What is a plasma? strong coupling, Debye screening, electron plasma oscillations, Langmuir wave propagation, sound waves

20th June 2006

OUTLINE

- Maxwell's equations, Lorentz force law, continuity equation.
- Definition of a plasma.
- Pressure gradient as a force per unit volume.
- Debye screening.
- Electron plasma oscillations.
- Langmuir waves.
- Geometric optics of short wavelength waves.
- (Ion) sound waves.

Electrodynamics with $4\pi = c = \epsilon_0 = 1$

$$\nabla \cdot \mathbf{B} = 0$$
$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$
$$\nabla \cdot \mathbf{E} = \rho_q$$
$$\nabla \times \mathbf{B} = \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}$$
$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0$$

for each species. $\rho_q = e\left(n_i - n_e\right)$ ... $\rho_m = \frac{m_i n_i + m_e n_e}{m_i + m_e}$

Plasma

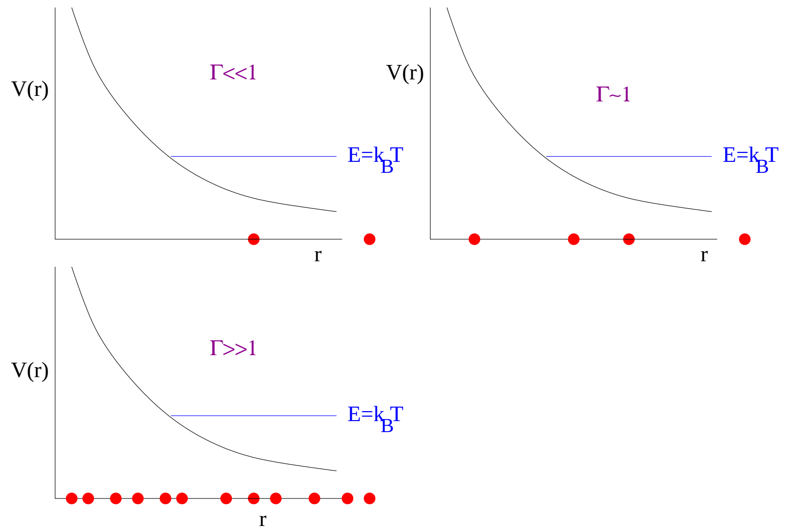
Average interparticle spacing (Wigner-Seitz radius) $\frac{4\pi}{3} n a_{ws}^3 = 1 \quad a_{ws} \sim n^{-1/3}$

$4\pi = \epsilon_0 = 1...$

Distance of closest approach $F(r) = -e^2/r^2 \quad V(r) = e^2/r \quad e^2/a_{dca} = \frac{1}{2} m v^2 \sim k_B T$

$$a_{dca} = e^2/k_B T$$

Coupling parameter is $\Gamma = \frac{a_{dca}}{a_{ws}}$. Average potential energy PE $\sim e^2/a_{ws}$. Average kinetic energy KE $\sim k_B T$

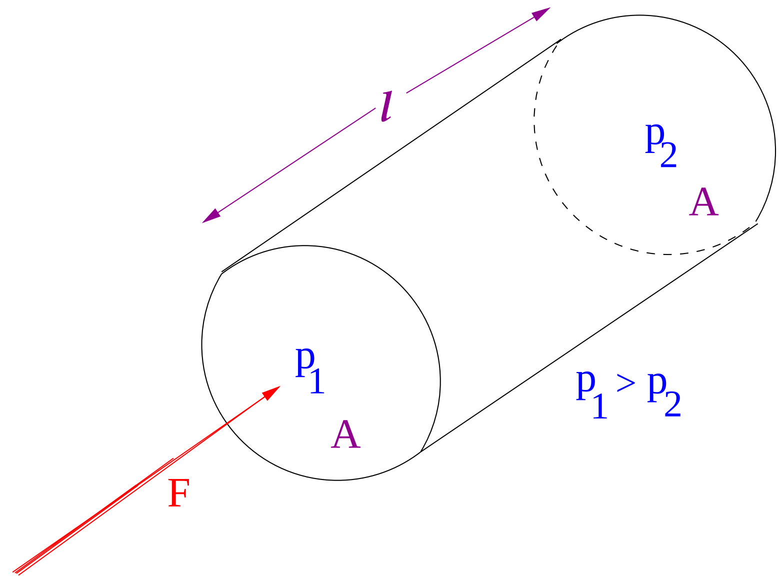


$$PE \ll KE \Rightarrow a_{dca} \ll a_{ws}$$

$$\Gamma \ll 1$$

$\Gamma \sim 1$, $\Gamma \gg 1$ ... strongly coupled plasma

Pressure gradient as a force



force $F = A\left(p_2 - p_1\right)$. Force/volume $= F/Al = \left(p_2 - p_1\right)/l \ \sim \ -\nabla p$

