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Scaling Universalities of kth-Nearest Neighbor Distances on Closed Manifolds

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Take N sites distributed randomly and uniformly on a smooth closed surface. We express the expected distance $\langle D_k(N)\rangle$ from an arbitrary point on the surface to its kth-nearest neighboring site, in terms of the function $A(l)$ giving the area of a disc of radius l about that point. We then find two universalities. First, for a flat surface, where $A(l) = \pi l^2$, $\langle D_k(N)\rangle$ is separable in k and N. All kth-nearest neighbor distances thus scale the same way in N. Second, for a curved surface, $\langle D_k(N)\rangle$ averaged over the surface is a topological invariant at leading and subleading order in a large N expansion. The $1/N$ scaling series then depends, up through $O(1/N)$, only on the surface's topology and not on its precise shape. We discuss the case of higher dimensions ($d > 2$), and also interpret our results using Regge calculus. © 1998 Academic Press

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1. INTRODUCTION

Many problems arising in applied mathematics involve the distance between neighboring sites in a space. One frequently wishes to calculate dis-

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tances to a nearest neighbor, second-nearest neighbor, and, more generally, kth-nearest neighbor. Examples in computational geometry and optimization abound, ranging from random packing of spheres to minimum spanning trees. Such problems also occur naturally in physics and have been considered in applications ranging from stellar dynamics [1], to interactions in liquid systems, to cellular objects such as foams and random lattices [2].

Here we consider the case of N sites placed randomly, with a uniform distribution, on a 2-D surface of fixed area. Let the random variable $D_k(N)$ represent the distance between a given point $\mathbf{x}$ and its kth-nearest site. The expectation value $\langle D_k(N)\rangle$ taken over the ensemble of randomly placed sites—and in fact all moments $\langle D_k^\alpha(N)\rangle$—then exhibit some surprising properties:

- When the surface is flat, $\langle D_k(N)\rangle$ may be written, up to corrections exponentially small in N, as

$$\langle D_k(N)\rangle \approx \frac{1}{\sqrt{\pi}}\,\frac{(k-1/2)!}{(k-1)!}\,\frac{N!}{(N+1/2)!}.$$

From this result, which we have reported in [3], we see that $\langle D_k(N)\rangle$ is separable in k and N; the N-dependence in $\langle D_k(N)\rangle$ is the same, regardless of k. Geometrically, the meaning of this universality is far from obvious. Furthermore, the property is not restricted to two dimensions, and turns out to be equally valid for flat spaces of any dimension.

- When the surface is curved, while the k-independence no longer holds, there is another universality: if $\langle D_k(N)\rangle$ is averaged over the entire surface and then expressed in terms of a $1/N$ series expansion, the leading and subleading coefficients of the series are topological invariants. Thus to $O(1/N)$, the large N scaling law depends not on the detailed shape of the surface but only on the surface's genus.

In this paper we explore these universalities. We start by expressing $\langle D_k(N)\rangle$ in terms of the area $A(l)$ of a disc of radius l on an arbitrary surface. We observe that, in the special case where $A(l)$ consists only of a power of l, $\langle D_k(N)\rangle$ exhibits the first universality, i.e., its scaling law in N does not depend on k. Then, for the general case, we give the relation between $A(l)$ and the Gaussian curvature of the surface, and we find the leading correction terms in the $1/N$ power series for $\langle D_k(N)\rangle$ as functions of Gaussian curvature. This results in the topological invariance at $O(1/N)$. We discuss higher order terms as well, and the case of higher dimensions. Finally, we show that a Regge calculus approach provides a simple means of obtaining this topological invariance for the case of polyhedral (nonsmooth) surfaces.

2. PRELIMINARIES

Take any point $\mathbf{x}$, and consider $P[D_k(N) = l]_{\mathbf{x}}$, the probability density for the point's kth-nearest neighboring site being located at a distance l from it. As seen in [3], this is equal to the probability density for having $k-1$ (out of N) sites *within* distance l, one site (out of $N-k+1$) *at* distance l, and the remaining $N-k$ sites *beyond* distance l. Let us choose units so that our surface has total area 1. Since sites are distributed uniformly over the surface, the probability of a site lying within distance l is then simply the area $A(l)_{\mathbf{x}}$ of a disc of radius l about point $\mathbf{x}$ on the surface. This is shown in Fig. 1.

