Polarization and Polarimetry

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GLOSSARY

Birefringence
The property of optically anisotropic materials, such as crystals, of having the phase velocity of propagation dependent on the direction of propagation and polarization. Numerically, birefringence is the refractive index difference between eigenpolarizations.

Diattenuation
The property of having optical transmittance depend on the incident polarization state. In diattenuators, the eigenpolarizations will have principal transmittances T_{max} and T_{min}, and diattenuation is quantified as (T_{max} - T_{min}/(T_{max} + T_{min}). Diattenuation may occur during propagation when absorption coefficients depend on polarization (also called dichroism) or at interfaces.

Eigenpolarization
A polarization state that propagates unchanged through optically anisotropic materials. Eigenpolarizations are orthogonal in homogeneous polarization elements.

Jones calculus
A mathematical treatment for describing fully polarized light. Light is represented by 2 x 1 complex Jones vectors and polarization components as 2 x 2 complex Jones matrices.

Mueller calculus
A mathematical treatment for describing completely, partially, or unpolarized light. Light is represented by the 4 x 1 real Stokes vector and polarization components as 4 x 4 real Mueller matrices.

Polarimetry
The measurement of the polarization state of light or the polarization properties (retardance, diattenuation, and depolarization) of materials.

Polarized light
A light wave whose electric field vector traces a generally elliptical path. Linear and circular polarizations are special cases of elliptical polarization. In general, light is partially polarized, and is a mixture of polarized light and unpolarized light.

Polarizer
A device with diattenuation approaching 1 that transmits one unique polarization state regardless of incident polarization.

Retardance
The optical phase shift between two eigenpolarizations.

Unpolarized light
Light of finite spectral width whose instantaneous polarization randomly varies over all states during the detection time. Not strictly a polarization state of light.

The Polarization state is one of the fundamental characteristics (along with intensity, wavelength, and coherence) required to describe light. The earliest recorded observation of polarization effects was reported by Bartholinus, who observed double refraction in calcite in 1669. Huygens demonstrated the concept of polarization by passing light through two calcite crystals in 1690. Today, the measurement, manipulation, and control of polarization plays an important role in optical sciences.

I. POLARIZATION STATES

Light can be represented as an electromagnetic wave that satisfies Maxwell's equations. A transverse electromagnetic wave has electric and magnetic field components that are orthogonal to the direction of propagation. As the wave propagates, the strengths of these transverse fields oscillate in space and time, and the polarization state is defined by the direction of the electric field vector **E**.

For our discussion, we will use a right-handed Cartesian coordinate system with orthogonal unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$. A monochromatic plane wave **E**(*z, t*) traveling in vacuum along the $\hat{z}$ direction with time *t* can be written as

$$\begin{aligned} E(z,t) = \mathrm{Re}\{&\hat{x}E_x \exp[i(\omega t - k_0 + \phi_x)] \\ &+ \hat{y}E_y \exp[i(\omega t - k_0 + \phi_y)]\} \end{aligned} \tag{1a}$$

$$\begin{aligned} E(z,t) = &\hat{x}E_x \cos(\omega t - k_0 + \phi_x) \\ &+ \hat{y}E_y \cos(\omega t - k_0 + \phi_y), \end{aligned} \tag{1b}$$

where ω is the angular optical frequency and E_x and E_y are the electric field amplitudes along the $\hat{x}$ and $\hat{y}$ axes, respectively. The free-space wavenumber is $k_o = 2\pi r / \lambda$ for wavelength λ, and ϕ_x, and ϕ_y are absolute phases. The difference in phase between the two component fields is then $\Delta\phi = \phi_y - \phi_x$. The direction of **E** and the polarization of the wave depend on the field amplitudes E_x and E_y and the phases ϕ_x and ϕ_y.

A. Linear Polarization

A wave is linearly polarized if an observer looking along the propagation axis sees the tip of the oscillating electric field vector confined to a straight line. Figure 1 depicts the wave propagation for two different linear polarizations when Eq.(1b) is plotted for $\phi_x = \phi_y = 0$. In Fig. 1, $E_y = 0$ and light is linearly polarized along the *x* axis; in the other example, light is polarized along the *y* axis when $E_x = 0$.

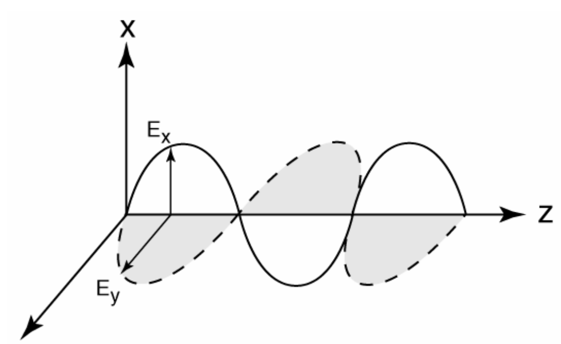

Figure 1 Two linear polarized waves. The electric field vector of *x*-polarized light oscillates in the *xz* plane. The shaded wave is *y*-polarized light in the *yz* plane.

For a field represented by Eqs. (l a) and (1b), light will be linearly polarized whenever $\Delta\phi = m\pi$, where *m* is an integer; the direction of linear polarization depends on the magnitudes of E_x and E_y. For example, if $E_x = E_y$, the vector sum of these orthogonal fields yields a wave polarized at 45° from the *x* axis. If $E_x = -E_y$ (or if $E_x = E_y$ and $\Delta\phi = \pi$), the light is linearly polarized at -45°. For in-phase component fields ($\Delta\phi = 0$), the linear polarization is oriented at an angle $\alpha = \tan^{-1}(E_y / E_x)$ with respect to the *x* axis.

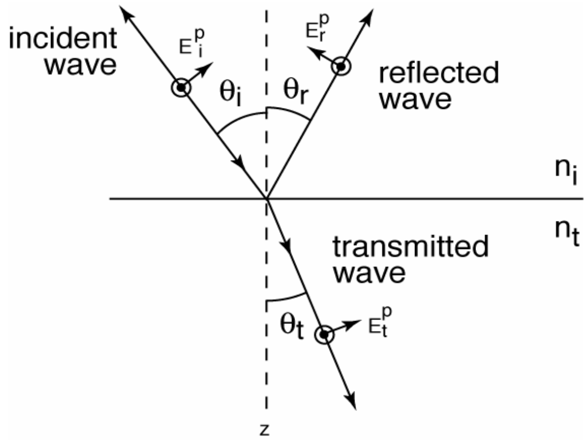

Figure 2 Light waves at a boundary. The plane of incidence coincides with the plane of the page. Incident, reflected, and transmitted *p*-polarized waves are in the plane of incidence. The corresponding *s*-polarizations (not shown) would be perpendicular to the plane of incidence.