$$nm\frac{\partial \mathbf{v}}{\partial t} = nq\mathbf{E} - \nabla p$$

if magnetic field $\mathbf{B} = 0$. ... Ideal gas $p = nk_BT$ for each species. Isothermal: $T = const.$

Debye length - Debye screening

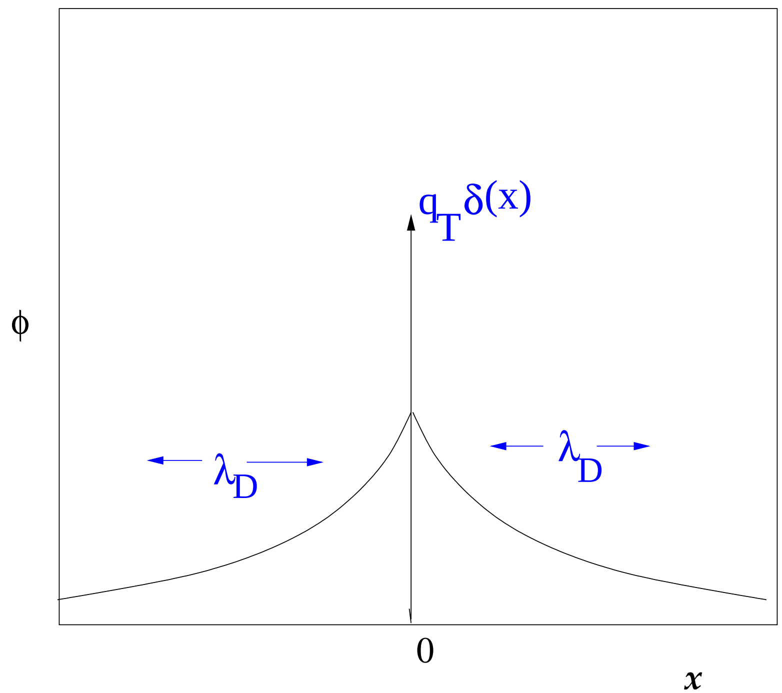
Test particle with charge q_t embedded in a plasma with isothermal T_e.
Electrons: $q = -e$

$$\nabla \cdot \mathbf{E} = e\,(n_i - n_e) + q_t \delta(\mathbf{x})$$
$$\mathbf{E} = -\nabla\phi \quad n_e e \nabla\phi - k_B T_e \nabla n_e = 0$$
$$n_e = n_i \exp\left(e\phi/k_B T_e\right)$$
$$-\nabla^2\phi = e n_i \left(1 - \exp\left(e\phi/k_B T_e\right)\right) + q_t \delta(\mathbf{x})$$

Take $e\phi/k_B T_e \ll 1$ and $1D$

$$\phi''(x) = \frac{n_i e^2}{k_B T_e}\phi - q_t \delta(x)$$

(Green's function) Then $\phi(x) \sim \exp\left(-|x|/\lambda_D\right)$ where $\lambda_D^2 = k_B T_e / n_i e^2$



Potential around a point charge q_T ... λ_D is the *Debye length.*

In 3D spherical geometry,

$$\phi(r) = \frac{q_T}{r} e^{-r/\lambda_D}$$

Bare charge for $r \ll \lambda_D$; very effectively screened (quasi-neutral) for $r \gg \lambda_D$.

Moving charge: screening is effective for slow motion, very weak for a fast moving test charge.