Dropping the argument $\mathbf{x}$ (in order to simplify the notation), we may then write

$$P[D_k(N) = l] = \binom{N}{k-1} [A(l)]^{k-1} \times \binom{N-k+1}{1} \frac{dA(l)}{dl} \times [1 - A(l)]^{N-k},$$

giving the expectation value (first moment)

$$\langle D_k(N) \rangle = \int_0^\infty P[D_k(N) = l]\, l\, dl = \frac{N!}{(N-k)!\,(k-1)!} \int_0^\infty l\,[A(l)]^{k-1}[1 - A(l)]^{N-k} \frac{dA(l)}{dl}\, dl.$$

Using the variable transformation $w = A(l)$, this may be written in terms of the inverse function $A^{-1}(w)$ as

$$\langle D_k(N) \rangle = \frac{N!}{(N-k)!\,(k-1)!} \int_0^1 A^{-1}(w)\, w^{k-1}(1-w)^{N-k}\, dw.$$

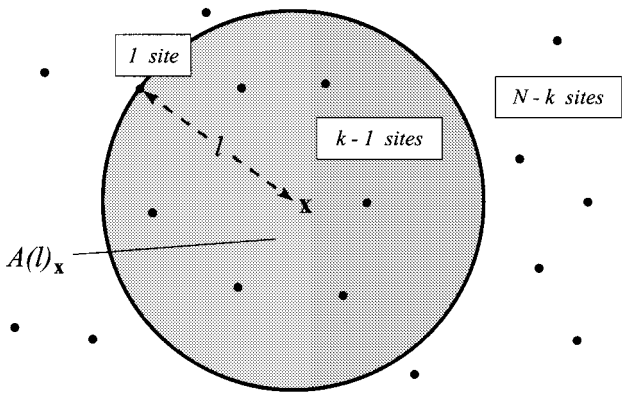

FIG. 1. A point $\mathbf{x}$ and the region within a distance l around it. Area $A(l)_{\mathbf{x}}$ is the probability of finding a site within distance l of $\mathbf{x}$. Here, $k-1$ sites lie within this region, the kth-nearest site lies at distance l exactly, and the remaining $N-k$ sites lie outside the region.

If $A^{-1}(w)$ admits a power series expansion in w,

$$A^{-1}(w) = w^{\gamma} \sum_{j=0}^{\infty} c_j w^j, \qquad \text{for some } \gamma \in [0, 1), \tag{1}$$

then

$$\langle D_k(N) \rangle = \frac{N!}{(N-k)!\,(k-1)!} \sum_{j=0}^{\infty} c_j \int_0^1 w^{k+j+\gamma-1}(1-w)^{N-k}\, dw. \tag{2}$$

Recognizing the integral as the Beta function $B(k+j+\gamma, N-k+1) = (k+j+\gamma-1)!\,(N-k)!/(N+j+\gamma)!$,

$$\langle D_k(N) \rangle = \sum_{j=0}^{\infty} c_j \frac{(k+j+\gamma-1)!}{(k-1)!} \frac{N!}{(N+j+\gamma)!}. \tag{3}$$

Several comments are in order concerning $\langle D_k(N) \rangle$. First, although we restrict ourselves to discussing the first moment of $D_k(N)$, we could in fact consider any moment $\langle D_k^{\alpha}(N) \rangle$ by taking $[A^{-1}(w)]^{\alpha}$ instead of $A^{-1}(w)$ in (1). Doing so would alter γ and the c_js, but would not change our results qualitatively. Second, there is no loss of generality in taking our total surface area to be unity; scaling this area by a constant (or even, as might seem intuitive in statistical physics, by N) would provide only a trivial scaling factor in our results. Third, we could imagine that the point $\mathbf{x}$ we consider is itself an $(N+1)$th site. This is simply a question of nomenclature: the problem of finding the expected distance *from an arbitrary point to its kth-nearest site*, for a system of N sites, is equivalent to the problem of finding the expected distance *between kth-nearest neighboring sites*, for a system of $N+1$ sites.

