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Interval Estimation in the Presence of Nuisance Parameters
2. Cousins and Highland Method

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Abstract

This note explores the Cousins-Highland method for setting limits. This method is a hybrid frequentist-Bayesian method and performs similarly to a full Bayesian treatment. However, it is found that the approximate formulae have limited validity.

1 Introduction

In 1992, Cousins and Highland [1] proposed a method of setting limits in cases where there is uncertainty on the efficiency. As part of the Statistics Committee's study of methods of handling nuisance parameters when setting limits [2], this note explores the Cousins-Highland method.

The Cousins-Highland method has been extended by others (for example, see reference [3]) to include uncertainties on nuisance parameters other than the efficiency and to use Feldman-Cousins for the frequentist part of the calculation. These methods are not covered here.

Section 2 describes the Cousins and Highland method. Section 3 gives results on studies of distribution of limits, coverage, and sensitivity for the Cousins-Highland method and compares them to the full Bayesian case. Section 4 contains the conclusions.

2 The Cousins and Highland Method

The Cousins and Highland method sets limits for a counting experiment where the primary measurement is a Poisson-distributed integer n with expected value μ, which is related to the physics quantity of interest R by a sensitivity S ($\mu = RS$). The sensitivity S contains factors such as efficiencies, acceptances, branching ratios, and integrated luminosity and, hence, may be larger than one. It is assumed that S is determined in an auxiliary experiment and has a mean value $\hat{S}$, an uncertainty σ, and a Bayesian probability density $W(S|\hat{S}, \sigma)$. Cousins and Highland do a second order expansion and average over W, assuming that the average of $S - \hat{S}$ is zero and the average of $(S - \hat{S})^2$ is σ^2. Any distribution for W that meets these conditions, such as a Gaussian, is covered by their formulae.

The Cousins and Highland paper only covers uncertainties in the sensitivity S. However, the method has been extended and generalized elsewhere (see, for example, Tegenfelt and Conrad [3] and a note on the CDF Statistics Committee Recommendations page [4]). In these methods, the probability distribution for the observed quantity is first multiplied by the Bayesian prior distributions for all nuisance parameters and then integrated over the nuisance parameters. Finally, the resulting function is treated as a probability distribution for the quantity of interest and is used in a Neyman construction of limits.

If S is exactly known, then the usual frequentist answer is that the upper limit on n at a given confidence level CL is the value of the Poisson parameter μ such that the probability to observe n_{obs} or fewer events is $1 - CL$, that is,

$$\sum_{n=0}^{n_{obs}} P(n|\mu) = 1 - CL, \tag{1}$$

where

$$P(n|\mu) = \frac{e^{-\mu}\mu^n}{n!}. \tag{2}$$

The limit on R is then μ/S.

Cousins and Highland handle an uncertainty on S by calculating a probability density function for n that is a weighted average of the Poisson distribution, where the weight is W, that is,

$$\begin{aligned} p(n|R) &= \int p(n|R,S)W(S|\hat{S},\sigma)dS \qquad (3) \\ &= \int \frac{e^{-RS}(RS)^n}{n!} W(S|\hat{S},\sigma)dS. \qquad (4) \end{aligned}$$

The upper limit on R is that value such that

$$\sum_{n=0}^{n_{obs}} p(n|R) = 1 - CL. \tag{5}$$

The Cousins and Highland paper presents approximate formulae for small σ in the case where there is no background, giving for the upper limit on R

$$\overline{R} = \frac{U_{n0}}{\hat{S}}\left[1 + (U_{n0} - n_{obs})\frac{\sigma^2}{2\hat{S}^2}\left\{1 + \left(\frac{(U_{n0} - n_{obs})\sigma}{2\hat{S}}\right)^2\right\}\right], \tag{6}$$

where $U_{n0} = \overline{R}_{n0}\hat{S}$ and $\overline{R}_{n0}$ is the upper limit in the case where $\sigma = 0$. Although this formula is convenient, the reader should bear in mind that several assumptions were made in its derivation [for example, $< S - \hat{S} >= 0$, no background, and $\sigma/\hat{S} \leq 1/(U_{n0} - n_{obs})$]. Cousins and Highlands also give first order approximate formulae for the case with background.