In general, linear polarization states are often defined by an orientation angle, though descriptive terms such as *x*- or *y*-polarized, or vertical or horizontal, may be used. However, when a wave is incident upon a boundary two specific linearly polarized states are defined. The *plane of incidence* (Fig. 2) is the plane containing the incident ray and the boundary normal. The linear polarization in the plane of incidence is called *p*-polarization and the field component perpendicular to the plane is *s*-polarized. This convention is used with the Fresnel equations (Section II.A) to determine the transmittance, reflectance, and phase shift when light encounters a boundary.

B. Circular Polarization

Another special case occurs when $E_x = E_y = E_o$ and the field components have a 90° relative phase difference [$\Delta\phi = (m + ½)\pi$]. If $\Delta\phi = \pi/2$, Eq.(1b) becomes

$$\begin{aligned} E^{rcp} &= E_0\left[\hat{x}\cos(\omega t - k_0 z) + \hat{y}\cos(\omega t - k_0 z + \pi/2)\right] \\ &= E_0\left[\hat{x}\cos(\omega t - k_0 z) - y\cos(\omega t - k_0 z)\right] \end{aligned} \tag{2}$$

As the wave advances through space the magnitude of $\mathbf{E}^{rcp}$ is constant but the tip of this electric field vector traces a circular path about the propagation axis at a frequency ω. A wave with this behavior is said to be right-circularly polarized.

Figure 3 shows the electric field vector for right-circular polarization when viewed at a fixed time ($t = 0$); here the field will trace a right-handed spiral in space. An observer looking toward the origin from a distant point ($z > 0$) would see the vector tip rotating counterclockwise as the field travels along *z*. In contrast, the same observer looking at a right-circularly polarized field at a *fixed* position (for example, $z = 0$) would see the vector rotation trace out a clockwise circle in the *xy* plane as time advances. This difference in the sense of rotation between space and time is often a source of confusion, and depends on notation (see Section I.F).

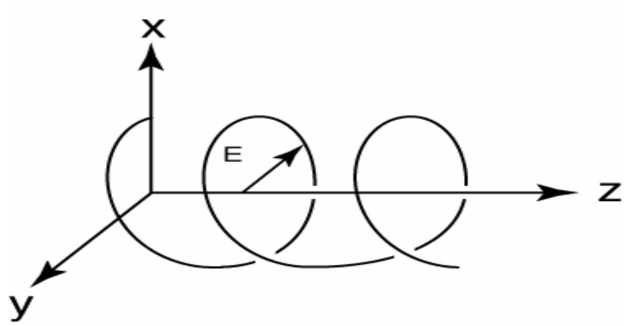

Figure 3 The electric field propagation for right-circular polarization, Eq. (2), when $t = 0$. At a fixed time, the tip of the electric field vector traces a right-handed corkscrew as the wave propagates along the +*z* direction.

When light is left-circularly polarized the field traces out a left-handed spiral in space at a fixed time and a counterclockwise circle in time at a fixed position. Equation (1b) describes left-circular polarization when $E_x = E_y = E_o$ and $\Delta\phi = -\pi/2$:

$$\begin{aligned} E^{lcp} &= E_0\left[\hat{x}\cos(\omega t - k_0 z) + \hat{y}\cos(\omega t - k_0 z - \pi/2)\right] \\ &= E_0\left[\hat{x}\cos(\omega t - k_0 z) + y\cos(\omega t - k_0 z)\right]. \end{aligned} \tag{3}$$

Right- and left-circular polarizations are orthogonal states and can be used as a basis pair for representing other polarization states, much as orthogonal linear states are combined to create circular polarization. Adding equal amounts of right- and left-circularly polarized light will yield a linearly polarized state. For example,

$$\tfrac{1}{2}E^{rcp} + \tfrac{1}{2}E^{lcp} = \hat{x}E_0\cos(\omega t - k_0 z). \tag{4}$$

In contrast, adding equal quantities of left- and right-circular polarization that are out of phase [by adding an additional π phase to both component fields in Eq. (2)] yields

$$-\tfrac{1}{2}E^{rcp} + \tfrac{1}{2}E^{lcp} = \hat{y}E_0\sin(\omega t - k_0 z). \tag{5}$$

In general, equal amounts of left- and right-circular polarization combine to produce a linear polarization with an azimuthal angle equal to half the phase difference.

C. Elliptical Polarization

For elliptically polarized light the electric field vector rotates at ω but varies in amplitude so that the tip traces out an ellipse in time at a fixed position *z*. Elliptical polarization is the most general state and linear and circular polarizations are simply special degenerate forms of elliptically polarized light. Because of this generality, attributes of this state can be applied to all polarization states.

The polarization ellipse (Fig. 4) can provide useful quantities for describing the polarization state. The azimuthal angle a of the semi-major ellipse axis from the *x* axis is given by

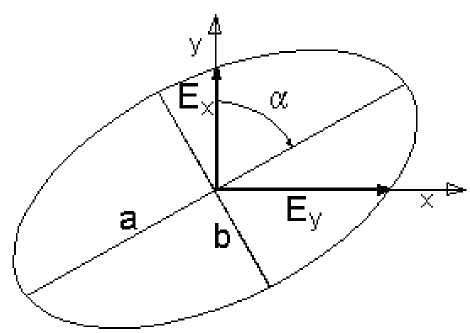

Figure 4 The polarization ellipse showing fields $\mathbf{E}_x$ and $\mathbf{E}_y$, ellipticity tan $|\varepsilon| = b/a$, and azimuthal angle α. The tip of the electric field **E** traces this elliptical path in the transverse plane as the field propagates down the *z* axis.

$$\tan(2\alpha) = \tan(\beta)\cos(\Delta\phi), \tag{6}$$

where $\tan(\beta) = E_y / E_x$ and $0 \le \beta \le \pi/2$. The ellipticity $\tan|\varepsilon| = b/a$, the ratio of the semi-minor and semi-major axes, is calculated from the amplitudes and phases of Eq. (1) as

$$\tan(\varepsilon) = \tan[\sin^{-1}(\sin 2\beta \sin\Delta\phi)/2]. \tag{7}$$

Polarization is right-elliptical when $0° < \Delta\phi < 180°$ and $\tan(\varepsilon) > 0°$ and left-elliptical when $-180° < \Delta\phi < 0°$

and tan(ε) < 0°.

