Operator Splitting Techniques for Radiation-Hydrodynamics

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Motivation

- The cutting-edge of high-performance scientific computing currently lies in the domain of multiphysics computation.
- Operator splitting has traditionally made the solution of multiphysics systems tenable by enabling each component of the physics to be solved in an essentially independent manner using standard solution techniques for each component.
- However traditional operator splitting can sometimes be inefficient and is only first-order accurate.
- Much research is currently being done to develop efficient second-order multiphysics time-integration methods.
- We are currently developing second-order splitting methods for radiation-hydrodynamics with the intent of achieving efficiency, accuracy, and compatibility with existing code architectures.

Overview

- The thermal radiation transport equation in static media
  - Angular, energy, and spatial discretization
  - Solution of the transport equation
    - Source Iteration
    - Linear Multifrequency Grey Acceleration (LMFGA)
    - Preconditioned Krylov adaptation of LMFGA
- The equations of nonrelativistic radiation-hydrodynamics
- A simplified model for nonrelativistic radiation-material coupling
- The concept of operator splitting
- A first-order splitting scheme using standard solution techniques
- A candidate second-order splitting scheme

Basic Equations

- The equations of thermal radiation transport consist of a transport equation for the angular intensity $I(\overrightarrow{r}, \overrightarrow{\Omega}, E, t)$:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} I + \sigma_t I = \frac{\sigma_s}{4\pi}\phi + \sigma_a B(T) \,,$$

  and an equation for the material temperature $T(\overrightarrow{r}, t)$:

$$C_v \frac{\partial T}{\partial t} = \int_0^\infty \sigma_a [\, \phi - 4\pi B(T) \,] \, dE \,.$$

- There are two other basic equations associated with the transport equation: the radiation energy and momentum equations.

Basic Equations

- The radiation energy equation is obtained by integrating the transport equation over all directions and energies:

$$\frac{\partial \mathcal{E}}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{\mathfrak{F}} = \int_0^\infty \sigma_a [\, 4\pi B(T) - \Phi \,]\, dE \, .$$

where the radiation energy density $(energy/volume)$ is given by

$$\mathcal{E} \equiv \frac{1}{c} \int_0^\infty \int_{4\pi} I\left(\overrightarrow{\Omega}, E\right)\, d\Omega\, dE \, ,$$

and the radiation flux $(energy - area - time)$ is given by

$$\overrightarrow{\mathfrak{F}} \equiv \int_0^\infty \int_{4\pi} \overrightarrow{\Omega}\, I\left(\overrightarrow{\Omega}, E\right)\, d\Omega\, dE \, .$$

Basic Equations

- Note from the definition of $\mathcal{E}$ that the radiation intensity integrated over angle and energy $(energy - area - time)$, which we denote by Φ, is equal to $c\mathcal{E}$.
- The radiation energy is an energy balance equation stating that the time rate of change of the radiation energy in a differential volume is equal to the energy sources minus the sinks.
- The radiation momentum equation is obtained by first multiplying the transport equation by $\overrightarrow{\Omega}/c$ and then integrating over all directions and energies:

$$\frac{1}{c^2}\frac{\partial \overrightarrow{\mathcal{F}}}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{\overrightarrow{\mathcal{P}}} + \int_0^\infty \frac{\sigma_a}{c} \overrightarrow{F}\, dE = 0\,,$$

Basic Equations

- where the radiation pressure $(energy/volume)$ is given by

$$\mathcal{P}_{i,j} \equiv \frac{1}{c} \int_0^{\infty} \int_{4\pi} \Omega_i \Omega_j I\left(\overrightarrow{\Omega}, E\right)\, d\Omega\, dE\,,$$

- The radiation momentum equation is a balance equation stating that the time rate of change of the radiation momentum in a differential volume is equal to the momentum sources minus the sinks.
- When we include material motion, the transport and hydrodynamics equations will be coupled.
- The radiation energy and momentum equations will be part of total (radiation plus material) energy and momentum conservation equations.

Nonlinearities

- Note that the radiative transfer equations have nonlinearities arising only from the temperature dependence of the material property coefficients and the Planck function.
- The heat capacity and scattering cross sections are generally a weak function of temperature, while the absorption cross sections are strong functions of temperature.
- The radiative transfer equations are generally solved using an approximate form of Newton's method.
- The method is approximate in that the non-linearities are usually not iterated to full consistency, and contributions to the Jacobian from the material property functions are neglected.
- Stability considerations require linearization of the Planck function but not the material property functions.

Discretization of the Transport Equation

- We use the S_n or discrete-ordinates angular discretization, which is basically a collocation technique.
- The collocation points correspond to quadrature points on the unit sphere and the integration over angle is carried out using the quadrature formula.
- There is only one energy discretization technique in use called the multigroup method, and it can be viewed as a Petrov-Galerkin method with an arbitrary piecewise discontinuous trial space and a piecewise-constant weighting space.
- The demands on transport spatial discretization schemes are extreme. For the most part, we use discontinuous-Galerkin methods, but both the gradient and interaction terms must be lumped to achieve adequate robustness.