We now turn to the properties of $A^{-1}(w)$, and their consequences on $\langle D_k(N) \rangle$.

3. FLAT SURFACES

On a flat surface, if we could neglect edge effects, the area included within distance l would simply be $A(l) = \pi l^2$. In that case, we would have $A^{-1}(w) = \sqrt{w/\pi}$, and so, from (1) and (3),

$$\langle D_k(N) \rangle = \frac{1}{\sqrt{\pi}} \frac{(k-1/2)!}{(k-1)!} \frac{N!}{(N+1/2)!}. \tag{4}$$

There would thus be a complete separation of the k-dependence and the N-dependence.

As we are working with a surface of fixed (unit) area, however, we cannot avoid considering edge or finite size effects. Let us restrict ourselves to the case where the surface is everywhere locally Euclidean within some minimum neighborhood of radius $l_0 > 0$. (The simplest example of this is a unit square with periodic boundary conditions, for which $l_0 = 1/2$. Clearly, many other constructions are possible.) Any required modification to the $A^{-1}(w)$ expression in (1) then concerns only w greater than $w_0 \equiv A(l_0)$. Correspondingly, (2) remains valid, up to remainder terms from the region of integration $w_0 \le w \le 1$. Since the $(1-w)^{N-k-1}$ term in the integral is bounded above by $(1-w_0)^{N-k-1}$ within this region, these remainder terms are exponentially small in N. Equation (4) is thus still correct *to all orders* in a $1/N$ series expansion and may be written as

$$\langle D_k(N)\rangle = \frac{1}{\sqrt{\pi}}\,\frac{(k-1/2)!}{(k-1)!}\,\frac{1}{\sqrt{N}}\left[1-\frac{3}{8N}+O\left(\frac{1}{N^2}\right)\right],$$

where all orders in the series are independent of k. We therefore see that the large N scaling law for kth-nearest neighbor distances on a 2-D flat surface without a boundary exhibits the universality in k to all orders in $1/N$.

The same holds true for flat manifolds of any dimension d. We assume there is some l_0 such that the volume included within distance $l < l_0$ is simply the volume of a d-dimensional ball:

$$A(l) = \frac{\pi^{d/2}\, l^d}{(d/2)!},$$

or

$$A^{-1}(w) = \frac{1}{\sqrt{\pi}}\left[w\left(\frac{d}{2}\right)!\right]^{1/d}.$$

As before, this condition allow us to write $\langle D_k(N)\rangle$ up to remainder terms that are exponentially small in N, so from (3),

$$\begin{aligned}\langle D_k(N)\rangle &\approx \frac{[(d/2)!]^{1/d}}{\sqrt{\pi}}\,\frac{(k-1+1/d)!}{(k-1)!}\,\frac{N!}{(N+1/d)!}\\ &= \frac{[(d/2)!]^{1/d}}{\sqrt{\pi}}\,\frac{(k-1+1/d)!}{(k-1)!}\,N^{-1/d}\left[1-\frac{1/d+1/d^2}{2N}+O\left(\frac{1}{N^2}\right)\right].\end{aligned}$$

Thus for flat spaces without a boundary, of any dimension d, the universality in k holds to all orders in $1/N$: $\langle D_k(N)\rangle$ is separable in k and N.

