Some Notes on Wiener Reconstruction

George B. Rybicki

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1 Stationary Processes

Wiener reconstruction of a curve from one-dimensional stationary data $y(t_i)$ at a discrete set of points t_i, $i = 1, 2, \ldots, n$ is treated in Press, Rybicki, & Hewitt ([1992]; PRH) and Rybicki & Press ([1992]; RP). The basic results as given in RP are

$$\widehat{s}_* = \mathbf{S}_*^T [\mathbf{S} + \mathbf{N}]^{-1} \mathbf{y} \tag{1}$$

for the reconstructed value [RP eq. (8)], and

$$\widehat{\sigma}_*^2 = \langle {s_*}^2 \rangle - \mathbf{S}_*^T [\mathbf{S} + \mathbf{N}]^{-1} \mathbf{S}_* \tag{2}$$

for the variance about this value [RP eq. (9)], that is, $\widehat{\sigma}_*^2 = \langle (s_* - \widehat{s}_*)^2 \rangle$. The notation is described in detail in RP, but briefly: $\mathbf{y}$ is the vector of measued values; $\mathbf{S}$ and $\mathbf{N}$ are the correlation matrices of signal and noise; $\mathbf{S}_*$ is the correlation vector between the signal at the reconstructed point and the measured points; and $\langle {s_*}^2 \rangle$ is the variance of the signal at the reconstructed point.

Using the above quantities, one can find the entire Wiener reconstruction of the curve by evaluating eq. (1) for each value of t in the desired range. Furthermore, one obtains error estimates for the reconstruction from eq. (2). These may be graphically presented as a "snake," the region between the two curves $\widehat{s}_* \pm \widehat{\sigma}_*$.

2 Weakly Nonstationary Processes

It is also of interest to consider weakly nonstationary processes, for which the mean is not well defined. As discussed in PRH and RP the revelant formulas for this case can be derived by assuming that the correlation function has an essentially infinite constant part. For the purpose of many numerical calculations, one can achieve this by simply adding a sufficiently large constant to the correlation function and and then using eqs. (1) and (2). However, it is of some interest to have special formulas for this case in which the formal limit has been taken analytically.

The detailed derivation is accomplished by first adding a finite constant λ to the correlation function; this is equivalent to making the replacements

$$\begin{aligned} \mathbf{S} &\rightarrow \mathbf{S} + \lambda \mathbf{E}\mathbf{E}^T \\ \mathbf{S}_* &\rightarrow \mathbf{S}_* + \lambda \mathbf{E} \\ \langle {s_*}^2 \rangle &\rightarrow \langle {s_*}^2 \rangle + \lambda \end{aligned} \tag{3}$$

where $\mathbf{E}$ is a vector of ones. The desired formulas are now obtained by taking the limit $\lambda \rightarrow \infty$.

After performing these operations, we find (see appendix) that formula (1) becomes

$$\widehat{s}_* = \mathbf{S}_*^T [\mathbf{S} + \mathbf{N}]^{-1} (\mathbf{y} - \bar{y}\mathbf{E}) + \bar{y} \tag{4}$$

where

$$\bar{y} = \frac{\mathbf{E}^T [\mathbf{S} + \mathbf{N}]^{-1} \mathbf{y}}{\mathbf{E}^T [\mathbf{S} + \mathbf{N}]^{-1} \mathbf{E}} \tag{5}$$

This is the result given in RP [RP eqs. (19) and (20)].