Solution of the Radiative Transfer Equations

- Traditional accelerated iterative solution techniques for the transport equation are closely related to multilevel or multigrid methods.
- In most instances, some type of diffusion operator is used to approximate a transport operator.
- For many years, it appeared that such methods were unconditionally effective as long as the diffusion equations were differenced in a manner consistent with the spatial discretization of the transport operator.
- Unfortunately, when discontinuous Galerkin methods are used for the transport equation, the consistent diffusion discretizations are of a mixed form and can be very expensive to solve.

Solution of the Radiative Transfer Equations

- A great deal of research effort has been spent over the last 20 years or so to find ways to either use simpler diffusion discretizations or solve the full discretizations in an approximate manner without significant loss of effectiveness.
- These efforts have met with limited success.
- Furthermore, over the last five years or so, it has been recognized that traditional acceleration techniques are not uniformly effective in multidimensional calculations even when consistent diffusion discretizations are used.
- In particular, it has been found that strong material inhomogeneities can degrade effectiveness and occasionally generate instabilities.

Solution of the Radiative Transfer Equations

- It has now become clear that by recasting traditional accelerated iteration schemes as preconditioned Krylov methods, far greater lattitude in the choice of diffusion discretization is possible, the degrading effects of strong material inhomogeneities can be significantly reduced, and any associated instabilities eliminated.
- Consequently, there is currently a great deal of research within the computational transport community devoted to preconditioned Krylov methods.

Solution of the Radiative Transfer Equations

- As previously noted, the radiative transfer equations are generally solved via an approximate form of Newton's method.
- After linearization, temporal discretization (backward-Euler), and energy discretization (multigroup), one can eliminate the temperature from the transport equation:

$$\overrightarrow{\Omega} \cdot \overrightarrow{\nabla} I_g + \sigma^*_{\tau,g} I = \frac{1}{4\pi}\sigma^*_{s,g}\phi_g + \frac{1}{4\pi}\nu\chi_g \sum_{k=1}^{G} \sigma^*_{a,k}\phi_k + \xi_g \quad , \quad g = 1, G,$$

- and obtain an intensity-dependent temperature equation:

$$T = T^* + \frac{\sum_{g=1}^{G} \sigma^*_{a,g}\left[\phi_g - 4\pi B^*_g\right] + \frac{C^*_v}{\Delta t^k}\left(T^n - T^*\right)}{\frac{C^*_v}{\Delta t^{n+\frac{1}{2}}} + \sum_{g=1}^{G} \sigma^*_{a,g} 4\pi \frac{\partial B^*_g}{\partial T}},$$

Solution of the Radiative Transfer Equations

- where

$$\sigma_\tau = \sigma_t + \tau \,,$$

$$\tau = \frac{1}{c\Delta t^{n+\frac{1}{2}}} \,,$$

$$\nu = \frac{\sum_{g=1}^{G} \sigma_{a,g}^{*} 4\pi \frac{\partial B_g^*}{\partial T}}{\frac{C_v^*}{\Delta t^{n+\frac{1}{2}}} + \sum_{g=1}^{G} \sigma_{a,g}^{*} 4\pi \frac{\partial B_g^*}{\partial T}}$$

$$\chi_g = \frac{\sigma_{a,g}^{*} \frac{\partial B_g^*}{\partial T}}{\sum_{k=1}^{G} \sigma_{a,k}^{*} \frac{\partial B_k^*}{\partial T}} \,,$$

$$\xi_g = \sigma_{a,g}^{*} B_g^* + \tau \psi_g^n -$$

$$\frac{1}{4\pi} \nu \chi_g \left[\sum_{k=1}^{G} \sigma_{a,k}^{*} 4\pi B_k^* + \frac{C_v^*}{\Delta t^{n+\frac{1}{2}}} \left(T^n - T^* \right) \right] .$$

Source Iteration

- The traditional method for solving the transport equation is a nested source iteration.
- Denoting the iteration index by ℓ, the inner iteration can be represented as follows:

$$\overrightarrow{\Omega} \cdot \overrightarrow{\nabla} I_g^{\ell+1} + \sigma_{\tau,g}^* I_g^{\ell+1} = \frac{1}{4\pi}\sigma_{s,g}^* \phi_g^\ell + \frac{1}{4\pi}\nu\chi_g \sum_{k=1}^{G} \sigma_{a,k}^* \phi_k \; + \xi_g \, ,$$

- and the outer iteration can be represented as follows:

$$\overrightarrow{\Omega} \cdot \overrightarrow{\nabla} I_g^{\ell+1} + \sigma_{\tau,g}^* I_g^{\ell+1} - \frac{1}{4\pi}\sigma_{s,g}^* \phi_g^{\ell+1} = \frac{1}{4\pi}\nu\chi_g \sum_{k=1}^{G} \sigma_{a,k}^* \phi_k^\ell \; + \xi_g \, .$$

Source Iteration

- The operator $\overrightarrow{\Omega} \cdot \overrightarrow{\nabla} + \sigma^*_{\tau,g}$ involves no angular or energy coupling.
- When spatially discretized it takes on a block lower-triangular form with a block corresponding to the intensities within a single spatial cell for a single direction and energy.
- This operator is easily inverted using a “wavefront” or “sweep” algorithm.
- The attenuation of errors in ϕ_g determines the convergence rate of the inner iteration process.
- The attenuation of errors in the absorption rate $f = \sum_{g=1}^{G} \sigma^*_{a,g} \phi_g$ determines the convergence rate of the outer iteration process.