3 Results

The Statistics Committee decided to test limit calculations using a Gamma distribution for the efficiency [2] (what Cousins and Highlands call S) because of the following properties: (1) it goes to zero for zero efficiency (thus, we don't have to worry about truncating), (2) it is the Bayesian posterior distribution for a counting experiment measurement of the efficiency (as is often the case in practice), and (3) it approaches a Gaussian when the uncertainty is much smaller than the mean value. This function is explored and exploited in reference [2].

To compare upper limit methods, the Statistics Committee decided to start with three cases: (1) true signal is 1 and the background is 0, (2) the true signal is 1 and the background is 3, and (3) the true signal is 10 and the background is 3. In all three cases, the true efficiency is 1 with a measurement uncertainty of 0.1, and the background has no uncertainty. The rest of this note will use notation consistent with reference [2] so that a direct comparison is easier. The signal level will be denoted as s and the efficiency as ϵ, compared to R and S, respectively, in the Cousins-Highland paper.

The Cousins and Highland method was studied with a simple Monte Carlo program. First, the efficiency was "measured" in a counting experiment. Then, a number of observed events was Poisson generated according to the true signal and true efficiency. For each pseudo-experiment, the limit was calculated according to the approximate formulae in the Cousins and Highland paper. Figures 1 to 3 show the distributions of limits, along with the distributions for a similar Bayesian calculation with constant prior for S. Two things to notice are (1) the upper limit is always reasonable, that is, there are no very small values and (2) the Cousins-Highland and Bayesian distributions are very similar.

Next the coverage (the fraction of pseudo-experiments in which the upper limit is greater than the true value of the signal) was determined for a variety of signal values s_t and efficiency values ϵ_t. The coverage is plotted versus $\epsilon_t s_t$ in figure 4. Since the Cousins-Highland method always produces a one-sided limit, the coverage must approach unity for small signal values. For values of $\epsilon_t s_t$ below about 10, the coverage of the Cousins-Highland method is very similar to (but a little below) that of the Bayesian method. Above 10, the Cousins and Highland method starts to rise again. This is due to the approximations used in the formula.

Next a numerical integration of an "extended" Cousins-Highland method (that is, averaging over the probability distribution W for the efficiency) was used, resulting in

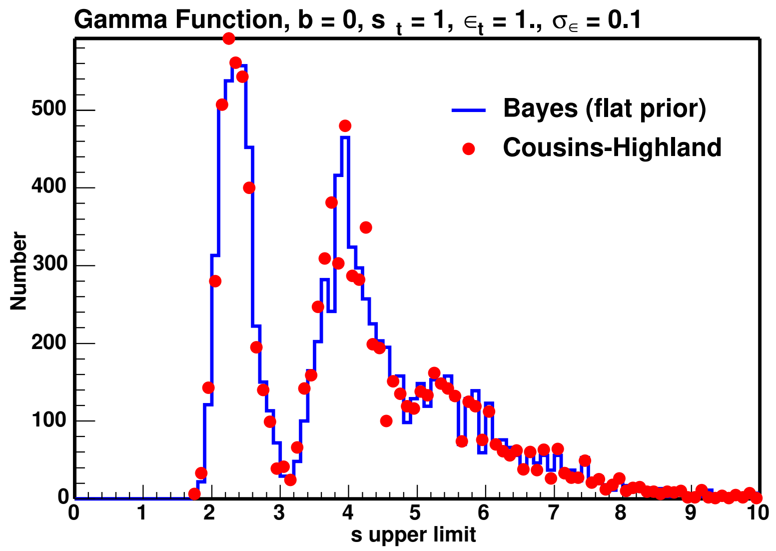

Figure 1: Distribution of limits for the case where the true signal is 1.0, the background is 0, the true efficiency is 1.0, the measurement uncertainty on the efficiency is 10%. The red histogram is the Cousins-Highland method and the blue histogram is a full Bayesian treatment with a flat prior. The peak between 2 and 3 is due to cases with zero observed events. The cases with other number of observed events give broader peaks that merge to form the rest of the distribution.