D. Unpolarized Light

Monochromatic, or single-frequency, light must necessarily be in some polarization state. Light that contains a band of wavelengths does not share this requirement.

Quasi-monochromatic light can be represented by modifying Eq.(1b) as

$$E(z,t) = \mathrm{Re}(\hat{x}E_x(t)\exp\{i[\omega_m t + \phi_x(t)]\} + \hat{y}E_y(t)\exp\{i[\omega_m t + \phi_y(t)]\}) \tag{8}$$

where ω_m is the mean frequency of an electric field with bandwidth $\Delta\omega < \omega_m$. Taking the real part of this complex analytic representation yields the true field. Whereas the field amplitudes $E_i(t)$ and phases $\phi_i(t)$ are constants for strictly monochromatic light, these quantities fluctuate irregularly when the light has finite bandwidth. The pairs of functions $E_i(t)$ and $\phi_i(t)$ have statistical correlations that depend on the spectral bandwidth of the light source. The coherence time $\tau \sim 2\pi/\Delta\omega$ describes the time scale during which the pairs of functions show similar time response. For some brief time $t \ll \tau$, $E_i(t)$ and $\phi_i(t)$ are essentially constant, and $\mathbf{E}(t)$ possesses some elliptical polarization state, but a later field $\mathbf{E}(t + \tau)$ will have a different elliptical polarization. Light is described as unpolarized, or natural, if the time evolutions of the pairs of functions are totally uncorrelated within the detection time, and any polarization state is equally likely during these successive time intervals.

While strictly monochromatic light cannot be unpolarized, natural light can be polarized into any desired elliptical state by passing it through the appropriate polarizer. Indeed, when unpolarized light is incident on a polarizer, the detected output intensity is independent of the polarization state transmitted by the polarizer. This occurs because a unique polarization exists for an infinitesimal time $t \ll \tau$ and the average projection of these arbitrary states on a given polarizer is ½ over the relatively long integration time of the detector. In the absence of dispersive effects, unpolarized light, when totally polarized by an ideal polarizer, will behave much like monochromatic polarized light.

It is often desirable to have unpolarized light, especially when the undesired polarization dependence of components degrades optical system performance. For example, the responsivity of photodetectors can exhibit polarization dependence and cause measurements of optical power to vary with the polarization even when intensity is constant. In some cases, pseudo-depolarizers are useful for modifying polarization to produce light that approximates unpolarized light (Section III.F). For quasi-monochromatic light, the orthogonal field components can be differentially delayed, or retarded, longer than τ, so that the fields become uncorrelated. Alternatively, repeatedly varying the polarization state over a time shorter than the detector response causes the measurement to include the influence of many polarization states. This method, known as polarization scrambling, can reduce some undesirable polarization effects by averaging polarizations.

The previous discussion implicitly assumes that the light has uniform properties over the wavefront. However, the polarization can be varied over the spatial extent of the beam using a spatially varying retardance. Further description of these methods and their limitations is found in the discussion on optical retarders.

E. Degree of Polarization

Light that is neither polarized nor unpolarized is partially polarized. The fraction of the intensity that is polarized for a time much longer than the optical period is called the degree of polarization P and ranges from $P = 0$ for unpolarized light to $P = 1$ when a light beam is completely polarized in any elliptical state. Light is partially polarized when $0 < P < 1$. Partially polarized light occurs when $E_i(t)$ and $\phi_i(t)$ are not completely uncorrelated, and the instantaneous polarization states are limited to a subset of possible states. Partially polarized light may also be represented as a sum of completely polarized and unpolarized components.

We can also define a degree of linear polarization (the fraction of light intensity that is linearly polarized) or a degree of circular polarization (the fraction that is circularly polarized). Degrees of polarization can be described formally using the coherency matrix or Stokes vector formalism described in Section IV.

F. Notation

The choice of coordinate system and the form of the field in Eqs. (1a) and (1b) is not unique. We have chosen a right-handed coordinate system such that the crossproduct is $\hat{x} + \hat{y} = \hat{z}$ and used fields with a time dependence $\exp[i\,(\omega t - kz)]$ rather than the complex conjugate $\exp[-i\,(\omega t - kz)]$. Both choices are equally valid, but may result in different descriptions of the same polarization states. Descriptions of circular polarization in particular are often contradictory because of the confusion arising from the use of varied conventions. In this article we follow the "Nebraska Convention" adopted in 1968 by the participants of the Conference on Ellipsometry at the University of Nebraska.

Also, the choice of the Cartesian basis set for describing the electric field is common but not obligatory. Any polarization state can be decomposed into a combination of any pair of orthogonal polarizations. Thus Eqs. (1a) and (1b) could be written in terms of right- and left-circular states or orthogonal elliptical states.

II. POLARIZERS

An ideal polarizer transmits only one unique state of polarization regardless of the state of the incident light. Polarizers may be delineated as linear, circular, or elliptical, depending on the state that is produced. Linear polarizers that transmit a linear state are the most common and are often simply called "polarizers." The transmission axis of a linear polarizer corresponds to the direction of the output light's electric field oscillation. This axis is fixed by the device, though polarizers can be oriented (rotated normal to the incident light) to select the azimuthal orientation of the output state. When linearly polarized light is incident on a linear polarizer, the transmittance T from the polarizer follows Malus's law,

$$T = \cos^2\theta, \tag{9}$$

where θ is the angle between the input polarization's azimuth and the polarizer's transmission axis. When the incident light is formed by a linear polarizer, Eq.(9) describes the transmission through two polarizers with angle θ between transmission axes. In this configuration the second polarizer is often called an analyzer, and the polarizer and analyzer are said to be crossed when the transmittance is minimized ($\theta = 90°$).

Since an ideal polarizer transmits only one polarization state it must block all others. In practice polarizers are not ideal, and imperfect polarizers do not exclude all other states. For an imperfect polarizer Malus's law becomes

$$T = (T_{\max} - T_{\min})\cos^2\theta + T_{\min}, \tag{10}$$

where T_{max} and T_{min} are called the principal transmittances, and transmittance T varies between these values. The *extinction ratio* T_{min} / T_{max} provides a useful measure of polarizer performance. Diattenuation is the dependence of transmittance on incident polarization, and can be quantified as $(T_{max} - T_{min})/(T_{max} + T_{min})$, where the maximum and minimum transmittances occur for orthogonal polarizations in homogeneous elements. (Homogeneous polarization elements have eigenpolarizations that are orthogonal and we consider such elements exclusively in this article.) Polarizers are optical elements that have a diattenuation approaching 1.