Source Iteration

- The inner iteration process can become arbitrarily slow to converge as $\sigma^*_{s,g} \longrightarrow \sigma^*_{\tau,g}$. This corresponds to scattering dominating absorption. However, this is rarely the case for laboratory high-energy density physics (HEDP) calculations, so we will not discuss inner iteration convergence acceleration techniques.
- The outer iteration can become arbitrarily slow to converge as $\nu \longrightarrow 1$ and $\tau \longrightarrow 0$. This physically corresponds to strong material-radiation coupling (small heat capacity and large absorption cross section), which is quite common in calculations for laboratory HEDP calculations.
- In general, outer source iteration without convergence acceleration is impractical in laboratory HEDP calculations.
- The linear multifrequency-grey technique is generally used to accelerate the convergence of the outer iterations.

Linear Multifrequency-Grey Acceleration

- Outer source iteration with LMFGA takes the following form:

$$\overrightarrow{\Omega} \cdot \overrightarrow{\nabla} I_g^{\ell+\frac{1}{2}} + \sigma_{\tau,g}^* I_g^{\ell+\frac{1}{2}} - \frac{1}{4\pi}\sigma_{s,g}^* \phi_g^{\ell+\frac{1}{2}} = \frac{1}{4\pi}\nu\chi_g f^\ell + \xi_g \,,$$

$$-\overrightarrow{\nabla} \cdot \langle D \rangle \overrightarrow{\nabla} \delta\Phi + \left[\langle \sigma_a \rangle (1-\nu) + \tau\right] \delta\Phi = f^{\ell+\frac{1}{2}} - f^\ell \,,$$

$$f^{\ell+1} = f^{\ell+\frac{1}{2}} + \langle \sigma_a \rangle \delta\Phi \,,$$

Linear Multifrequency-Grey Acceleration

- where

$$\langle D \rangle = \sum_{g=1}^{G} \frac{\varsigma_g}{3\sigma^*_{\tau,g}} ,$$

$$\langle \sigma_a \rangle = \sum_{g=1}^{G} \sigma^*_{a,g} \varsigma_g ,$$

$$\varsigma_g = \frac{\frac{\chi_g}{\sigma^*_{\tau,g}}}{\sum_{k=1}^{G} \frac{\chi_k}{\sigma^*_{\tau,k}}} .$$

Linear Multifrequency-Grey Acceleration

- The scheme appears to be unconditionally effective in 1-D but can apparently become unstable in strongly heterogeneous multidimensional problems.
- This has motivated us to develop a preconditioned Krylov method based upon the multifrequency-grey acceleration technique.

Krylov Methods

- The details of Krylov methods are not of importance for this discussion, but some knowledge is required.
- Suppose one wants to solve a linear system of the following basic form:

$$\mathbf{M}\overrightarrow{x} = \overrightarrow{y},$$

  where $\mathbf{M}$ is a matrix, $\overrightarrow{x}$ is the solution vector, and $\overrightarrow{y}$ is the source vector.
- To use a Krylov solver, one must be able to evaluate the *action* of $\mathbf{M}$ on an arbitrary vector, $\overrightarrow{z}$, i.e., given $\overrightarrow{z}$, one must compute

$$\overrightarrow{v} = \mathbf{M}\overrightarrow{z}.$$

- From the viewpoint of the user, the performance of a matrix-vector multiply is all that is required per Krylov iteration.

Krylov Methods

- In many instances, the matrix $\mathbf{M}$ is dense and the action of $\mathbf{A}$ must be calculated in an indirect manner.
- Characterizing the convergence of Krylov methods for general matrices remains an open problem.
- However, there is one simple rule that can be followed: convergence will improve as the domain of the eigenvalues becomes smaller and as the domain moves away from the origin.
- Preconditioning can be used to improve convergence.
- Left preconditioning consists of multiplying a matrix equation from the left with a "preconditioning matrix" that yields a new equation with the same solution but better convergence properties:

$$\mathbf{G}\mathbf{M}\overrightarrow{x} = \mathbf{G}\overrightarrow{y} \ .$$

An LMFG-Preconditioned Krylov Method

- In general, there is a simple way to define a preconditioned Krylov method based upon a traditional accelerated iteration scheme.
  - Express the accelerated iteration process in the form of Richardson iteration.
  - The system solved by Richardson iteration is the preconditioned system that should be solved with the Krylov method.
- Richardson iteration for the following linear system

$$\mathbf{M}\overrightarrow{x} = \overrightarrow{y}\,,$$

takes the following form:

$$x^{\ell+1} = \overrightarrow{x}^{\ell} + \overrightarrow{y} - \mathbf{M}\overrightarrow{x}^{\ell}\,.$$