However, the weakly nonstationary result corresponding to formula (2) is not given in RP. It can be shown (see the appendix) that the desired formula is

$$\widehat{\sigma}_*^2 = \langle {s_*}^2 \rangle - \mathbf{S}_*^T [\mathbf{S} + \mathbf{N}]^{-1} \mathbf{S}_* + \frac{\left(\mathbf{S}_*^T [\mathbf{S} + \mathbf{N}]^{-1} \mathbf{E} - 1 \right)^2}{\mathbf{E}^T [\mathbf{S} + \mathbf{N}]^{-1} \mathbf{E}} \tag{6}$$

It is clear that the formulas (4–6) for the weakly nonstationary case must be invariant to the substitutions (3) for any finite λ, since such a substitution cannot change the limit. (This can also be proved directly, but the proof is lengthy.) In particular, one may choose λ to be equal to $-\langle {s_*}^2 \rangle$, so that,

$$\begin{aligned} \mathbf{S} &\rightarrow -\mathbf{V} \\ \mathbf{S}_* &\rightarrow -\mathbf{V}_* \\ \langle {s_*}^2 \rangle &\rightarrow 0 \end{aligned} \tag{7}$$

where $\mathbf{V}$ is the structure function matrix, etc. Then eqs. (4)–(6) become

$$\widehat{s}_* = -\mathbf{V}_*^T [-\mathbf{V} + \mathbf{N}]^{-1} (\mathbf{y} - \bar{y}\mathbf{E}) + \bar{y} \tag{8}$$

where

$$\bar{y} = \frac{\mathbf{E}^T[-\mathbf{V}+\mathbf{N}]^{-1}\mathbf{y}}{\mathbf{E}^T[-\mathbf{V}+\mathbf{N}]^{-1}\mathbf{E}} \tag{9}$$

and

$$\widehat{\sigma}_*^2 = -\mathbf{V}_*^T[-\mathbf{V}+\mathbf{N}]^{-1}\mathbf{V}_* + \frac{\left(\mathbf{V}_*^T[-\mathbf{V}+\mathbf{N}]^{-1}\mathbf{E}+1\right)^2}{\mathbf{E}^T[-\mathbf{V}+\mathbf{N}]^{-1}\mathbf{E}} \tag{10}$$

Thus the calculation of the Wiener reconstruction for weakly nonstationary processes can be done using only the structure function.

A Appendix: Proof of Equations (4) and (6)

It is convenient here to define matrices $\mathbf{A}$ and $\mathbf{C}$ by $\mathbf{C} = \mathbf{A}^{-1} = \mathbf{S}+\mathbf{N}$. After making the substitutions (3) the quantities $\widehat{s}_*$ and $\widehat{\sigma}_*^2$ become

$$\widehat{s}_* = [\mathbf{S}_*^T + \lambda\mathbf{E}^T][\mathbf{C}+\lambda\mathbf{E}\mathbf{E}^T]^{-1}\mathbf{y} \tag{11}$$

and

$$\widehat{\sigma}_*^2 = [\langle {s_*}^2\rangle + \lambda] - [\mathbf{S}_*^T + \lambda\mathbf{E}^T][\mathbf{C}+\lambda\mathbf{E}\mathbf{E}^T]^{-1}[\mathbf{S}_* + \lambda\mathbf{E}] \tag{12}$$

The quantity $[\mathbf{C}+\lambda\mathbf{E}\mathbf{E}^T]^{-1}$ can be expressed by means of the well-known Woodbury formula

$$[\mathbf{C}+\lambda\mathbf{E}\mathbf{E}^T]^{-1} = \mathbf{A} - \frac{\lambda}{1+\lambda g^{-1}}\mathbf{A}\mathbf{E}\mathbf{E}^T\mathbf{A} = \mathbf{A} - \frac{g}{1+g\lambda^{-1}}\mathbf{A}\mathbf{E}\mathbf{E}^T\mathbf{A} \tag{13}$$

where

$$g \equiv (\mathbf{E}^T\mathbf{A}\mathbf{E})^{-1} \tag{14}$$

is a scalar. Equation (13) can now be expanded to second order in the small parameter λ^{-1} as follows:

$$[\mathbf{C}+\lambda\mathbf{E}\mathbf{E}^T]^{-1} = \mathbf{M}_0 + \lambda^{-1}\mathbf{M}_1 + \lambda^{-2}\mathbf{M}_2 + \ldots \tag{15}$$

where

$$\mathbf{M}_0 = \mathbf{A} - g\mathbf{A}\mathbf{E}\mathbf{E}^T\mathbf{A}, \qquad \mathbf{M}_i = (-g)^{i+1}\mathbf{A}\mathbf{E}\mathbf{E}^T\mathbf{A}, \qquad i \geq 1 \tag{16}$$