It may be interesting to consider a slight variation on the problem, giving this universality *exactly* and not only to all orders of an expansion. Take the

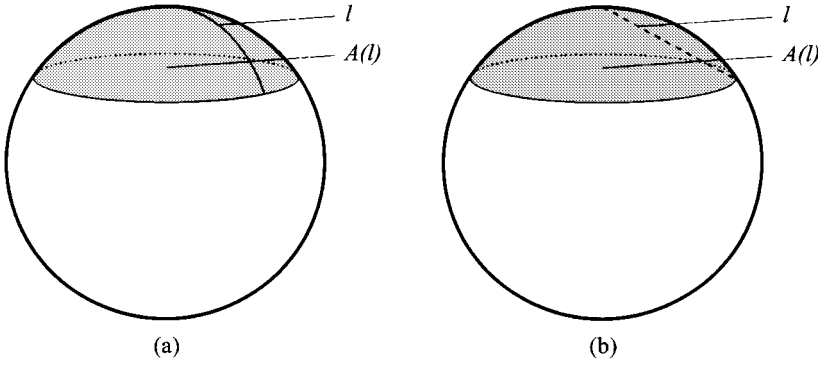

FIG. 2. A spherical surface, where $A(l)$ denotes the area of a spherical cap spanned by l: (a) l represents the arc length distance; (b) l represents the chord length distance.

case of a spherical surface embedded in 3-D Euclidean space, with the usual measure of area over the sphere, but with a peculiar sort of "distance": rather than the conventional choice of the arc length (geodesic) metric, use the chord length. (See Fig. 2.) For a chord of length l originating at a pole of the sphere, the area of the spherical cap spanned by it is simply $A(l) = \pi l^2$. The kth-nearest neighbor distance properties using chord length "distance" on this curved surface then appear analogous to those on a flat surface. There is, however, one important distinction. The relevant threshold w_0 for edge effects is in this case $w_0 = \pi(2R)^2$, where R is the radius of the sphere. Since $\pi(2R)^2$ is precisely the total surface area of the sphere, it is by construction equal to 1. Equation (2) thus requires no corrections at all, and so the universality in (4) is exact.

4. CURVED SURFACES

Now consider the case of a surface with intrinsic curvature, with the distance defined in terms of a metric, i.e., along geodesics of the surface. Let us begin with a spherical surface, this time letting l represent arc length; the area of the spherical cap spanned by an arc originating at a pole of the sphere (see Fig. 2a) is given by

$$A(l)_{\text{sphere}} = 2\pi R^2\left[1 - \cos\frac{l}{R}\right]$$
$$= 4\pi R^2 \sin^2\frac{l}{2R}.$$

Since the total surface area $4\pi R^2$ is normalized to 1,

$$A(l)_{\text{sphere}} = \sin^2\sqrt{\pi}\,l,$$

so

$$A^{-1}(w)_{\text{sphere}} = \frac{\sin^{-1}\sqrt{w}}{\sqrt{\pi}}$$
$$\simeq \sqrt{\frac{w}{\pi}} \sum_{j=0}^{\infty} \frac{w^j}{2j+1} \frac{(2j)!}{2^{2j}(j!)^2}. \tag{5}$$

As in the case of the chord length "distance," this $A^{-1}(w)$ expression is exact everywhere for $0 \le w \le 1$. Equations (1) and (3) then require no corrections, and we find

$$\langle D_k(N)\rangle_{\text{sphere}} = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{2j+1} \frac{(2j)!}{2^{2j}(j!)^2} \frac{(k+j-1/2)!}{(k-1)!} \frac{\sqrt{N}\,N!}{(N+j+1/2)!}$$
$$= \frac{1}{\sqrt{\pi}} \frac{(k-1/2)!}{(k-1)!} \frac{1}{\sqrt{N}} \left[1 + \frac{4k-7}{24N} + O\left(\frac{1}{N^2}\right)\right]. \tag{6}$$

Clearly, the k universality does not apply here: the $O(1/N)$ coefficient explicitly contains k.

Another sort of universality, however, is found when we turn to the more general case of an arbitrary closed surface, i.e., an abstract 2-D manifold with no boundary. Given a smooth surface, we may introduce a system of curvilinear coordinates u and v and write (at least piecewise on the surface) the differential length element ds in the conformal, orthogonal form (see, e.g., [4]):

$$ds^2 = f(u, v)\,[du^2 + dv^2]. \tag{7}$$

The Gaussian curvature $K(u, v)$ of the surface is then expressed in terms of the function $f(u, v)$ by

$$K = \frac{1}{2f^3}\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 - f\frac{\partial^2 f}{\partial u^2} - f\frac{\partial^2 f}{\partial v^2}\right]. \tag{8}$$

