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SCATTERING FROM A PENNY-SHAPED CRACK IN A POROELASTIC MEDIUM

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ABSTRACT

One common method of modeling fracturing is to consider a background medium containing a distribution of aligned 'penny-shaped' cracks and to calculate the long-wavelength overall elastic properties. If the system of pores and fractures in a fluid-saturated rock is interconnected, then fluid flow between pores and fractures must be taken into account. In this paper we consider the scattering that takes place when a plane longitudinal wave, harmonic in time, propagating in the positive direction of the *z*-axis, impinges at normal incidence on a penny-shaped crack imbedded in a poroelastic medium. We solve a Fredholm integral equation of the second kind in the limits of high and low frequencies to obtain the scattered field. The effective properties of a medium containing a distribution of such scatterers are obtained using the 'forward-amplitude theorem of multiple scattering'.

INTRODUCTION

Naturally fractured reservoirs are becoming increasingly important for oil and gas exploration in many areas of the World. Nelson (2001) recognized four different types of fractured reservoir. In type I reservoirs, the matrix has little porosity or permeability, the fractures providing the essential storage capacity and permeability. In type II reservoirs, the rock matrix has low permeability, but may have low, moderate or even high porosity. In these reservoirs the fractures provide the permeability required to make production economic. In type III reservoirs, the rock matrix has good porosity and permeability, while the fractures help to assist production. Finally, in type IV reservoirs the fractures do not provide additional porosity or permeability. From an economic point of view the impact of fractures in Type I and II reservoirs is the most significant, because of the low matrix permeability of such reservoirs.

Characterization of a reservoir requires a prediction of how fluid flow will affect its elastic properties. For wave propagation at low frequencies, in a porous reservoir this effect can be expressed using Gassmann's relations (Gassmann, 1951) which provide explicit analytical expressions for the effective elastic moduli of a fluid-saturated rock as functions of the properties of the dry skeleton and the saturating fluid. If aligned fractures are introduced, fluid flow will affect the elastic anisotropy of the rock and may result in significant frequency-dependent attenuation and dispersion (Thomson, 1995; Hudson et al., 1996; Hudson et al., 2001; Tod, 2001).

One common method of modeling fracturing is to consider a background medium containing a distribution of aligned 'penny-shaped' cracks and to calculate the long-wavelength overall elastic properties. This approach has been taken for elasticity (Hudson, 1980; Mori and Tanaka, 1973). In this paper we consider the same problem for a distribution of aligned penny-shaped cracks imbedded in a medium governed by Biots equations of poroelasticity (Biot, 1956; Biot, 1962).

A SINGLE CRACK

The fundamental ingredient required to calculate the effective properties of a medium containing a distribution of cracks is knowledge of the wave scattering that takes place due to the presence of a single crack. In this paper we consider the scattering that takes place when a plane longitudinal wave, harmonic in time, propagating in the positive direction of the *z*-axis, impinges at normal incidence on a penny-shaped crack (Figure 1) imbedded in an infinite medium governed by Biots equations of poroelasticity.

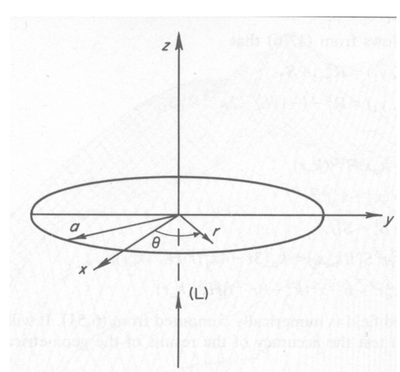

Figure 1. Geometry of the scattering problem (after Achenbach et al., 1982).

Robertson (1967) solved the same problem for the simpler elastic case and we closely follow his method of solution. A cylindrical co-ordinate system is adopted due to the axisymmetrical nature of the problem. In the following, we non-dimensionalise all quantities with respect to the shear modulus, shear wave speed and crack radius.