Most interfaces with nonnormal optical incidence will exhibit some linear diattenuation since the Fresnel reflection and transmission coefficients depend on the polarization. High-performance polarizers exploit these effects to achieve very high diattenuations by differentially reflecting and transmitting orthogonal polarizations. In contrast, dichroism is a material property in which diattenuation occurs as light travels through the medium. Most commercial polarizers exploit dichroism, polarizationdependent reflection or refraction in birefringent crystals, or polarization-dependent reflectance and transmittance in dielectric thin-film structures.

A. Fresnel Equations

Maxwell's equations applied to a plane wave at an interface between two dielectric media provide the relationship among incident, transmitted, and reflected wave amplitudes and phases. Figure 2 shows the electric fields and wavevectors for a wave incident upon the interface between two lossless, isotropic dielectric media. The plane of incidence contains all three wavevectors and is used to define two specific linear polarization states; *p*-polarized light has its electric field vector within the plane of incidence, and *s*-polarized light is perpendicular to this plane. The law of reflection $\theta_i = \theta_r$ provides the direction of the reflected wave. The refraction angle is given by Snell's law,

$$n_i \sin\theta_i = n_t \sin\theta_t. \tag{11}$$

Fresnel's equations yield the amplitudes of the transmitted field E_t and reflected field E_r as fractions of the incident field E_i. For *p*-polarized light in isotropic, homogeneous, dielectric media, the amplitude reflectance r_p is

$$r_p = \left(\frac{E_r}{E_i}\right)_p = \frac{n_t \cos\vartheta_i - n_i \cos\theta_t}{n_i \cos\vartheta_t + n_t \cos\theta_i} \tag{12}$$

and amplitude transmittance tp is

$$t_p = \left(\frac{E_t}{E_i}\right)_p = \frac{2n_i \cos\vartheta_i}{n_i \cos\vartheta_t + n_t \cos\theta_i}. \tag{13}$$

For s-polarized light, the corresponding Fresnel equations are

$$r_S = \left(\frac{E_r}{E_i}\right)_s = \frac{n_i \cos\vartheta_i - n_t \cos\theta_t}{n_i \cos\vartheta_i + n_t \cos\theta_t} \tag{14}$$

and

$$t_s = \left(\frac{E_t}{E_i} \right)_s = \frac{2 n_i \cos \vartheta_i}{n_i \cos \vartheta_i - n_t \cos \theta_t}. \tag{15}$$

The Fresnel reflectance for cases $n_i/n_t = 1.5$ and $n_t/n_i = 1.5$ is shown in Fig. 5. At an incidence angle $\theta_B = \tan^{-1}(n_t/n_i)$(for $n_i > n_t$), known as the Brewster angle, $r_p = 0$ and p-polarized light is totally transmitted. In a *pile-of-plates* polarizer, plates of glass are oriented at the Brewster angle so that only s-polarized light is reflected from each plate, and the successive diattenuations from each plate increase the degree of polarization of transmitted light.

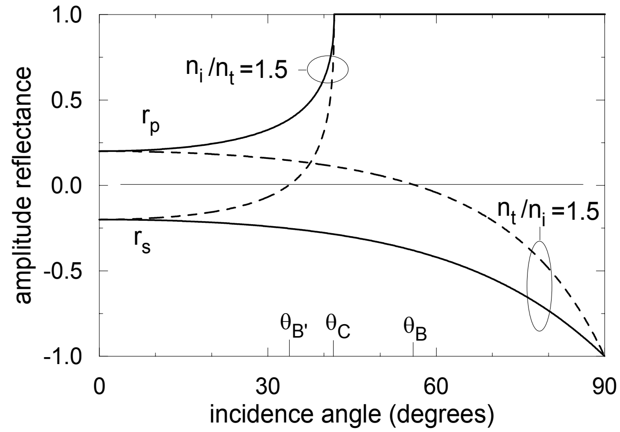

Figure 5 Fresnel reflectances for p-polarized (solid curve) and s-polarized (dashed) light for cases $n_i/n_t = 1.5$ and $n_t/n_i = 1.5$ The amplitude reflectance is 0 for p-polarized light at the Brewster angle θ_B and is one for all polarizations when the incidence angle is $\theta \geq \theta_C$.

When $n_i > n_t$, both polarizations may be completely reflected if the incidence angle is larger than the critical angle θ_c,

$$\theta_c = \sin^{-1} \frac{n_t}{n_i}. \tag{16}$$

When $\theta_i \geq \theta_c$, the light undergoes *total internal reflection* (TIR). For these incidence angles no net energy is transmitted beyond the interface and an *evanescent* field propagates along the direction θ_t. The reflectance can be reduced from 1 if the medium beyond the interface is thinner than a few wavelengths and followed by a higher refractive index material. The resulting *frustrated total internal reflection* allows energy to flow across the interface, leading to nonzero transmittance. For this reason, TIR devices using glass-air interfaces must be kept free of contaminants that may frustrate the TIR. Birefringent crystal polarizers obtain very high extinction ratios by transmitting one linear polarization while forcing the orthogonal polarization to undergo TIR.

B. Birefringent Crystal Polarizers

Birefringent polarizers spatially separate an incident beam into two orthogonally polarized beams. In a conventional polarizer, the undesired polarization is eliminated by directing one beam into an optical absorber so that a single polarization is transmitted. Alternatively, a polarizing beamsplitter transmits two distinct orthogonally polarized beams that are angularly separated or displaced.

In birefringent materials, the incident polarization is decomposed into two orthogonal states called principal polarizations or eigenpolarizations. When the eigenpolarizations travel at the same velocity (and see the same refractive index), the direction of propagation is called an optic axis (see Section III.A). When light does not travel along an optic axis, the eigenpolarizations see different refractive indices and thus propagate at different velocities through the material.

When light enters or exits a birefringent material at a nonnormal angle θ that is not along an optic axis, the eigenpolarizations refract at different angles, undergoing what is termed double refraction. Also, each eigenpolarization may encounter different reflectance or transmittance at interfaces (since Fresnel coefficients depend on the refractive indices), and diattenuation results. Complete diattenuation occurs if one eigenpolarization undergoes total internal reflection while the other eigenpolarization is transmitted.