Electron plasma oscillations

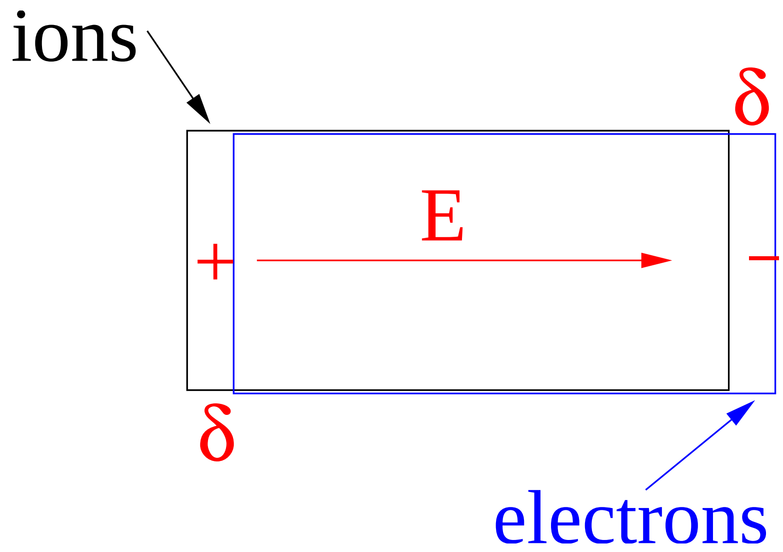
Linearize: $n_e(x,t) = n_{e0} + \widetilde{n}_e e^{ikx - i\omega t}$ ... Immobile ions (Justify later – the frequency is so high that the ions hardly move)

$$-i\omega n_e m_e \widetilde{v}_e = iken_e\widetilde{\phi}$$
$$k^2\widetilde{\phi} = -e\widetilde{n}_e$$
$$-i\omega\widetilde{n}_e + ikn_e\widetilde{v}_e = 0$$

Gives

$$\omega^2 = \omega_{pe}^2$$

Plasma frequency. Electron plasma oscillations. Note $\omega \neq \omega(k)$. The oscillations just sit there oscillating, but do not propagate.



Electron plasma oscillations of a displaced bunch of electrons $m_e d^2\delta/dt^2 = -n_e e^2 \delta \quad \omega^2 = \omega_{pe}^2$.

Finite T_e: Langmuir waves

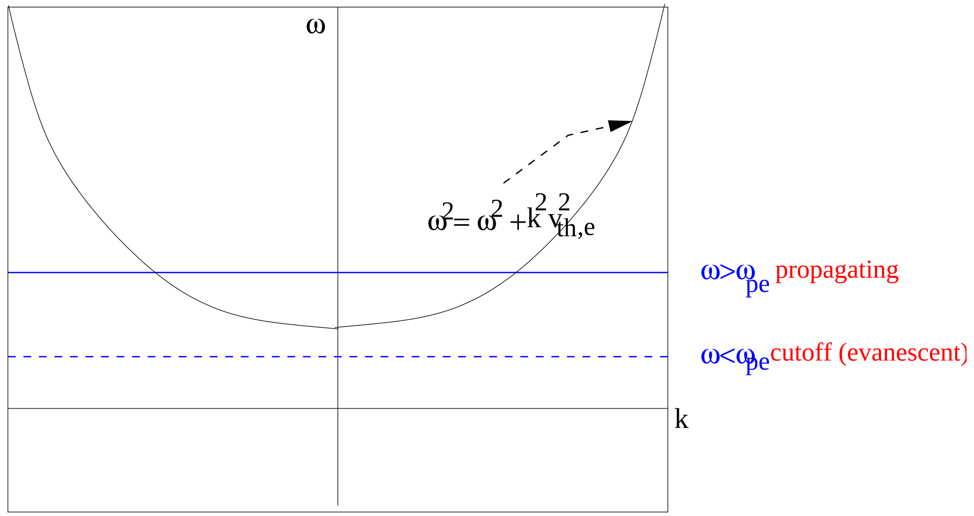
$$
\begin{gathered}
-i\omega n_e m_e \widetilde{v}_e = iken_e\widetilde{\phi} - k_B T_e ik\widetilde{n}_e/n_e \\
k^2\widetilde{\phi} = -e\widetilde{n}_e \\
-i\omega\widetilde{n}_e + ikn_e\widetilde{v}_e = 0
\end{gathered}
$$

Gives ($v_{te}^2 = k_B T_e/m_e$)