An LMFG-Preconditioned Krylov Method

- The preconditioned system corresponding to the LMFGA method is

$$\mathbf{CB}f = \mathbf{C}\sum_{g=1}^{G}\sigma_{a,g}^{*}\mathbf{PA}_g^{-1}\xi_g\,,$$

$$\mathbf{B} = \left[\mathbf{I} - \sum_{g=1}^{G}\sigma_{a,g}^{*}\mathbf{PA}_g^{-1}\frac{1}{4\pi}\nu\chi_g\right]\,,$$

$$\mathbf{A}_g \equiv \overrightarrow{\Omega}\cdot\overrightarrow{\nabla} + \sigma_{\tau,g}^{*} - \frac{1}{4\pi}\sigma_{s,g}\mathbf{P}\,,$$

$$\mathbf{P}\langle\cdot\rangle = \int_{4\pi}\langle\cdot\rangle\, d\Omega\,,$$

$$\mathbf{C} \equiv \left(\mathbf{I} + \langle\sigma_a\rangle\mathbf{H}^{-1}\nu\right)\,,$$

$$\mathbf{H} \equiv -\overrightarrow{\nabla}\cdot\langle D\rangle\overrightarrow{\nabla} + \left[\langle\sigma_a\rangle(1-\nu)+\tau\right]\,.$$

An LMFG-Preconditioned Krylov Method

- The main advantage of solving the equation for f as opposed to ϕ_g or ψ_g is that the rank of the f-equation is far less than that of the equations corresponding to the other possible unknowns.
- As one would expect, forming the action of the operator is completely analogous to peforming an LMFGA iteration.
- The one-group equations are solved by a Krylov method preconditioned with source iteration.
- The diffusion equation itself is solved via a precondioned Krylov method.
- Thus this is a nested Krylov method.
- This scheme has not yet been tested. A multigroup diffusion version has been tested and found to be very effective.

A Generic Transport Equation

- An understanding of the radiation-hydrodynamics coupling is facilitated by expressing the lab-frame transport equation with material motion in generic form as follows:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} I = Q \,,$$

where Q can take on different forms depending upon the model equation.

- The corresponding radiation energy equation is

$$\frac{\partial \mathcal{E}}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{F}} = S_{re} \,,$$

where

$$S_{re} = \int_0^\infty \int_{4\pi} Q \; d\Omega \, dE \,.$$

A Generic Transport Equation

- The corresponding radiation momentum equation is

$$\frac{1}{c^2}\frac{\partial \overrightarrow{\mathcal{F}}}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{\overrightarrow{\mathcal{P}}}_g = \overrightarrow{S}_{rm} \,,$$

where

$$\overrightarrow{S}_{rm} = \int_0^\infty \int_{4\pi} \frac{\overrightarrow{\Omega}}{c} Q \, d\Omega \, dE \,.$$

The Nonrelativistic Rad-Hydro Equations

- We are now ready to write down the non-relativistic radiation-hydrodynamics equations.
- Conservation of Mass:

$$\frac{\partial}{\partial t}\rho + \overrightarrow{\nabla} \cdot \left(\rho \overrightarrow{u}\right) = 0\,,$$

where ρ is the mass density,

- Conservation of Material Momentum:

$$\frac{\partial}{\partial t}\left(\rho \overrightarrow{u}\right) + \overrightarrow{\nabla} \cdot \left(\rho \overrightarrow{u} \otimes \overrightarrow{u}\right) + \overrightarrow{\nabla} p = -\overrightarrow{S}_{rm}\,,$$

where $\overrightarrow{u}$ is the material velocity and p is the pressure,

The Nonrelativistic Rad-Hydro Equations

- Conservation of Total Material Energy:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \overrightarrow{\nabla}\cdot\left[\left(\frac{1}{2}\rho u^2 + \rho e + p\right)\overrightarrow{u}\right] = -S_{re}\,,$$

  where e is the material specific internal energy. The pressure and temperature are given by the equation of state, $p = p(\rho, e)$, and $T = T(\rho, e)$.

- The remaining equation is the transport equation, which also yields the radiation energy and momentum equations.

- It is easily shown that if the material and radiation momentum equations are summed, a conservation equation for the total momentum is obtained; and if the total material and radiation energy equations are summed, a conservation equation for the total energy is obtained.

The Nonrelativistic Rad-Hydro Equations

- In some hydro methods, an internal material energy equation is used instead of a total energy equation:

$$\frac{\partial}{\partial t}(\rho e) + \overrightarrow{\nabla} \cdot \left(\rho e \overrightarrow{u}\right) + p \overrightarrow{\nabla} \cdot \overrightarrow{u} = S_{re} - \overrightarrow{S}_{rm} \cdot \overrightarrow{u} .$$

- The term on the right is equal to the total energy exchange rate minus the kinetic energy exchange rate and thus is equal to the internal energy exchange rate.

A Simplified Lab-Frame Transport Equation

- Our primary intent is to accurately compute the exchange of energy and momentum between the radiation and material fields with nonrelativistic material motion, which implies red and blue shifts that are too small to be resolved by the group structure.
- We have derived an approximate lab-frame transport equation in accordance with these constraints that has the following properties when coupled with the hydrodynamic equations:
  - Total (radiation plus material) energy and momentum are conserved.
  - The correct equilibrium solutions for radiation energy density, radiation flux, and radiation pressure are obtained to O(u/c)
  - The equilibrium-diffusion limit for radiation-hydrodynamics is preserved to O(u/c).