Most birefringent polarizers are made from calcite, a naturally occurring mineral. Calcite is abundant in its polycrystalline form, but optical-grade calcite required for polarizers is rare, which makes birefringent polarizers more costly than most other types. Calcite transmits from below 250 nm to above 2 μm and is used for visible and nearinfrared applications. Other birefringent crystals, such as magnesium fluoride (with transmittance from 140 nm to 7 μm), can be used at some wavelengths for which calcite is opaque.

Prism polarizers are composed of two birefringent prisms cut at an internal incidence angle that transmits only one eigenpolarization while totally internally reflecting the other (Fig. 6). The prisms are held together by a thin cement layer or may be separated by an air gap and externally held in place for use with higher power laser beams. The transmitted beam contains only one eigenpolarization since the orthogonal polarization is completely reflected. The prisms are aligned with parallel optic axes, so that this transmitted beam undergoes very small deviations, usually less than 5 min of arc. Often the reflected beam also contains a small amount of the transmitted eigenpolarization since nonzero reflectance results if the refractive indices of the cement and transmitted eigenpolarization are not exactly equal. Because the reflected beam has poorer extinction, it is usually eliminated by placing an indexmatched absorbing layer on the side face toward which light is reflected.

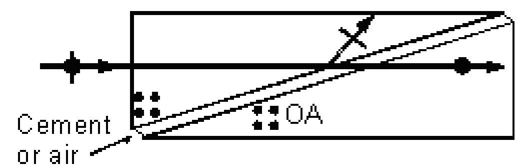

Figure 6 Glan-Thompson prism polarizer. At the interface, *p*-polarized light reflects (and is typically absorbed by a coating at the side of the prism) and *s*-polarized light is transmitted. The optic axes (shown as dots) are perpendicular to the page.

Glan prism polarizers are the most common birefringent crystal polarizer. They exhibit superior extinction; extinction ratios of 10^{-5}-10^{-6} are typical, and extinctions below 10^{-7} are possible. The small residual transmittance can arise from material imperfection, scattering at the prism faces, or misalignment of the optic axes in each prism of the polarizer.

Because total internal reflection requires incidence angles larger than θ_c, the polarizer operates over a limited range of input angles that is often asymmetric about normal incidence. The semi-field angle is the maximum angle for which output light is completely polarized regardless of the rotational orientation of the polarizer (that is, for any azimuthal angle of output polarization). The field angle is twice the semi-field angle. The field angle depends on the refractive index of the intermediate layer (cement or air) and the internal angle of the contacted prisms. Since the incidence angle at the contacting interface depends in part on the refractive index when light is nonnormally incident on the polarizer, the field angle is wavelength dependent.

Birefringent crystal polarizing beamsplitters transmit two orthogonal polarizations. Glan prism polarizers can act as beamsplitters if the reflected beam exits through a polished surface, though extinction is degraded. Polarizing beamsplitters with better extinction separate the beams through refraction at the interface. In Rochon prisms, light linearly polarized in the plane normal to the prism

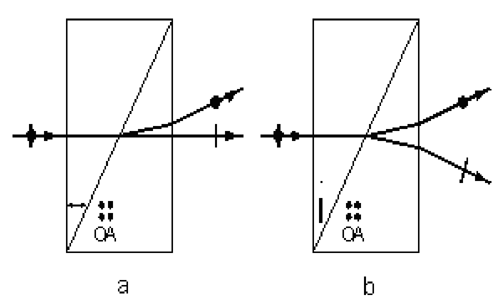

Figure 7 (a) Rochon and (b) Wollaston polarizers. The directions of the optic axes are shown in each prism (as dots for axes perpendicular to the page and as a two-arrow line for the axes in the plane of the page).

is transmitted undeviated, while the orthogonal polarization is deviated by an angle dependent on the prism wedge angle and birefringence (Fig. 7a). Sénarmont polarizing beam splitters are similar, but the polarizations of the deviated and undeviated beams are interchanged. Wollaston polarizers (Fig. 7b) deviate both output eigenpolarizations with nearly equal but opposite angles when the input beam is normally incident. For all these polarizers, the deviation angle depends on the wedge angle and varies with wavelength.

C. Interference Polarizers

The Fresnel equations show that the transmittance and reflectance of obliquely incident light will depend on the polarization. Dielectric stacks made of alternating high and low-refractive index layers with quarter-wave optical thickness can be tailored to provide reflectances and transmittances with large diattenuation. Optical thickness depends on incidence angle, and polarizers based on quarter-wave layers are sensitive to incidence angle and wavelength. Designs that increase the wavelength range do so at the expense of input angle range, and vice versa. Polarizing beamsplitter cubes are made by depositing the stack on the hypotenuse of a right-angle prism and cementing the coated side to the hypotenuse of a second prism.

The extinction of these devices is limited by the defects in the coating layers or the optical quality of the optical substrate material through which light must pass. The state of polarization may also be altered by the birefringence in the substrate. Commercial thin-film polarizers are available with an extinction of about 10^{-5}.

D. Dichroic Polarizers

Some molecules are optically anisotropic, and light polarized along one molecular direction may undergo greater absorption than perpendicularly polarized light. When these molecules are randomly oriented, this molecular-level diattenuation will average out as the light propagates through the thickness, and bulk diattenuation may not be observed. However, linear polarizers can be made by orienting dichroic molecules or crystals in a plastic or glass matrix that maintains a desired alignment of the transmission axes. Extinction ratios between 10^{-2} and 10^{-5} are possible in oriented dichroics in the visible and near-infrared regions.

Dichroic sheet polarizers are available with larger areas and at lower cost than other polarizer types. Also, the acceptance angle, or maximum input angle from normal incidence that does not result in degraded extinction, is typically large in dichroics because diattenuation occurs during bulk propagation rather than at interfaces. However, the maximum transmittance of these polarizers may be significantly less than unity since the transmission axis may also absorb light. Because absorbed light will heat the material and may cause damage at high power, incident powers are limited.

III. RETARDERS

Retarders are devices that induce a phase difference, or retardation, between orthogonally polarized components of a light wave. Linear retarders are the most common and produce a retardance $\Delta\phi = \phi_y - \phi_x$ [using the notation of Eqs. (1a) and (1b)] between orthogonal linear polarizations. Circular retarders cause a phase shift between rightand left-circular polarizations and are often called rotators because circular retardance changes the azimuthal angle of linearly polarized light. Because the polarization state of light is determined by the relative amplitudes and phase shifts between orthogonal components, retarders are useful for altering and controlling a wave's polarization. In fact, an arbitrary polarization state can be converted to any other state using an appropriate retarder.