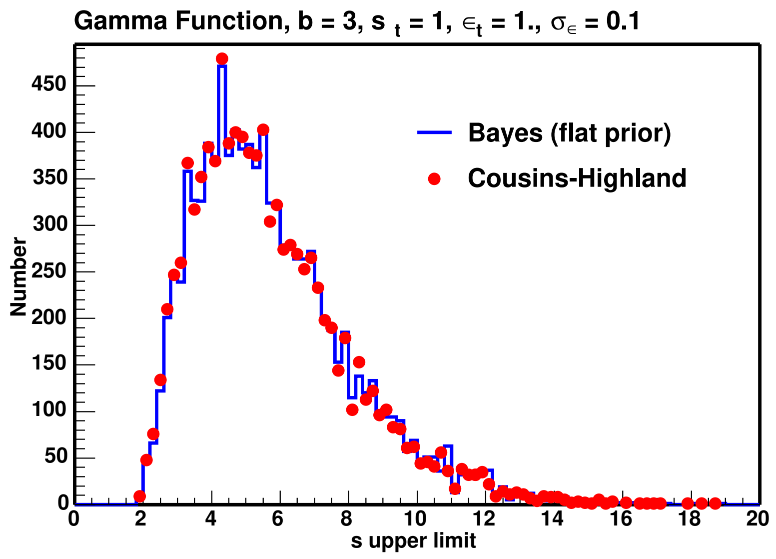

Figure 2: Distribution of limits for the case where the true signal is 1.0, the background is 3.0, the true efficiency is 1.0, the measurement uncertainty on the efficiency is 10%. The red histogram is the Cousins-Highland method and the blue histogram is a full Bayesian treatment with a flat prior.

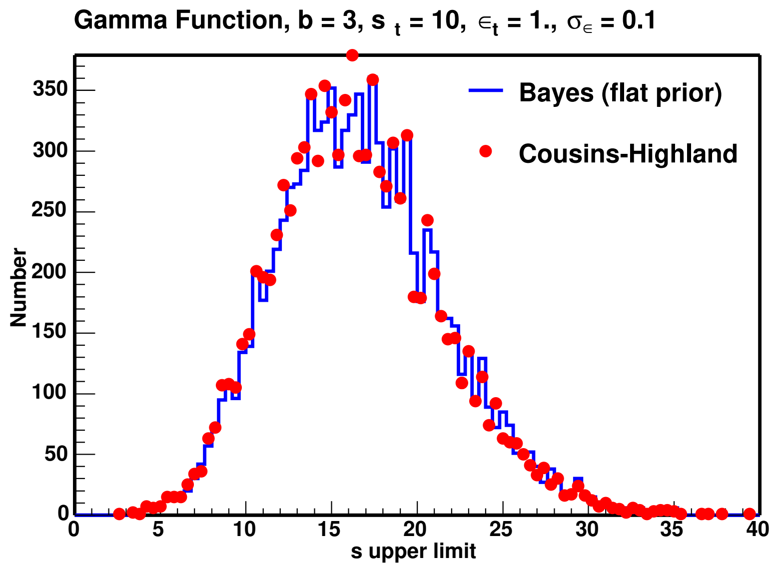

Figure 3: Distribution of limits for the case where the true signal is 10.0, the background is 3.0, the true efficiency is 1.0, the measurement uncertainty on the efficiency is 10%. The red histogram is the Cousins-Highland method and the blue histogram is a full Bayesian treatment with a flat prior.

