) illustrates another simple model used to demonstrate self organized criticality. This figure shows an active state in a toy model for wild fires in a slowly growing forest. Here each cell can be either dead (black), a tree (green) or a fire (yellow). At each time step new trees are born on neighboring cells with a small probability, taken as 1/32 per time step for this figure. This probability sets a scale, and should be taken to zero as the size of the system goes to infinity. Finally, any existing fire goes out leaving an empty cell, while spreading to all neighboring trees. When run, this model displays fire fronts separated by periods of tree regrowth. If the tree density gets too large, there is a catastrophic fire, while if it is too low, the fire tends to go out. The steady state has phenomena occurring at a wide range of length scales. As in the real world, if one artificially puts out fires, then trees grow to a high density state which is particularly prone to a huge catastrophic fire.

If self organized criticality is indeed ubiquitous, then it should appear in the simplest cellular automaton models. Thus motivated, we studied the famous "game of life" automaton.[14] Here simulations also suggested power law distributions for avalanche times and volumes.[7] This paper has been somewhat controversial, with other work indicating a possible cutoff at rather large avalanche size.[15] In Fig. (3) I show the distribution of avalanche lengths from a sequence of 25,000 avalanches on a 512 by 512 lattice and 6000 avalanches on a 1024 by 1024 lattice. In each case the initial lattice was obtained by relaxing a random configuration and then discarding 1000 avalanches. An avalanche is started by placing a random new live cell. Cases where the new cell dies immediately with no other changes of the lattice are not counted as an avalanche. The avalanche is considered terminated if the lattice is either identical to one two time steps back, or for two successive steps has the same total number of live cells as it had 12 steps previously. These conditions are meant to eliminate isolated oscillators. While oscillators of arbitrarily long period can exist, they appear to be rare and did not occur in these simulations. Note that the distribution is approximately a power law for over 4 orders of magnitude. The drop off for avalanches of length about 10^4 may be a finite size effect or, if ref. [15] is correct, may be a fundamental cutoff. Fig. (3) hints that the 1024 by 1024 site lattice scales somewhat further than the 512 by 512 system, thus supporting the point of view in Ref. [7].

In short, self-organized criticality represents a general concept, complementary to chaos, attempting to describe how real systems may automatically exhibit complex phenomena over a wide range of scales. Indeed, from galaxy clustering at the largest scales to new particles at the smallest, physics never seems to get boring.

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