What is $A(l)$ on this surface? To find out, we first determine the manifold's geodesic lines. For ds given by (7), we may use the geodesic equation [4]:

$$\frac{d^2u}{ds^2} + \frac{1}{2f}\frac{\partial f}{\partial u}\left[\left(\frac{du}{ds}\right)^2 - \left(\frac{dv}{ds}\right)^2\right] + \frac{1}{f}\frac{\partial f}{\partial v}\frac{du}{ds}\frac{dv}{ds} = 0. \tag{9}$$

Let us expand u and v as functions of distance s from an initial point, along a fixed geodesic:

$$u(s) = u_0 + su_0' + \frac{s^2}{2}u_0'' + \cdots \quad \text{and} \quad v(s) = v_0 + sv_0' + \frac{s^2}{2}v_0'' + \cdots, \tag{10}$$

where $u_0 \equiv u(0)$, $u_0' \equiv u'(0)$, etc., and likewise for v. Then, expanding $f(u, v)$ in terms of u and v and substituting (10),

$$\begin{aligned} f(u, v) = f(u_0, v_0) &+ s\left[u_0' f_u(u_0, v_0) + v_0' f_u(u_0, v_0)\right] \\ &+ s^2\left[\frac{u_0''}{2} f_u(u_0, v_0) + \frac{v_0''}{2} f_v(u_0, v_0) + \frac{(u_0')^2}{2} f_{uu}(u_0, v_0)\right. \\ &\left. + u_0' v_0' f_{uv}(u_0, v_0) + \frac{(v_0')^2}{2} f_{vv}(u_0, v_0)\right] + O(s^3), \end{aligned}$$

where subscripts on f denote partial derivatives.

Using (7) and (9), we can solve for all but three of the coefficients in (10). Let us choose u_0, v_0, and u_0' to be these three. Now, consider the area $A(l)$ about the point (u_0, v_0). For ds given in (7), the differential surface element will be $d\mu = f\, du\, dv$, so

$$\begin{aligned} A(l) &= \int f\, du\, dv \\ &= \int f\, J\left(\frac{u, v}{s, u_0'}\right) ds\, du_0' \\ &= \int f \left|\frac{\partial u}{\partial s}\frac{\partial v}{\partial u_0'} - \frac{\partial u}{\partial u_0'}\frac{\partial v}{\partial s}\right| ds\, du_0'. \end{aligned} \tag{11}$$

The limits of integration over s are 0 and l; the limits of integration over u_0', which may be found from (7), are $-\sqrt{1/f(u_0, v_0)}$ and $\sqrt{1/f(u_0, v_0)}$. In evaluating the Jacobian some care must be taken, as a sign ambiguity allows two solutions for the coefficients in (10). $A(l)$ will be the sum of (11) evaluated at each of the two solutions, ultimately causing all odd powers of l to vanish.

The result, after lengthy algebraic manipulations, may be written as

$$A(l) = \pi l^2\left[1 - \frac{l^2}{24}\frac{1}{f^3}\left(f_u^2 + f_v^2 - f f_{uu} - f f_{vv}\right) + O(l^4)\right], \tag{12}$$

where, in order to avoid cluttering the notation, we have omitted the (u_0, v_0) arguments at which all functions are to be evaluated. For the leading correction term in $A(l)$ given in (12), we recognize the expression (8) for the Gaussian curvature K. In retrospect, this is not surprising. By symmetry, the correction series to πl^2 can contain only even powers of l; if we consider $A(l)$ as a geometric expansion about a flat space approximation, the only obvious form of scalar curvature with dimensions l^{-2} will be K itself [5].

With some perseverance, one may carry the expansion in (12) to higher orders, obtaining

$$A(l) = \pi l^2 \Bigg[1 - \frac{l^2}{12} K + \frac{l^4}{720} (2K^2 - 3\nabla^2 K) - \frac{l^6}{161280} \big(8K^3 - 3[10(\nabla K)^2 + 14K\nabla^2 K - 5\nabla^4 K]\big) + O(l^8) \Bigg], \quad (13)$$

where ∇ is the gradient operator. We thus obtain a series expansion giving the area of a disc on a smooth 2-D surface, in a form that depends only on intrinsic quantities, i.e., not on the choice of coordinate system.