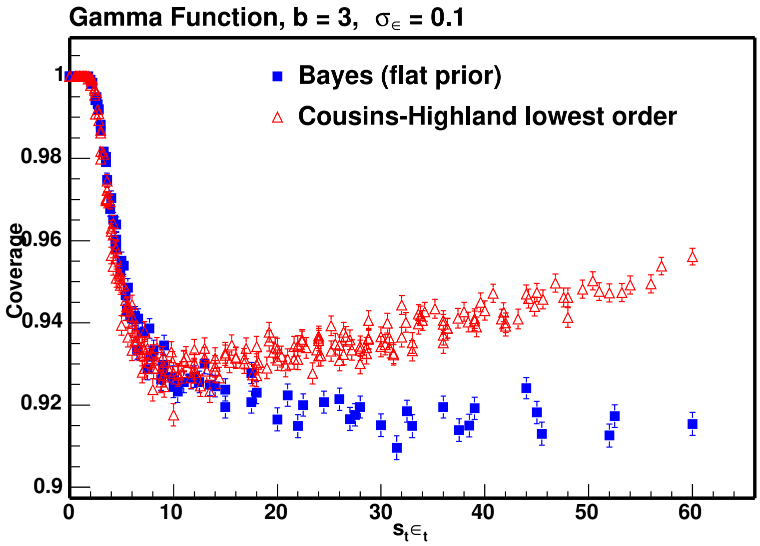

Figure 4: Coverage as a function of the product of the true efficiency and the true signal. The red points are the Cousins-Highland first order approximation, and the blue points are a full Bayesian treatment with a flat prior. The background is 3.0 and the measurement uncertainty on the efficiency is 10% for all points.

figure 5. Here, the results are always well-behaved and in good agreement with the Bayesian results. This indicates that the "exteneded" Cousins-Highland method is reasonable (at least for the cases considered here), but that the approximation formulae (particularly the first order ones) can't always be trusted, even in cases where the Cousins-Highland paper indicate they should be valid.

Another measure of a method is its sensitivity, defined as the average upper limit for a true signal value of zero. Figure 6 shows the sensitivity as a function of the background level for the full Cousins-Highland and the Bayesian methods. They are quite close, with the Cousins-Highland sensitivity being very slightly smaller.

4 Conclusion

The "extended" Cousins-Highland method with a full analytic or numeric integration of the formulae gives results very similar to those of a full Bayesian method. However, the approximate formulae, while easier to apply, break down in some cases. The only way we see to be sure that the approximations are working is to do the full calculation.

The Cousins-Highland is a mixed frequentist-Bayesian method, in the sense that

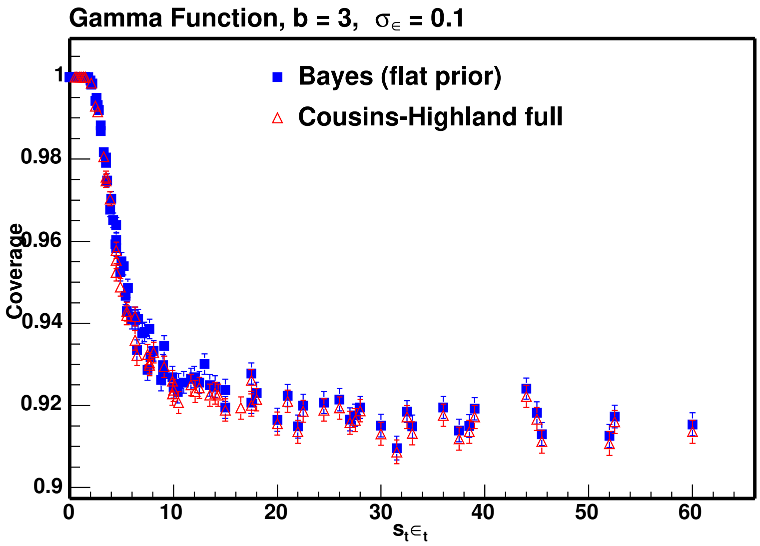

Figure 5: Coverage as a function of the product of the true efficiecny and the true signal. The red points are the "extended" Cousins-Highland method (numerically integrated), and the blue points are a full Bayesian treatment with a flat prior. The background is 3.0 and the measurement uncertainty 10% for all points.