Boundary Conditions

Solving the single scattering problem amounts to obtaining a solution to Biots equations of poroelasticity that satisfies imposed conditions at the boundary between crack and medium. The behaviour of the medium in the neighbourhood of the crack is the same as that produced in a semi-infinite poroelastic medium $z \ge 0$ when its free surface $z = 0$ is subject to the following boundary conditions:

$$\sigma_{rz} = 0 \qquad 0 \le r < \infty \qquad (1)$$

$$U_z = 0 \qquad 0 \le r < \infty \qquad (2)$$

$$u_z = 0 \qquad 1 < r < \infty \qquad (3)$$

$$\sigma_{zz} + p_f = -ik_1\left(H - \alpha M\right)u_0 \qquad 0 \le r < 1 \qquad (4)$$

where σ_{rz} is the shear stress acting on the crack surface, U_z is the *z*-component of absolute fluid displacement, u_z is the *z*-component of solid displacement, σ_{zz} is the component of stress acting normal to the crack surface, p_f is the hydrostatic pressure of the crack-filling fluid, k_1 is the wavenumber of the incident wave, H is the saturated compressional-wave modulus, α and M are Biots parameters accounting for compressibility of the two-phased material and u_0 represents the amplitude of the incident wave. It should be noted that we seek the scattered field here, not the total field. These conditions constitute a mixed boundary value problem because stresses are prescribed within the crack, and displacements outside the crack. Condition (1) expresses the fact that there is no component of stress in the plane of the crack because the incident wave is longitudinal and impinges at normal incidence. Condition (2) states that there is no absolute fluid displacement in the scattered field perpendicular to the crack, and arises from the symmetry of the problem. Condition (3) also arises from symmetry considerations. Perhaps the most interesting boundary condition is (4), which defines the relationship between crack deformation and the resultant flow of fluid into or out of the crack (it is a permeable crack boundary). Because the penny-shaped crack has infinitesimally small thickness, we can neglect the compression of the crack-filling fluid that takes place during deformation. It follows that the change in volume of the crack must be equal to the volume of fluid that flows into or out of the crack during deformation. This approach relies on the assumption that when fluid is displaced from the crack, there is enough surrounding pore space to accommodate it. When expressed mathematically, this series of assumptions leads to (4).

The Scattered Field

Because the general solution to Biots equations in cylindrical co-ordinates are obtained using Hankel integral transform techniques, application of the boundary conditions (1)-(4) yields a set of integral equations whose solution gives the unknown scattered field amplitude co-efficients. There are three such co-efficients: $A_1(y)$ (representing the fast compressional wave), $A_2(y)$ (slow compressional wave) and $C(y)$ (shear wave), where y is the radial wavenumber. Because boundary conditions (1) and (2) are valid for all r, we can express two of the unknown amplitude functions in terms of the third, say $A_1(y)$, and reduce the problem to the solution of only two dual integral equations of the form

$$\int_0^\infty y\left[1 + T(y)\right]B(y)J_0(yr)\,dy = -p_0 \qquad 0 \le r < 1 \qquad (5)$$

$$\int_0^\infty B(y)J_0(yr)\,dy = 0 \qquad 1 < r < \infty \qquad (6)$$

where $T(y)$ is a known transfer function between stress and displacement arising from the mixed nature of the boundary conditions, $B(y)$ is the unknown amplitude function associated with the scattered field, J_0 is the zero order Bessel function of the first kind and $p_0 = ik_1\left(H - \alpha M\right)u_0$ is a constant representing the magnitude of the incident wave. It can be shown (Noble, 1963) that solving equations (5) and (6) is equivalent to solving the Fredholm equation of the second kind

$$\varphi(x) + \frac{1}{\pi}\int_0^\infty R(x,y)T(y)\varphi(y)\,dy = -p_0 S(y) \qquad (7)$$

where

$$R(x,y) = \frac{\sin(x-y)}{x-y} - \frac{\sin(x+y)}{x+y} \qquad (8)$$

$$S(y) = \frac{\sin y - y\cos y}{y^2} \tag{9}$$

and

$$B(y) = \frac{2}{\pi}\int_0^1 \varphi(\xi)\sin(\xi y)\,d\xi \tag{10}$$

For frequencies such that the fast compressional wave wavelength is much larger than the crack radius, T takes the form

$$T(y) = \frac{M}{H(L-1)}\left[\frac{T_1 + 2yq_2\alpha T_2}{2yq_2k_2^2}\right] \tag{11}$$

where

$$T_1 = \left(2\alpha y^2 + Lk_2^2\right)^2 \tag{12}$$

$$T_2 = k_2^2(\alpha - 2L) - 2\alpha y^2 \tag{13}$$

$$q_2 = \sqrt{y^2 + k_2{}^2} \tag{14}$$

L is the dry compressional-wave modulus, and k_2 is the slow wave wavenumber. We derive an analytical solution of (7) only for the low and high frequency asymptotic behaviour of (11). In the limit of low frequencies, (11) reduces to

$$T_{low} = \frac{M\left(2L^2 + 3\alpha^2 - 4\alpha L\right)k_2^2}{4H(L-1)y^2} \tag{15}$$