A Simplified Lab-Frame Transport Equation

- The simplified equation takes the form of the transport equation in a static medium plus a P_1-like correction term:

$$\frac{1}{c}\frac{\partial I_g}{\partial t} + \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} I_g + \sigma_{t,g} I_g = \frac{\sigma_{s,g}}{4\pi}\phi_g + \sigma_{a,g} B_g +$$

$$\frac{1}{4\pi} C_{0,g} + \frac{3}{4\pi} \overrightarrow{C}_{1,g} \cdot \overrightarrow{\Omega} \,,$$

where

$$C_{0,g} = -\sigma_{t,g} \left(\overrightarrow{F}_g - \frac{4}{3}\phi_g \frac{\overrightarrow{u}}{c} \right) \cdot \frac{\overrightarrow{u}}{c} \,,$$

$$\overrightarrow{C}_{1,g} = \sigma_{t,g} \frac{4}{3} \phi_g \frac{\overrightarrow{u}}{c} \,.$$

A Simplified Lab-Frame Transport Equation

- The corresponding radiation energy and radiation momentum equations are respectively:

$$\frac{\partial \mathcal{E}}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{F}} = \sum_{g=1}^{G} \sigma_{a,g} [\, 4\pi B_g - \phi_g \,] -$$

$$\sum_{g=1}^{G} \sigma_{a,g} \left(\overrightarrow{F}_g - \frac{4}{3} \phi_g \frac{\overrightarrow{u}}{c} \right) \cdot \frac{\overrightarrow{u}}{c} \, ,$$

and

$$\frac{1}{c^2} \frac{\partial \overrightarrow{\mathcal{F}}_g}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{\overrightarrow{\mathcal{P}}}_g = - \sum_{g=1}^{G} \frac{\sigma_{a,g}}{c} \left(\overrightarrow{F}_g - \frac{4}{3} \phi_g \frac{\overrightarrow{u}}{c} \right) \, ,$$

Operator Splitting

- To demonstrate the concept of splitting, we first consider the following system:

$$\frac{\partial f}{\partial t} = (\mathbf{A} + \mathbf{B})\, f\,,$$

where f is the solution and $\mathbf{A}$ and $\mathbf{B}$ are linear operators.

- Let us further assume that we know how to efficiently invert $\mathbf{A}$ and $\mathbf{B}$ individually but not the sum of $\mathbf{A}$ and $\mathbf{B}$.

- Then we can efficiently solve this system as follows:

$$\frac{f^* - f^n}{\Delta t} = \mathbf{A} f^*\,,$$

$$\frac{f^{n+1} - f^*}{\Delta t} = \mathbf{B} f^{n+1}\,.$$

Operator Splitting

- Note that that each step of the process is fully implicit ensuring stability.
- If we add the two equations, we get a consistent time discretization:

$$\frac{f^{n+1} - f^n}{\Delta t} = \mathbf{A} f^* + \mathbf{B} f^{n+1} \,.$$

- Traditional operator splitting can sometimes require very small time steps and is only first-order accuate, but the method has been the workhorse of multiphysics computation for decades.
- A great deal of current research is devoted to the development of splitting techniques that are both second order accurate and efficient.
- Alternatively, current research includes simultaneous solution of the equations using Jacobian-free Newton-Krylov methods with operator splitting for preconditioning.

Solution of the Rad-Hydro Equations

- We next demonstrate a way to use operator splitting to solve the radiation-hydrodynamics equations with first-order accuracy in time using standard solution techniques for the hydro equations together with standard techniques for the static radiative transfer equations.

Solution of the Rad-Hydro Equations

- The first step is to explicitly solve the hydrodynamics equations as if there were no radiation:

$$\frac{1}{\Delta t}\left(\rho^* - \rho^n\right) + \overrightarrow{\nabla} \cdot \left(\rho \overrightarrow{u}\right)^n = 0\,,$$

$$\frac{1}{\Delta t}\left[\left(\rho \overrightarrow{u}\right)^* - \left(\rho \overrightarrow{u}\right)^n\right] + \overrightarrow{\nabla} \cdot \left(\rho \overrightarrow{u} \otimes \overrightarrow{u}\right)^n + \overrightarrow{\nabla} p^n = \overrightarrow{0}\,,$$

$$\frac{1}{\Delta t}\left[\left(\frac{1}{2}\rho u^2 + \rho e\right)^* - \left(\frac{1}{2}\rho u^2 + \rho e\right)^n\right] +$$
$$\overrightarrow{\nabla} \cdot \left[\left(\frac{1}{2}\rho u^2 + \rho e + p\right)\overrightarrow{u}\right]^n = 0\,.$$

Solution of the Rad-Hydro Equations

- The second step is to implicitly solve the radiative transfer equations "almost" as if there were no hydrodynamics:

$$\frac{1}{c\Delta t}\left(I_g^* - I_g^n\right) + \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} I_g^* + \sigma_{t,g}^n I_g^* = \frac{\sigma_{s,g}^n}{4\pi}\phi_g^* +$$

$$\sigma_{a,g}^n \left[B_g^n + \frac{\partial B_g^n}{\partial T}\left(T^{<n+1>} - T^n\right)\right], \quad g = 1, G,$$

$$\frac{C_v^n}{\Delta t}\left(T^{<n+1>} - T^n\right) =$$

$$\sum_{g=1}^{G} \sigma_{a,g}^n \left[\phi_g^* - 4\pi B_g^n - 4\pi \frac{\partial B_g^n}{\partial T}\left(T^{<n+1>} - T^n\right)\right] +$$

$$\frac{1}{\Delta t}\left[(\rho e)^* - (\rho e)^n\right].$$

Solution of the Rad-Hydro Equations

- Note that the last term on the right side of the temperature equation effectively adds the effect of the internal energy advection and work from the pressure gradient that was computed in the first step.
- Since it simply represents an explicit source term, it does not affect the applicability of the standard transport solution technique.