We may invert (13) to obtain the power series

$$A^{-1}(w) = \sqrt{\frac{w}{\pi}} \Bigg[1 + \frac{K}{24\pi} w + \frac{9K^2 + 4\nabla^2 K}{1920\pi^2} w^2 + \frac{15K^3 + 14K\nabla^2 K - 2(\nabla K)^2 + \nabla^4 K}{21504\pi^3} w^3 + O(w^4) \Bigg]. \quad (14)$$

As an example, take the special case of a spherical surface, where the Gaussian curvature is a constant $K = 1/R^2$, or $K = 4\pi$ for a unit surface. All derivatives of K then vanish, leaving

$$A^{-1}(w)_{\text{sphere}} = \sqrt{\frac{w}{\pi}} \Bigg[1 + \frac{w}{6} + \frac{3w^2}{40} + \frac{5w^3}{112} + O(w^4) \Bigg],$$

from which we recover the first few terms of our earlier result (5).

Given an expression for $A^{-1}(w)$ on a general 2-D surface, we may now find $\langle D_k(N) \rangle$ using (1) and (3). It is helpful at this point to define the reduced variable $\langle \tilde{D}_k(N) \rangle$ by dividing out the leading asymptotic (large N) behavior from $\langle D_k(N) \rangle$. Recalling the notation of (1), $A^{-1}(w) = w^\gamma \sum_j c_j w^j$ for some $\gamma \in [0, 1)$, define

$$\langle \tilde{D}_k(N) \rangle = \langle D_k(N) \rangle \frac{1}{c_0} \frac{(k-1)!}{(k+\gamma-1)!} N^\gamma = N^\gamma \sum_{j=0}^{\infty} \frac{c_j}{c_0} \frac{(k+j+\gamma-1)!}{(k+\gamma-1)!} \frac{N!}{(N+j+\gamma)!}, \quad (15)$$

so that $\lim_{N\to\infty} \langle \tilde{D}_k(N) \rangle = 1$ (this is seen from Stirling's law). $\langle \tilde{D}_k(N) \rangle$ then corresponds to the $1/N$ power series appearing in $\langle D_k(N) \rangle$, giving the corrections to its leading (large N) behavior. It is here that the interesting universality properties emerge. Consider $\int \langle \tilde{D}_k(N) \rangle \, d\mu$, the average over the entire surface, obtained—via (1) and (15)—from precisely the average of the series coefficients in (14). Examine, in particular, the $O(w)$ term

of (14). By the Gauss–Bonnet theorem [4], $\int K\,d\mu = 2\pi\chi$ on any closed surface, where χ is the Euler characteristic of the surface, a topological invariant. Up to leading corrections, then,

$$\int A^{-1}(w)\,d\mu = \sqrt{\frac{w}{\pi}}\left[1 + \frac{\chi}{12}w + O(w^2)\right],$$

giving

$$\begin{aligned}\int \langle \tilde{D}_k(N)\rangle\,d\mu &= \frac{\sqrt{N}\,N!}{(N+1/2)!}\left[1 + \frac{\chi}{12}\,\frac{k+1/2}{N+3/2} + O\left(\frac{1}{N^2}\right)\right] \\ &= 1 + \frac{\chi(2k+1)-9}{24N} + O\left(\frac{1}{N^2}\right). \qquad (16)\end{aligned}$$

We thus discover a different sort of universality from the one we had in the case of flat space. To $O(1/N)$, the scaling law for kth-nearest neighbor distances depends only on the surface's topology, and not on its detailed properties.

The Euler characteristic χ for a surface is related to its genus g by $\chi = 2(1-g)$. Taking the torus as one example, $g = 1$, so $\chi = 0$ and the k-dependence in (16) once again disappears, at least at $O(1/N)$. This is to be expected: a flat space with periodic boundary conditions has, after all, the topology of a torus. And conversely, because of the topological invariance, all tori behave like flat space to $O(1/N)$. Taking the spherical surface as another example, $g = 0$, so $\chi = 2$ and we recover from (16) the power series in (6).