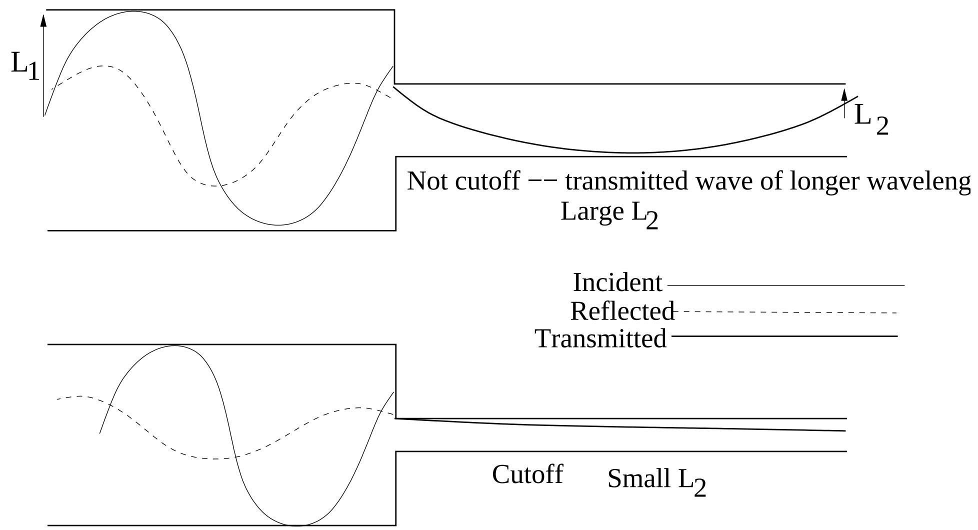
$$
\omega^2 = \omega_{pe}^2 + k^2 v_{te}^2 = \omega_{pe}^2\left(1 + k^2\lambda_D^2\right)
$$

Now $\omega = \omega(k)$.

Cutoff for $\omega < \omega_{pe}$. Just like waveguide cutoff $\omega^2 = \pi^2 c^2/L^2 + k^2c^2$. Cutoff frequency ω_{pe} in the plasma, $\pi c/L$ in the waveguide.

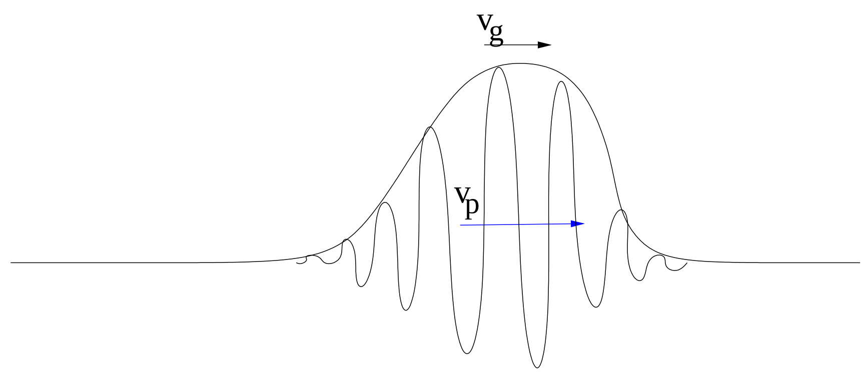


ω
ω² = ω² + k² v²th,e
ω>ωpe propagating
ω<ωpe cutoff (evanescent)
k



Group velocity (velocity of a wavepacket – a superposition of waves with k's close)

$$v_g = \frac{d\omega(k)}{dk} \neq 0$$



Individual sinusoidal waves travel at phase velocity
Wave packet travels at group velocity
Illustration for $v_p > v_g$

Non−dispersive wave (ω / k = const.) −− wavepacket propagates intact.

$v_p = v_g$

These waves now propagate. Also note: $\omega^2 = 0 \rightarrow$ ... $\frac{1}{1+k^2\lambda_D^2} \longleftarrow$ $Fourier\,Transform \longrightarrow e^{-|x|/\lambda_D}$ ... Debye screening

Geometric optics (WKB)