Solution of the Rad-Hydro Equations

- The third step is to explicitly solve for the final radiation intensities and material temperatures adding the effect of the hydrodynamic coupling.
- Note that this step is effectively explicit and involves only local updates in each cell.

$$\frac{1}{c\Delta t}\left(I_g^{n+1} - I_g^*\right) =$$

$$-\frac{1}{4\pi}\sigma_{t,g}^n\left(\overrightarrow{F}_g^* - \frac{4}{3}\phi_g^*\frac{\overrightarrow{u}^n}{c}\right)\cdot\frac{\overrightarrow{u}^n}{c} +$$

$$\frac{3}{4\pi}\sigma_{t,g}^n\frac{4}{3}\phi_g^*\frac{\overrightarrow{u}^n}{c}\cdot\overrightarrow{\Omega}, \quad g = 1, G,$$

Solution of the Rad-Hydro Equations

- The fourth step is to compute the final hydrodynamics variables including the effect of the radiation coupling.
- Note that this step is effectively explicit and involves only local updates in each cell.

$$\rho^{n+1} = \rho^* \,,$$

$$\frac{1}{\Delta t}\left[\left(\rho \overrightarrow{u}\right)^{n+1} - \left(\rho \overrightarrow{u}\right)^{*}\right] = \sum_{g=1}^{G} \sigma_{t,g}^{n} \left(\overrightarrow{F}_g^{*} - \frac{4}{3}\phi_g^{*}\frac{\overrightarrow{u}^{n}}{c}\right) \,,$$

Solution of the Rad-Hydro Equations

$$\frac{1}{\Delta t}\left[\left(\frac{1}{2}\rho u^2+\rho e\right)^{n+1}-\left(\frac{1}{2}\rho u^2+\rho e\right)^{*}\right]=$$

$$\sum_{g=1}^{G}\sigma_{a,g}^{n}\left[\phi_g^{*}-4\pi B_g^{n}-4\pi\frac{\partial B_g^{n}}{\partial T}\left(T^{<n+1>}-T^{n}\right)\right]+$$

$$\sum_{g=1}^{G}\sigma_{t,g}^{n}\left(\overrightarrow{F}_g^{*}-\frac{4}{3}\phi_g^{*}\frac{\overrightarrow{u}^{n}}{c}\right)\cdot\frac{\overrightarrow{u}^{n}}{c}.$$

Solution of the Rad-Hydro Equations

- The fifth and final step is to use the equation-of-state to obtain the final pressures and temperatures:

$$p^{n+1} = p(\rho^{n+1}, e^{n+1}) \, ,$$

$$T^{n+1} = T(\rho^{n+1}, e^{n+1}) \, .$$

Solution of the Rad-Hydro Equations

- The equation for the total change in material density is:

$$\frac{1}{\Delta t}\left(\rho^{n+1} - \rho^{n}\right) + \overrightarrow{\nabla} \cdot \left(\rho \overrightarrow{u}\right)^{n} = 0\,,$$

- The equation for the total change in material momentum is:

$$\frac{1}{\Delta t}\left[\left(\rho \overrightarrow{u}\right)^{n+1} - \left(\rho \overrightarrow{u}\right)^{n}\right] + \overrightarrow{\nabla} \cdot \left(\rho \overrightarrow{u} \otimes \overrightarrow{u}\right)^{n} + \overrightarrow{\nabla} p^{n} =$$

$$\sum_{g=1}^{G} \sigma_{t,g}^{n} \left(\overrightarrow{F}_{g}^{*} - \frac{4}{3}\phi_{g}^{*} \frac{\overrightarrow{u}^{n}}{c}\right)\,,$$

Solution of the Rad-Hydro Equations

- The equation for the total change in material energy is:

$$\frac{1}{\Delta t}\left[\left(\frac{1}{2}\rho u^2+\rho e\right)^{n+1}-\left(\frac{1}{2}\rho u^2+\rho e\right)^{n}\right]+$$

$$\overrightarrow{\nabla}\cdot\left[\left(\frac{1}{2}\rho u^2+\rho e+p\right)\overrightarrow{u}\right]^n=$$

$$\sum_{g=1}^{G}\sigma_{a,g}^{n}\left[\phi_g^*-4\pi B_g^n-4\pi\frac{\partial B_g^n}{\partial T}\left(T^{<n+1>}-T^k\right)\right]+$$

$$\sum_{g=1}^{G}\sigma_{t,g}^{n}\left(\overrightarrow{F}_g^*-\frac{4}{3}\phi_g^*\frac{\overrightarrow{u}^n}{c}\right)\cdot\frac{\overrightarrow{u}^n}{c}\,.$$