A. Linear Birefringence

In optically anisotropic materials, such as crystals, the phase velocity of propagation generally depends on the direction of propagation and polarization. The optic axes are propagation directions for which the phase velocity is independent of the azimuth of linear polarization. For other propagation directions, two orthogonal eigenaxes perpendicular to the propagation define the linear polarizations of waves that propagate through the crystal with constant phase velocity. These eigenpolarizations are linear states whose refractive indices are determined by the crystal's dielectric tensor and propagation direction. Light polarized in an eigenpolarization will propagate through an optically anisotropic material with unchanging polarization, while light in other polarization states will change with distance as the beam propagates. Uniaxial crystals and materials that behave uniaxially are commonly used in birefringent retarders and polarizers. These crystals have a single optic axis, two principal refractive indices n_o and n_e, and a linear birefringence $\Delta n = n_e - n_o$. When light travels parallel to the optic axis, the eigenpolarizations are degenerate, and all polarizations propagate with index n_o. For light traveling in other directions, one eigenpolarization has refractive index n_o and the other's varies with direction between n_o and n_e (and equals n_e when the propagation is perpendicular to the optic axis).

B. Waveplates

Waveplates are linear retarders made from birefringent materials. Rewriting Eq. (1a) for propagation through a birefringent medium of length L yields

$$\begin{aligned} E(z=L,t) = \mathrm{Re}\{E_x \exp[i(\omega t - k_0 n_x L)] \\ + E_y \exp[i(\omega t - k_0 n_y L)]\}, \end{aligned} \tag{17}$$

where the x and y directions coincide with eigenpolarizations and the absolute phases are initially equal (at $z = 0$, $\phi_x = \phi_y = 0$). The retardance $\Delta\phi = k_o(n_x - n_y)L$ is the relative phase shift between eigenpolarizations and depends on the wavelength, the propagation distance, and the difference between the refractive indices of the eigenpolarizations. If the z axis is an optic axis, then $n_x = n_y = n_o$, and there is no retardance; if is $\hat{z}$ perpendicular to an optic axis, the retardance is $\Delta\phi = \pm k_o(n_o - n_e)L$. In general, the retardance over a path of length L in a material with birefringence Δn is given by

$$\Delta\phi = 2\pi\Delta n \mathrm{L} / \lambda . \tag{18}$$

Retardance may be specified in radians, degrees $[\Delta\phi = 360° \cdot (n_o - n_e)L/\lambda 0]$, or length $[\Delta\phi = n_o - n_e)L]$.

A waveplate that introduces a π-radian or 180° phase shift between the eigenpolarizations is called a *half-wave plate*. Upon exiting the plate, the two eigenpolarizations have a $\lambda/2$ relative delay and are exactly out of phase. A half-wave plate requires a birefringent material with thickness given by

$$L_{\lambda/2} = \frac{\lambda_0 (2m+1)}{2|n_o - n_e|}. \tag{19}$$

where the waveplate order m is a positive integer that need not equal 0 since additional retardances of 360° do not affect the phase relationship. *Quarter-wave plates* are another common component and provide phase shifts of 90° or $\pi / 2$.

The eigenaxis with the lower refractive index (n_o in positive uniaxial crystals such as quartz, and n_e in negative uniaxial crystals such as calcite) is called the fast axis of the retarder due to the faster phase velocity and is often marked by the manufacturer. The eigenaxes can be identified by rotating the retarder between crossed polarizers until the transmittance is minimized. When the polarizer transmission axis coincides with the retarder eigenaxis, the input polarization matches the eigenpolarization, and the light travels through the crystal unchanged until blocked by the analyzer. An input different from the eigenpolarization will exit the crystal in a different polarization state and will not be completely blocked by the analyzer.

Waveplates are commonly made using quartz, mica, or plastic sheets that are stretched to produce an anisotropy that gives rise to birefringence. At visible wavelengths, n_e - n_o ~0.009 for quartz, and the corresponding zeroth-order (m = 0) quarter-wave plate thickness of ~40 µm poses a severe manufacturing challenge. Mica can be cleaved into thin sections to obtain zeroth-order retardance, but the resulting waveplate usually has poorer spatial uniformity. Polymeric materials often have lower birefringence and can be most easily fabricated into zeroth-order waveplates.

In many applications, retardance of integral multiples of 2π is unimportant, and multiple-order ($m \geq 0$) waveplates are often lower in cost because the increased thickness eases fabrication. However, this approach can result in increased retardance errors. For example, retardance depends on the wavelength [explicitly in Eq. (18) or through dispersion]. Also, retardance can change with temperature or with nonnormal incidence angles- that vary the optical thickness and propagation direction. Retardance errors arising from changes in wavelength, temperature, or incidence angle linearly increase with thickness and make multiple-order waveplates unadvisable in applications that demand accurate retardance.

Compound zeroth-order waveplates represent a compromise between manufacturability and performance when true zeroth-order waveplates are not easily obtained. When two similar waveplates are aligned with orthogonal optic axes, the phase shifts in each waveplate have opposite sign and the combined retardance will be the difference between the two retardances. Compound zeroth-order retarders are made by combining two multipleorder waveplates in this way so that the net retardance is less than 2π. For example, two multiple-order waveplates with retardance $\Delta\phi_1 = 20\pi + \pi/2$ and $\Delta\phi_2 = -20\pi$ can be combined to yield a compound zeroth-order quarterwave plate. Compound zeroth-order waveplates exhibit the same wavelength and temperature dependence as zeroth-order waveplates since retardance errors are proportional to the difference of plate thicknesses. However, input angle dependence is the same as in a multiple-order waveplate with equivalent total thickness.

C. Compensators

A compensator is a variable linear retarder that can be adjusted over a continuous range of values (Fig. 8). In a Babinet compensator, two wedged plates of birefringent material are oriented with their optic axes perpendicular. In this arrangement, the individual wedges impart opposite signs of retardance, and the net retardance is the difference between the individual magnitudes. The magnitudes depend on the thickness of each wedge traversed by the optical beam. Typically one wedge is fixed and the other translated by a micrometer drive so that this moving wedge presents a variable thickness in the beam path, and the net retardance depends on the micrometer adjustment. The use of two wedges eliminates the beam deviation and the output beam is collinear to the input.