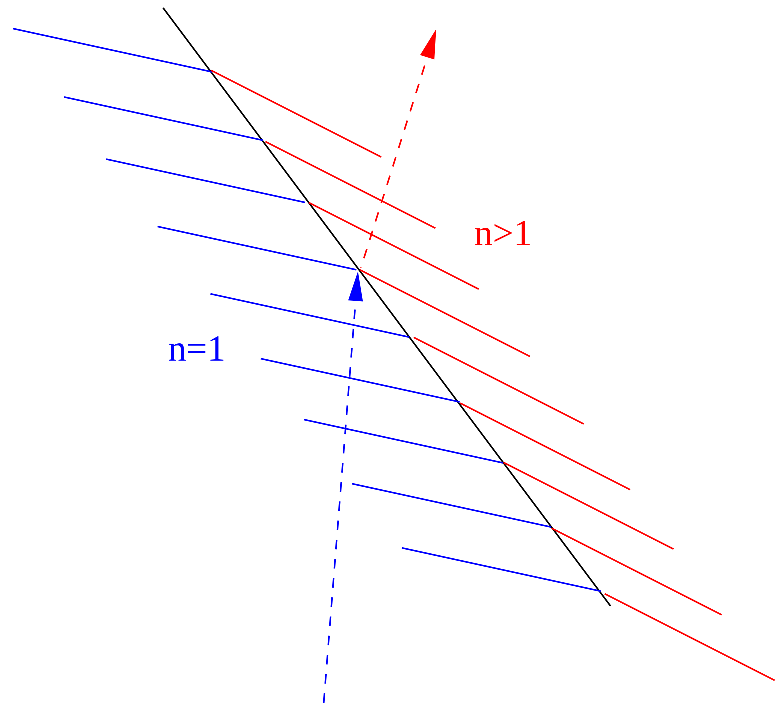
$n_e = n_e(x) \rightarrow \omega = \omega(x, k)$ Geometric optics is OK if $|k| \gg |\nabla n_e(x)|/n_e$ (WKB). The *ray equations* are:

$$\frac{dx}{dt} = \frac{\partial \omega(x,k)}{\partial k} \quad (a)$$
$$\frac{dk}{dt} = -\frac{\partial \omega(x,k)}{\partial x} \quad (b)$$

(Hamiltonian) Why (b)? Antenna excites a single ω, which stays fixed.

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial x}\frac{dx}{dt} + \frac{\partial \omega}{\partial k}\frac{dk}{dt}$$

$$= \frac{\partial \omega}{\partial x}\frac{\partial \omega}{\partial k} + \frac{\partial \omega}{\partial k}\left(-\frac{\partial \omega}{\partial x}\right) = 0$$

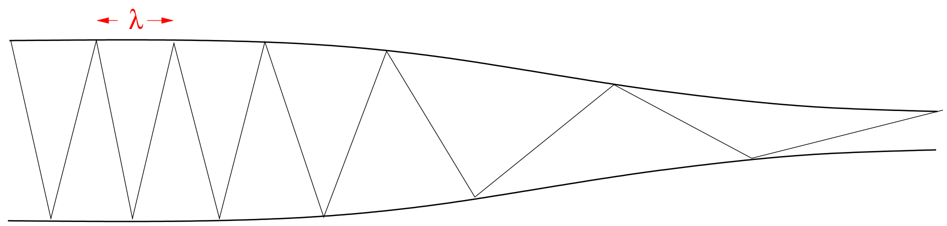


$\omega = \frac{kc}{n(x)}$ ... $k = \omega n(x)/c$ ... k increases as $n(x)$

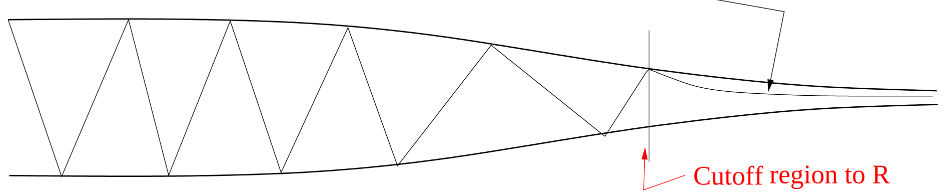
For $\omega = kc/n(x)$, $dx/dt = \partial\omega/\partial k = c/n(x)$ and $dk/dt = -\partial\omega/\partial x = kcn'(x)/n(x)^2$ gives $dk/dx = kn'(x)/n(x)$, same as $k(x) = \omega n(x)/c$ ...

$dk/dx = \omega n'(x)/c = kn'(x)/n$.