Solution of the Rad-Hydro Equations

- The equation for the total change in radiation intensity is:

$$\frac{1}{c\Delta t}\left(I_g^{n+1} - I_g^{n}\right) + \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} I_g^{*} + \sigma_{t,g}^{n} I_g^{*} =$$

$$\frac{\sigma_{s,g}^{n}}{4\pi}\phi_g^{*} + \sigma_{a,g}^{n}\left[B_g^{n} + \frac{\partial B_g^{n}}{\partial T}\left(T^{<n+1>} - T^{n}\right)\right] -$$

$$\frac{1}{4\pi}\sigma_{t,g}^{n}\left(\overrightarrow{F}_g^{*} - \frac{4}{3}\phi_g^{*}\frac{\overrightarrow{u}^{n}}{c}\right) \cdot \frac{\overrightarrow{u}^{n}}{c} -$$

$$\frac{3}{4\pi}\sigma_{t,g}^{n}\frac{4}{3}\phi_g^{*}\frac{\overrightarrow{u}^{n}}{c} \cdot \overrightarrow{\Omega}, \quad g = 1, G.$$

Solution of the Rad-Hydro Equations

- The equations for the total change in radiation energy and momentum are respectively:

$$\frac{1}{c\Delta t}\left(\mathcal{E}^{n+1}-\mathcal{E}^{n}\right)+\overrightarrow{\nabla}\cdot\overrightarrow{\mathcal{F}}^{n+1}=$$

$$-\sum_{g=1}^{G}\sigma^{n}_{a,g}\left[\phi^{*}_{g}-4\pi B^{n}_{g}-4\pi\frac{\partial B^{n}_{g}}{\partial T}\left(T^{<n+1>}-T^{k}\right)\right]-$$

$$\sum_{g=1}^{G}\sigma^{n}_{t,g}\left(\overrightarrow{F}^{*}_{g}-\frac{4}{3}\phi^{*}_{g}\frac{\overrightarrow{u}^{n}}{c}\right)\cdot\frac{\overrightarrow{u}^{n}}{c},$$

and

$$\frac{1}{c^{2}\Delta t}\left(\overrightarrow{\mathcal{F}}^{n+1}-\overrightarrow{\mathcal{F}}^{n}\right)+\overrightarrow{\nabla}\cdot\overset{\Longrightarrow}{\mathcal{P}}_{g}=-\sum_{g=1}^{G}\sigma^{n}_{t,g}\left(\overrightarrow{F}^{*}_{g}-\frac{4}{3}\phi^{*}_{g}\frac{\overrightarrow{u}^{n}}{c}\right).$$

Solution of the Rad-Hydro Equations

- This algorithm conserves mass, total momentum (radiation plus material), and total energy (radiation plus material).
- It is very nearly equivalent to an unsplit scheme, i.e., to simultaneous solution of all our *semi-linearized* equations.
- The only reason this is not so is that the radiation equations are solved in two separate implicit steps.
- Let us combine steps two and three into a single step as follows.

Solution of the Rad-Hydro Equations

$$\frac{1}{c\Delta t}\left(I_g^{n+1}-I_g^n\right)+\overrightarrow{\Omega}\cdot\overrightarrow{\nabla}I_g^{n+1}+\sigma_{t,g}^n I_g^{n+1}=\frac{\sigma_{s,g}^n}{4\pi}\phi_g^{n+1}+$$

$$\sigma_{a,g}^n\left[B_g^n+\frac{\partial B_g^n}{\partial T}\left(T^{<n+1>}-T^n\right)\right]-,$$

$$\frac{1}{4\pi}\sigma_{t,g}^n\left(\overrightarrow{F}_g^{n+1}-\frac{4}{3}\phi_g^{n+1}\frac{\overrightarrow{u}^n}{c}\right)\cdot\frac{\overrightarrow{u}^n}{c}+$$

$$\frac{3}{4\pi}\sigma_{t,g}^n\frac{4}{3}\phi_g^{n+1}\frac{\overrightarrow{u}^n}{c}\cdot\overrightarrow{\Omega},\quad g=1,G,$$

Solution of the Rad-Hydro Equations

$$\frac{C_v^n}{\Delta t}\left(T^{<n+1>} - T^n\right) =$$

$$\sum_{g=1}^{G} \sigma_{a,g}^n \left[\phi_g^{n+1} - 4\pi B_g^n - 4\pi \frac{\partial B_g^n}{\partial T}\left(T^{<n+1>} - T^n\right)\right] +$$

$$\frac{1}{\Delta t}\left[(\rho e)^* - (\rho e)^n\right] .$$

- With this modification the equations for the total changes are as follows.

Solution of the Rad-Hydro Equations

$$\frac{1}{\Delta t}\left(\rho^{n+1} - \rho^{n}\right) + \overrightarrow{\nabla}\cdot\left(\rho\,\overrightarrow{u}\right)^{n} = 0\,,$$

$$\frac{1}{\Delta t}\left[\left(\rho\,\overrightarrow{u}\right)^{n+1} - \left(\rho\,\overrightarrow{u}\right)^{n}\right] + \overrightarrow{\nabla}\cdot\left(\rho\,\overrightarrow{u}\otimes\overrightarrow{u}\right)^{n} =$$

$$\sum_{g=1}^{G}\sigma_{t,g}^{n}\left(\overrightarrow{F}_{g}^{n+1} - \frac{4}{3}\phi_g^{n+1}\frac{\overrightarrow{u}^{n}}{c}\right)\,,$$