We now expand the following quantity, which appears in the both formulas (11) and (12),

$$[\mathbf{S}_*^T+\lambda\mathbf{E}^T][\mathbf{C}+\lambda\mathbf{E}\mathbf{E}^T]^{-1} = (\mathbf{E}^T\mathbf{M}_1+\mathbf{S}_*^T\mathbf{M}_0)+\lambda^{-1}(\mathbf{E}^T\mathbf{M}_2+\mathbf{S}_*^T\mathbf{M}_1)+O(\lambda^{-2}) \tag{17}$$

where we have use the results $\mathbf{E}^T\mathbf{M}_0 = \mathbf{M}_0\mathbf{E} = 0$. Therefore as $\lambda \to \infty$,

$$\widehat{s}_* = (\mathbf{E}^T\mathbf{M}_1 + \mathbf{S}_*^T\mathbf{M}_0)\mathbf{y} \tag{18}$$

Substituting for $\mathbf{M}_i$, using the definition of g, we find

$$\begin{aligned} \widehat{s}_* &= g\mathbf{E}^T\mathbf{A}\mathbf{y} + \mathbf{S}_*^T\mathbf{A}\mathbf{y} - g\mathbf{S}_*^T\mathbf{A}\mathbf{E}\mathbf{E}^T\mathbf{y} \\ &= \bar{y} + \mathbf{S}_*^T\mathbf{A}y - \mathbf{S}_*^T\mathbf{A}\bar{y}\mathbf{E} \\ &= \mathbf{S}_*^T\mathbf{A}(\mathbf{y} - \bar{y}\mathbf{E}) + \bar{y} \end{aligned} \tag{19}$$

which proves eq. (4).

Returning to eq. (17), we find that

$$\begin{aligned} [\mathbf{S}_*^T + \lambda\mathbf{E}^T][\mathbf{C} + \lambda\mathbf{E}\mathbf{E}^T]^{-1}[\mathbf{S}_* + \lambda\mathbf{E}] = \lambda(\mathbf{E}^T\mathbf{M}_1\mathbf{E}) \\ + (\mathbf{E}^T\mathbf{M}_2\mathbf{E} + \mathbf{E}^T\mathbf{M}_1\mathbf{S}_* + \mathbf{S}_*^T\mathbf{M}_1\mathbf{E} + \mathbf{S}_*^T\mathbf{M}_0\mathbf{S}_*) + O(\lambda^{-1}) \end{aligned} \tag{20}$$

Substitution into eq. (12), we note that the term of order λ cancels due to $\mathbf{E}^T\mathbf{M}_1\mathbf{E} = 1$. Thus, in the limit $\lambda \to \infty$,

$$\begin{aligned} \widehat{\sigma}_*^2 &= \langle {s_*}^2 \rangle - \mathbf{S}_*^T\mathbf{A}\mathbf{S}_* + g - g\mathbf{E}^T\mathbf{A}\mathbf{S}_* - g\mathbf{S}_*^T\mathbf{A}\mathbf{E} + \mathbf{S}_*^T\mathbf{A}\mathbf{E}\mathbf{E}^T\mathbf{S}_* \\ &= \langle {s_*}^2 \rangle - \mathbf{S}_*^T\mathbf{A}\mathbf{S}_* + g(\mathbf{S}_*^T\mathbf{A}\mathbf{E} - 1)^2 \end{aligned} \tag{21}$$

which proves eq. (6). [Note that to obtain this expression the expansion of Eq. (15) was required to second order.]

References

[1992] Press, W.H., Rybicki, G.B., Hewitt, J.N. 1992, ApJ, 385, 404

[1992] Rybicki, G.B., Press, W.H. 1992, Astrophys. Journ. 398, 169