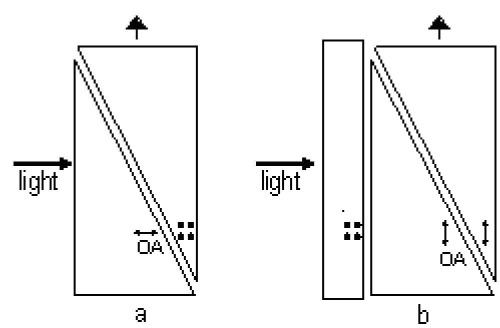

Figure 8 (a) Babinet and (b) Soleil-Babinet compensators. One wedge moves in the direction of the vertical arrow to adjust the retardance. The direction of the optic axes are shown using notation from Fig. 7.

The Babinet compensator has the disadvantage that the retardance varies across the optical beam because the relative thicknesses of each wedge and corresponding net retardance vary over the beam in the direction of wedge travel. This can be overcome using a Soleil (or BabinetSoleil) compensator. In this device the two wedged pieces have coincident optic axes and translation of the moving wedge changes the total thickness and retardance of the combined retarder. The total thickness of this two-wedge piece is now constant over the useful aperture. A parallel plate of fixed retardance is placed after the wedge, in the same manner as a compound zeroth-order retarder, to improve performance.

D. Rhombs

Retarders can also be fabricated of materials that do not exhibit birefringence. The phase shift between *s*- and *p*-polarized waves that occurs at a total internal reflection (Section II, Fresnel equations) can be exploited to obtain a linear retarder. When light is incident at angles larger than the critical angle, the retardance at the reflection is

$$\Delta\phi = \phi_p - \phi_s = 2\tan^{-1}\left[\frac{\cos\theta_i\sqrt{\sin^2\theta_i - (n_i/n_t)^2}}{\sin^2\theta_i}\right] \tag{20}$$

and depends on the incidence angle and refractive indices. A Fresnel rhomb is a solid parallelogram fabricated so that a beam at normal incidence at the entrance face totally reflects twice within the rhomb to provide a net retardance of $\pi/2$. This retarder is, however, very sensitive to the incidence angle and laterally displaces the beam. Concatenating two Fresnel rhombs (Fig. 9) provides collinear output and can greatly reduce the sensitivity of retardance to incident angle since retardance changes at the first pair of reflections are partially canceled by the second pair.

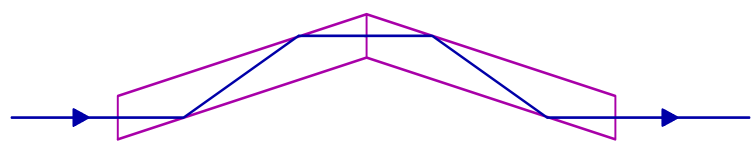

Figure 9 Two Fresnel rhombs concatenated to form a Fresnel double rhomb.

Total-internal-reflection retarders are less sensitive to wavelength variation than waveplates whose retardance increases with L/λ since the rhomb retardance does not depend on the optical path length. Wavelength dependence is limited only by the material dispersion $dn/d\lambda$, which contributes small retardance changes. Thus, rhomb devices are more nearly achromatic than waveplates and can be operated over ranges of 100 nm or more. Rhomb devices are much larger than waveplates, and the clear aperture has practical limits since increasing cross section requires a proportional increase in length. Performance can also be compromised by the presence of birefringence in the bulk glass. Birefringence, arising from stresses in material production or optical fabrication, can lead to spatial variations and path-length dependence, and limit retardance stability to several degrees if not mitigated.

E. Circular Retarders

Some materials can exhibit circular birefringence, or *optical activity*, in which the eigenpolarizations are right- and left-circular and the retardance is a phase shift between these two circular states. Circular retarders are often called rotators because incident linear polarization will generally exit at a different azimuthal angle that depends on the *rotary power* (circular retardance per unit length) and thickness. A material that rotates linearly polarized light clockwise (as viewed by an observer facing the light source) is termed dextrorotary or right-handed, while counterclockwise rotation occurs in levorotary, or left-handed, materials. The sense of rotation is fixed with respect to the propagation direction; if the beam exiting an optically active material is reflected back through the material, the polarization will be restored to the initial azimuth. Thus a double pass through an optically active material will cause no net rotation of linear polarization.

Crystalline quartz exhibits optical activity that is most evident when propagation is along the optic axis and retardance is absent. The property is not limited to crystalline materials, however; molecules that are chiral (that lack plane or centrosymmetry and are not superposable on their mirror image) can yield optical activity. *Enantiomers* are chiral molecules that share common molecular formulas and ordering of atoms but differ in the three-dimensional arrangement of atoms; separate enantiomers have equal rotary powers but differ in the sense of rotation. Liquids and solutions of chiral molecules such as sugars may be optically active if an excess of one enantiomer is present.

In solution, each enatiomeric form will rotate light, and the net rotation depends on the relative quantities of dextrorotary and levorotary enantiomers. Mixtures with equal quantities of enantiomers present are called racemic and the net rotation is zero. Most naturally synthesized organic chiral molecules, for example, sugars and carbohydrates, occur in only one enatiomeric form. Saccharimetry, the measurement of the optical rotary power of sugar solutions, is used to determine the concentration of sugar in single-enantiomer solutions.

F. Electrooptic and Magnetooptic Effects

In some materials, retardance can be induced by an electric or magnetic field. These effects are exploited to create active devices that produce an electrically controllable retardance.

Crystals that are not centrosymmetric may exhibit a linear birefringence proportional to an applied electric field called the linear electrooptic effect or Pockels effect. In these materials, applied fields cause an otherwise isotropic crystal to behave uniaxially (and uniaxial crystals to become biaxial). Crystal symmetry determines the direction of the optic axes and the form of the electrooptic tensor. The magnitude of the induced birefringence thus depends on the polarization direction, the applied field strength and direction, and the material.

The electrically induced birefringence can be appreciable in some materials, and the Pockels effect is widely used in retardance modulators, phase modulators, and amplitude modulators. Modulators are often characterized by their half-wave voltage V_π, or the voltage needed to cause a 180° phase shift or retardance. V_π can vary from ~10 V in waveguide modulators to hundreds or thousands of volts in bulk modulators.

The Kerr, or quadratic, electrooptic effect occurs in solids, liquids, or gases and has no symmetry requirements. In this effect, the linear birefringence magnitude is proportional to the square of the applied electric field and the induced optic axis is parallel to the field direction. The effect is typically smaller than the Pockels effect and is often negligible in Pockels materials.