Solution of the Rad-Hydro Equations

$$\frac{1}{\Delta t}\left[\left(\frac{1}{2}\rho u^2+\rho e\right)^{n+1}-\left(\frac{1}{2}\rho u^2+\rho e\right)^{n}\right]+$$

$$\overrightarrow{\nabla}\cdot\left[\left(\frac{1}{2}\rho u^2+\rho e+p\right)\overrightarrow{u}\right]^{n}=$$

$$\sum_{g=1}^{G}\sigma_{a,g}^{n}\left[\phi_g^{n+1}-4\pi B_g^n-4\pi\frac{\partial B_g^n}{\partial T}\left(T^{<n+1>}-T^n\right)\right]+$$

$$\sum_{g=1}^{G}\sigma_{t,g}^{n}\left(\overrightarrow{F}_g^{n+1}-\frac{4}{3}\phi_g^{n+1}\frac{\overrightarrow{u}^{n}}{c}\right)\cdot\frac{\overrightarrow{u}^{n}}{c}.$$

Solution of the Rad-Hydro Equations

$$\frac{1}{c\Delta t}\left(I_g^{n+1} - I_g^n\right) + \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} I_g^* + \sigma_{t,g}^n I_g^* =$$

$$\frac{\sigma_{s,g}^n}{4\pi}\phi_g^{n+1} + \sigma_{a,g}^n\left[B_g^n + \frac{\partial B_g^n}{\partial T}\left(T^{<n+1>} - T^n\right)\right] -$$

$$\frac{1}{4\pi}\sigma_{t,g}^n\left(\overrightarrow{F}_g^{n+1} - \frac{4}{3}\phi_g^{n+1}\frac{\overrightarrow{u}^n}{c}\right)\cdot\frac{\overrightarrow{u}^n}{c} -$$

$$\frac{3}{4\pi}\sigma_{t,g}^n\frac{4}{3}\phi_g^{n+1}\frac{\overrightarrow{u}^n}{c}\cdot\overrightarrow{\Omega}, \quad g = 1, G,$$

Solution of the Rad-Hydro Equations

$$\frac{1}{c\Delta t}\left(\mathcal{E}^{n+1}-\mathcal{E}^{n}\right)+\overrightarrow{\nabla}\cdot\overrightarrow{\mathcal{F}}^{n+1} =$$

$$-\sum_{g=1}^{G}\sigma_{a,g}^{n}\left[\phi_g^{n+1}-4\pi B_g^n-4\pi\frac{\partial B_g^n}{\partial T}\left(T^{<n+1>}-T^k\right)\right]-$$

$$\sum_{g=1}^{G}\sigma_{t,g}^{n}\left(\overrightarrow{F}_g^{n+1}-\frac{4}{3}\phi_g^{n+1}\frac{\overrightarrow{u}^{n}}{c}\right)\cdot\frac{\overrightarrow{u}^{n}}{c},$$

and

$$\frac{1}{c^2\Delta t}\left(\overrightarrow{\mathcal{F}}^{n+1}-\overrightarrow{\mathcal{F}}^{n}\right)+\overrightarrow{\nabla}\cdot\overrightarrow{\overrightarrow{\mathcal{P}}}_g = -\sum_{g=1}^{G}\sigma_{t,g}^{n}\left(\overrightarrow{F}_g^{n+1}-\frac{4}{3}\phi_g^{n+1}\frac{\overrightarrow{u}^{n}}{c}\right).$$

Solution of the Rad-Hydro Equations

$$p^{n+1} = p(\rho^{n+1}, e^{n+1})\,,$$
$$T^{n+1} = T(\rho^{n+1}, e^{n+1})\,.$$

Solution of the Rad-Hydro Equations

- The difficulty with combining the second and third steps is that the Krylov solution technique for the radiation equations is no longer directly applicable.
- There is more than one way to incorporate the material-motion terms in the solution process.
- The best way is probably to include them in the one-group solves.
- The Krylov vector for each group will have to be expanded to include the fluxes in addition to the angle-integrated intensities.
- Because the material-motion corrections terms are basically perturbative in the nonrelativistic limit, one can expect the Krylov method preconditioned with source iteration to remain highly effective.
- This is the great advantage of using preconditioned Krylov methods.

Solution of the Rad-Hydro Equations

- Finally, we note that being able to combine the original steps 2 and 3 paves the way for a second-order accurate scheme based upon splitting.
- Our second-order accurate scheme uses the same basic hydrodynamics and transport solution technology as the first-order scheme but applies it in the form of a predictor-corrector method.
- The predictor step is identical to the four-step first-order scheme with averaging of quantities at indexes $n+1$ and n to obtain predicted quantities at index $n+\frac{1}{2}$.
- The hydrodynamic solutions in the corrector step remain explicit but with predicted values at $n+\frac{1}{2}$ replacing certain quantities at n.

Solution of the Rad-Hydro Equations

- Two independent solutions of the transport and temperature equations are performed in the corrector step using the trapezoidal/BDF-2 scheme.
- Our second-order scheme remains to be theoreticaly analyzed and computationally tested.
- It will eventually be implemented in a code to model laser-driven radiative shocks.
- We hope to report on its effectiveness relative to the first-order method for the case of grey diffusion rather than multigroup transport within a few months.