Waveguide analogy: $L(x)$ needs to vary continuously for WKB to be valid. As $L(x)$ decreases, k decreases (wavelength increases)



Wavelength increases as wave propagates to right and L(x) decreases.
WKB is valid if L(x) decreases slowly; if cutoff is reached, this is like a turning point [V(x)=E] in quantum mechanics − there is an evanescent wave to the right



Waveguide with discontinuous L(x)** satisfied WKB (trivially) except at the step in L(x), where WKB is not valid (matching formulas used instead)

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Just one more wave (promise!)

Sound wave or ion-acoustic wave.

Quasineutrality holds if $\omega \ll \omega_{pe}$ and $k\lambda_D \ll 1$. $n_e = n_i$ – equilibrium *and* perturbation

Slow wave – ions can move too. For simplicity take $T_i = 0$:

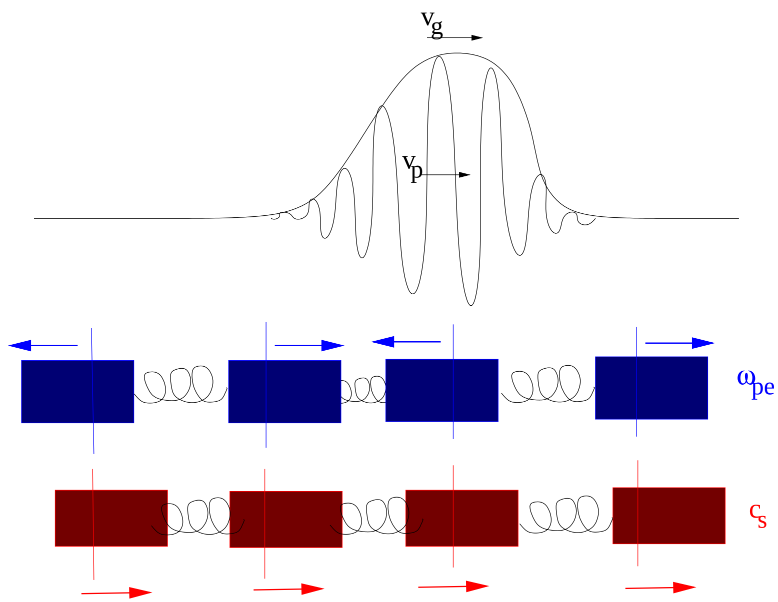
$$-i\omega n m_e \widetilde{v}_e = iken\widetilde{\phi} - k_B T_e ik\widetilde{n}_e/n_e$$
$$-k^2\widetilde{\phi} = e\left(\widetilde{n}_e - \widetilde{n}_i\right) \qquad \widetilde{n}_e = \widetilde{n}_i \equiv \widetilde{n}$$
$$-i\omega\widetilde{n} + ikn\widetilde{v}_i = 0$$

$$-i\omega n m_i \widetilde{v}_i = -iken\widetilde{\phi}$$

Substituting,

$$\omega^2 = k^2 c_s^2 = k^2 \frac{T_e}{m_i}$$

N.b. T_e/m_i. $\omega = \pm k c_s$ – two propagating but **non-dispersive** waves ($v_p = v_g$).



Sound wave

- Another electrostatic longitudinal wave.
- Electrons move with ions (nearly).
- $\widetilde{n}_e = \widetilde{n}_i$ – quasineutrality: small charge separation gives electric field $-ik\widetilde{\phi}$ but so small that charge separation is negligible elsewhere. $-ik\widetilde{\phi}$ holds electrons and ions together.
- Slow wave, so electron inertia is negligible.
- Sound wave in neutral fluid: $\omega^2 = k^2 T/m = k^2 c_s^2$. ... $c_s^2 = T/m$ (isothermal) or $c_s^2 = \gamma T/m \quad \gamma = 5/3$ (adiabatic). Collisions $\longleftrightarrow$ electric field.

Discussion

1. $\Gamma \sim 1$ or greater ... strongly coupled plasma. Very cold and/or dense plasma. Collisions are 'happening all the time'. ICF applications.

2. Debye length is also associated with sheaths around electrostatic probes.

3. Langmuir waves are electrostatic ($\widetilde{\mathbf{B}} = 0$) and longitudinal. If they are excited in a plasma, they cannot escape. Electromagnetic waves from an antenna can convert to Langmuir waves and then remain trapped (e.g. for plasma heating).

4. Sound waves are quasi-neutral. The electric field from a ***small*** charge imbalance holds the electrons and ions together. Low frequency.