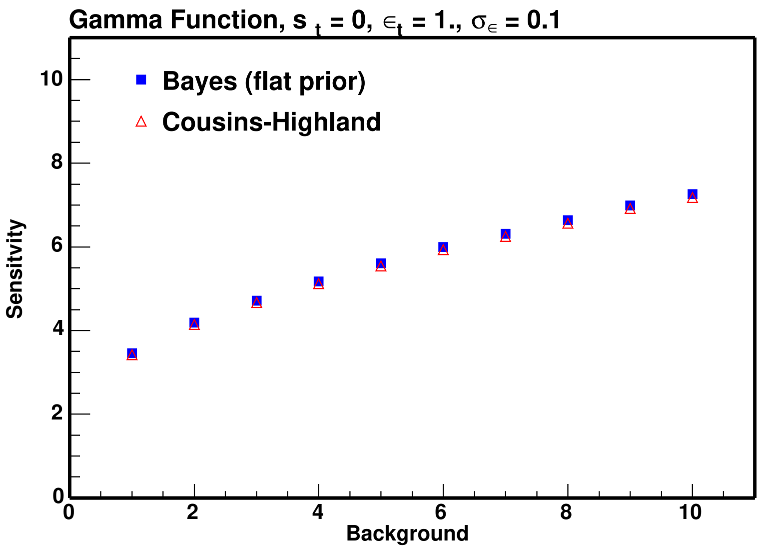

Figure 6: Sensitivity as a function of the background. The red points are the Cousins-Highland method, and the blue points are a full Bayesian treatment with a flat prior. The background is 3.0 and the measurement uncertainty on the efficiency is 10% for all points.

the Poisson distribution is averaged over the efficiency distribution (a very Bayesian thing to do), and then a limit is determined in a frequentist manner.

The Cousins-Highland paper deals primarily with measurement uncertainties on the efficiency, not on the background or other nuisance parameters. The basic idea could be extended to other cases (for example, averaging the probability distribution over background values in addition to averaging over the efficiency). This has not been explored for this note.

The Statistic Committee is in the process of evaluating various limit methods and do not yet have a recommended method or methods. However, we do recommend against the Cousins-Highland method for several reasons. First, the approximate formulae cannot be trusted, so a full calculation of the integrals (either analytically or numerically) is necessary. Once this is done, the amount of work is no less than for the full Bayesian treatment and gives similar behavior for upper limits as a Bayesian method with constant priors. The Cousins-Highland paper only discusses efficiency as a nuisance parameter. The method could be extended to other types of nuisance parameters, but the behavior was unexplored in the Cousins and Highland paper. Tegenfeldt and Conrad [3] looked at extensions of the Cousins and Highland method for cases with background uncertainties.

In comparison, the Bayesian method has the following assets: (1) it is, at least in principle, well defined for any number and type of nuisance parameters, (2) performs at least as well as Cousins-Highland, and (3) is no more difficult to calculate than the full Cousins-Highland-like method, that is, when approximations for small σ are not made.

References

[1] R. D. Cousins and V. L. Highland, Nucl. Instr. and Meth. **A320**, 331-335 (1992).

[2] CDF Statistics Committee, "Limit Calculations in the Presence of Nuisance Paramters," CDF note 7117.

[3] Conrad, Botner, Hallgren, and Pérez de los Heros, hep-ex/0202013 and hep-ex/0206034. Rolke, López, Conrad, and James, physics/0403059. Tegenfeldt and Conrad, physics/0408039.

[4] Giovanni Punzi, "Including systematic uncertainties in Confidence Limits", on the CDF Statistics Committee web page at http://www-cdf.fnal.gov/physics/statistics/notes/Syst_in_limits_report.pdf.