The properties of $\int \langle \tilde{D}_k(N)\rangle\,d\mu$ are far less clear at higher orders in $1/N$. Using the divergence theorem and integration by parts, we may obtain from (14)

$$\begin{aligned}\int A^{-1}(w)\,d\mu = \int \sqrt{\frac{w}{\pi}}\Bigg[1 &+ \frac{K}{24\pi}\,w + \frac{3K^2}{640\pi^2}\,w^2 \\ &+ \frac{15K^3 + 16K\,\nabla^2 K}{21504\pi^3}\,w^3 + O(w^4)\Bigg]\,d\mu. \qquad (17)\end{aligned}$$

If we looked only at terms up through $O(w^2)$, we might believe that this series is simply, by analogy with (5), the expansion of $\int (2/\sqrt{K}) \cdot \sin^{-1}\sqrt{Kw/4\pi}\,d\mu$. Unfortunately, starting at $O(w^3)$ we see this is not true, since the contributions of curvature and its gradients do not all vanish in the average over the surface! Furthermore, even for terms in (17) of the form $\int K^n\,d\mu$, at $n > 1$ there is no straightforward equivalent to the Gauss–Bonnet theorem; the theorem is a direct consequence of the integrand's linearity. Thus, for a general 2-D surface, a simplified form does not appear to exist for the terms in $\int \langle \tilde{D}_k(N)\rangle\,d\mu$ beyond $O(1/N)$. More

particularly, the *only* case in which $\int \langle \tilde{D}_k(N) \rangle \, d\mu$ would be independent of k beyond $O(1/N)$ is if the curvature is identically equal to 0, i.e., a flat surface.

Let us briefly consider the case of curved higher-dimensional manifolds. The calculation is now far more complicated, as it is no longer possible to write the metric tensor in a conformal form as we did in (7). In addition, whereas in 2-D the only intrinsic scalar quantity describing curvature is the Gaussian curvature K, for $d > 2$ there are $d(d-1)(d-2)(d+3)/12$ different such quantities [5]. However, all of them except K itself have dimensions of higher order than l^{-2}. It thus seems reasonable to conjecture that, as we argued in 2-D, the $O(l^2)$ correction term in $A(l)$ can only involve K. (Indeed, we have verified that this is true in 3-D.) In that case, we may rely on the example of the spherical surface—easily generalized to d dimensions—to provide us with the initial terms for a general manifold:

$$A(l) = \frac{\pi^{d/2}}{(d/2)!} \, l^d \left[1 - \frac{d(d-1)}{d+2} \, \frac{K}{6} \, l^2 + O(l^4) \right],$$

or

$$A^{-1}(w) = \left[\frac{(d/2)!}{\pi^{d/2}} \right]^{1/d} w^{1/d} \left[1 + \frac{d-1}{d+2} \, \frac{K}{6} \left(\frac{(d/2)!}{\pi^{d/2}} \right)^{2/d} w^{2/d} + O(w^{4/d}) \right]. \tag{18}$$

Note that $A^{-1}(w)$ now contains a series in $w^{2/d}$ rather than in w. Appropriately modifying (1), it may be shown that $\langle \tilde{D}_k(N) \rangle$ is in general given by a series in $1/N^{1/d}$ for odd d, and $1/N^{2/d}$ for even d.

Consider, then, the average $\int A^{-1}(w) \, d\mu$ over the manifold. The higher-dimensional generalization of the Gauss–Bonnet theorem involves an integrand of dimensions l^{-d} [6]. The corresponding coefficient in the power series in (18) is at $O(l^d)$, or $O(w)$; this is the only term that could yield a topological invariant. If d is odd, we see in (18) that the series contains no such term. If d is even, the series for $A^{-1}(w)$ does contain an $O(w)$ term, but unlike in 2-D it is no longer the *leading* correction. And while it will still contribute to $\int \langle \tilde{D}_k(N) \rangle \, d\mu$ at $O(1/N)$, again, in higher dimensions $O(1/N)$ is no longer the leading correction term in $\int \langle \tilde{D}_k(N) \rangle \, d\mu$. The only potential candidate for a topological invariant is thus a higher-order correction term. We cannot rule out the possibility of obtaining such an invariant at $O(1/N)$, but the $O(w)$ term in $A^{-1}(w)$ is in general a complicated one involving many different curvature scalars, and so this is far from obvious. We leave it as an open question.