whereas for high frequencies,

$$T_{high} = \frac{ML^2k_2}{2H(L-1)y} \tag{16}$$

The effective media theory we intend to employ in the next section requires only the far-field scattering amplitude in the forward direction, *f(0)*. The asymptotic far-field amplitudes can be obtained by solving (7) using the approximations (15) and (16) and are

$$f_{low}(0) = \frac{ik_1L(5 - 2D_{low})(H - \alpha M)}{3\pi H(L-1)} \tag{17}$$

where

$$D_{low} = \frac{M\left(2L^2 + 3\alpha^2 - 4\alpha L\right)k_2^2}{4H(L-1)} \tag{18}$$

and

$$f_{high}(0) = \frac{ik_1(H - \alpha M)}{12\pi^2H^2(L-1)^2MLk_2} \tag{19}$$

A DISTRIBUTION OF CRACKS

Multiple Scattering

To model the behaviour of a poroelastic medium containing a distribution of cracks, we make use of the 'forward-amplitude theorem of multiple scattering' derived by Waterman and Truell (1961). This theorem characterizes the behaviour of a scattering medium through the complex propagation constant κ which is specified explicitly in terms of the number of scatterers per unit volume (n_0) and the far-field amplitude *f(θ)* obtained for a single scatterer. For low crack densities, this expression takes the form

$$\frac{\kappa}{k_1} \approx 1 + \frac{2\pi n_0 f(0)}{k_1^2} \tag{20}$$

κ gives a complete description of the effect that the scatterers have on the propagation of the incident wave through the medium. The modified phase velocity (v) and attenuation (a) that the incident wave experiences due to scattering can be determined from the real and imaginary components of κ as follows:

$$v = \frac{1}{\mathrm{Re}(\kappa)} \tag{21}$$

$$a = \frac{2\,\mathrm{Re}(\kappa)}{\mathrm{Im}(\kappa)} \tag{22}$$

Numerical Results

In this section we analyse the dependence of velocity and attenuation on frequency numerically. We solve (7) numerically using MATLAB and obtain the effective propagation constant for a distribution of cracks using equations (20)-(22). Table 1 contains the values of the relevant constants used in the calculations. Figure 2 is a plot of Frequency vs Dispersion. The low and high frequency approximations are correct in the limit of low and high frequencies, respectively, but are incorrect between these two limits. Figure 3 is a plot of Frequency vs Attenuation.

Table 1. Numericalintegration constants.

Data Type	Value
Grain bulk modulus	$37\times10^{9}\,GPa$
Grain shear modulus	$44\times10^{9}\,GPa$
Fluid bulk modulus	$2.25\times10^{9}\,GPa$
Porosity	30%

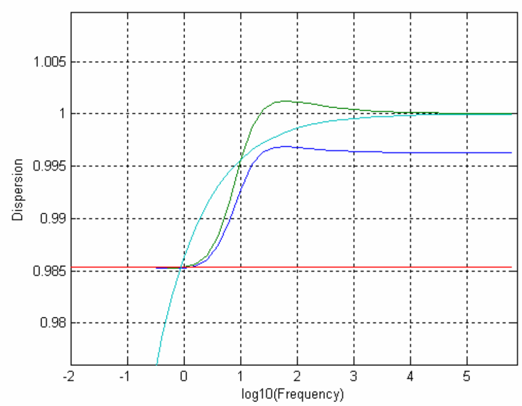

Figure 2. Frequency vs Dispersion. The blue and green curves are the numerical solutions to (7) using the low and high frequency approximations, respectively. The red and teal curves are the low and high frequency asymptotes, respectively.

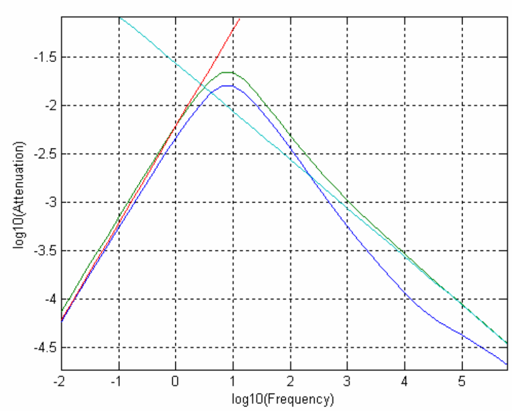

Figure 3. Frequency vs Attenuation. The blue and green curves are the numerical solutions to (7) using the low and high frequency approximations, respectively. The red and teal curves are the low and high frequency asymptotes, respectively.

Similarly, the asymptotic behaviour is correct.

ACKNOWLEDGMENTS

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