5. REGGE CALCULUS

We have remarked that from a physical point of view it is natural, in the 2-D case, for the leading corrections in $A(l)$ to contain only the Gaussian curvature, as this represents the leading deviation from planarity. Consequently, only the mean curvature—or, using the Gauss–Bonnet theorem, the Euler characteristic χ—matters in the $O(1/N)$ term of $\int \langle \tilde{D}_k(N) \rangle \, d\mu$. We have seen using differential methods (geodesics) that this physical picture is indeed correct. These methods apply to a smooth closed surface. For polyhedral surfaces, which are not smooth, we may in fact obtain a similar result more easily, using the nondifferential method of Regge calculus. Consider a polyhedron with a number of vertices, edges, and faces. Following the work of Regge [7] and others since then [8], we observe that the curvature is concentrated at the vertices and is measured by a deficit angle: if θ_i is the sum of the angles incident on vertex i, the deficit angle at that vertex is $\Delta_i = 2\pi - \theta_i$. It may then be shown that the Gauss–Bonnet theorem, on polyhedra, reduces to Euler's relation $2\pi\chi = \sum_i \Delta_i$.

Let P be a polyhedron with a fixed number of vertices, and consider the problem of finding the large N scaling series $\int \langle \tilde{D}_k(N) \rangle \, d\mu$ on P. As $N \to \infty$, corrections to the flat space value about a given point $\mathbf{x}$ arise only when $\mathbf{x}$ is near one of the vertices, because only in that case can curvature (i.e., the deficit angle) enter into the local calculation of $A(l)$ about $\mathbf{x}$. It is then sufficient to understand the corrections associated with one vertex at a time. Consider the neighborhood of a vertex i. $A(l)$ receives a correction from the flat space value, and, by a simple geometric construction, one can see that this correction is exactly proportional to the deficit angle Δ_i. Correspondingly, the leading correction term both in $A^{-1}(w)$ and in the scaling series will be proportional to Δ_i for small deficit angles. Now sum over all the vertices $i = 1, \ldots, N$, assuming that all the deficit angles are small. We then find that the $O(1/N)$ term in $\int \langle \tilde{D}_k(N) \rangle \, d\mu$ is proportional to χ, and we recover the topological invariant derived in the case of a smooth manifold.

A word of caution is necessary, however. It is tempting at this point to take the limit where P becomes a smooth manifold, expecting to recover (16). Unfortunately this will not work; a direct computation shows that the limit does not commute with the limit $N \to \infty$ taken above, and the coefficient thus obtained at $O(1/N)$ will not be the correct one.

6. CONCLUSIONS

Given N sites distributed randomly and uniformly on a surface with no boundaries, we have considered the properties of mean distances to neighboring sites. When the surface is flat, we have seen that the expression

for the mean kth-nearest site distance is separable in k and N. The scaling law in N for mean kth-nearest neighbor distances is thus the same, regardless of k. This universality applies equally well to higher moments of the distances, and to Euclidean manifolds in dimensions greater than 2. For surfaces with curvature, while this general property is no longer valid, we have found that when the kth-nearest neighbor distance is written as a large N expansion, averaged over the surface, the leading correction term in the series is a topological invariant. The scaling series thus depends, to $O(1/N)$, on the genus of the manifold but not on its other properties.

Although we have considered these universalities only for the moments of point-to-point distances, similar properties hold for higher-order simplices such as areas of triangles associated with nearby points. The problem is thus a natural one to consider further in the context of random triangulations, foams, or other physical problems [2] tightly connected to geometry.

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