The Faraday effect is an induced circular birefringence that is proportional to an applied magnetic field. It is often called Faraday rotation because the circular birefringence rotates linearly polarized light by an angle proportional to the field. The Faraday effect can occur in all materials, though the magnitude is decreased by birefringence.

In contrast to optical activity, the sense of Faraday rotation is determined by the direction of the magnetic field. Thus, a double-pass configuration in which light exiting a Faraday rotator reflects and propagates back through the material will yield twice the rotation of a single pass. This property is exploited in optical isolators, or components that transmit light in only one direction. In the simplest isolators, a 45° Faraday rotator is placed between polarizers with transmission axes at 0° and 45°. In the forward direction, light linearly polarized at 0° is azimuthally rotated 45° to coincide with the analyzer axis and is fully transmitted; backward light input at 45° rotates to 90° and is completely blocked by the polarizer at 0°.

Faraday mirrors, made by combining a 45° Faraday rotator with a plane mirror, have the extraordinary property of "unwinding" polarization changes caused by propagation. Polarized light that passes through an arbitrary retarder, reflects off a Faraday mirror, and retraces the input path will exit with a fixed polarization for all magnitudes or orientations of the retarder so long as the retardance is unchanged during the round-trip time. When the input light is linearly polarized, the return light is always orthogonally polarized for all intervening retardances. These devices find applications in fiber optic systems since bend-induced retardance is difficult to control in an ordinary optical fiber.

G. Pseudo-Depolarizers

Conversion of a polarized, collimated light beam into a beam that is truly unpolarized is difficult. Methods for obtaining truly unpolarized light rely on diffuse scattering, such as passing light through ground glass plates or an integrating sphere. These methods result in light propagating over a large range of solid angles and decrease the irradiance, or power per unit area, away from the depolarizer. The loss is often unacceptable when a collimated beam is needed.

Approximations to the unpolarized state can be created using pseudo-depolarizers that produce a large variety of states over time, wavelength, or the beam cross section. As described in Section I, temporal decorrelation requires that the beam propagate through a retardance that is much larger than the light's coherence length $L_c = c\tau \approx 2\pi c / \Delta\omega$. If nonmonochromatic, linearly polarized light bisects the axes of a waveplate with sufficiently large retardance, the two linear eigenpolarizations will emerge with a relative phase shift that rapidly and arbitrarily changes on the order of the coherence time. At any moment the instantaneous output state will be restricted to a point on the Poincaré sphere (see Section IV) along the great circle connecting the ±45° and circular polarization states. When the detector is slower than τ, the averaged response will include the influence of all these states.

Lyot depolarizers are configurations of two retarders that perform this temporal decorrelation for any input polarization state. These are commonly made by concatenating thick birefringent plates that act as high-order waveplates or by connecting lengths of polarization-maintaining (PM) fiber. PM fiber has about one wavelength of retardance every few millimeters, and can be obtained in lengths sufficient to decorrelate multimode laser light.

A polarized light beam can also be converted to a beam with a spatial distribution of states to approximate unpolarized light, without the requirements on spectral bandwidth. For example, the retardance across a wedged waveplate is not spatially uniform, and an incident beam will exit with a spatially varying polarization. When detected by a single photodetector, the influence of all the states will be averaged in the output response. These methods often satisfy needs for unpolarized light, but clearly depend on the details and requirements of the application.

IV. MATHEMATICAL REPRESENTATIONS

Several methods have been developed to facilitate the representation of polarization states, polarization elements, and the evolution of polarization states as light passes through components. Using quasi-monochromatic fields, the 2 x 2 coherency matrix can be used to represent polarizations and determine the degree of polarization of light. The four-element stokes vector describes the state of light using readily measurable intensities and can be related to the coherency matrix. Mueller calculus represents optical components as real 4 x 4 matrices; when combined with Stokes vectors it provides a quantitative description of the interaction of light and optical components. In contrast, Jones calculus represents light using two-element electric field vectors. Jones calculus cannot describe partially polarized or unpolarized light, but retains phase information so that coherent beams can be properly combined. Finally, the Poincaré Sphere is a pictorial representation that is useful for conceptually understanding the interaction between retarders and polarization states. A brief discussion introduces each of these methods.

A. Coherency Matrix

Using Eq. (8), we can define orthogonal field components of a quasi-monochromatic plane wave $E_x = E_x(t) \exp[i(\omega t - k_o z + \phi_x(t))]$ and likewise for E_y. The coherency matrix **J** is given by

$$J = \begin{bmatrix} \langle E_x E_x^* \rangle \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle \langle E_y E_y^* \rangle \end{bmatrix} = \begin{bmatrix} J_{xx} J_{xy} \\ J_{yx} J_{yy} \end{bmatrix}, \tag{21}$$

where the angle brackets denote a time average and the asterisk denotes the complex conjugate. The total irradiance I is given by the trace of the matrix, $\mathrm{Tr}(\mathbf{J}) = J_{xx} + J_{yy}$, and the degree of polarization is

$$P = \sqrt{1 - \frac{4|J|}{(J_{xx} + J_{yy})^2}} \tag{22}$$

where |J| is the determinant of the matrix. Recalling the notation for elliptical light, one can find the azimuthal angle α of the semi-major ellipse axis from the x axis and the ellipticity angle ε of the polarized component as

$$\begin{aligned} \alpha &= \frac{1}{2}\tan^{-1}\left[\frac{J_{xx} + J_{yy}}{J_{xx} - J_{yy}}\right] \\ \varepsilon &= \frac{1}{2}\tan^{-1}\left[\frac{-i(J_{xy} - J_{yx})}{P(J_{xx} - J_{yy})}\right]. \end{aligned} \tag{23}$$

Partially polarized light can be decomposed into polarized and unpolarized components and expressed using coherency matrices as $\mathbf{J} = \mathbf{J}_p + \mathbf{J}_u$. Thus the state of the polarized portion of light can be extracted from the coherency matrix even when light is partially polarized. The coherency matrix representation of several states is provided in Table I.

B. Mueller Calculus

In Mueller calculus the polarization state of light is represented by a four-element Stokes vector S. The Stokes parameters s_0 s_1 s_2 and s_3 are related to the coherency matrix elements or the quasi-monochromatic field representation through

$$
\begin{aligned}
s_0 &= J_{xx} + J_{yy} = \left\langle E_x(t)^2 \right\rangle + \left\langle E_y(t)^2 \right\rangle \\
s_1 &= J_{xx} - J_{yy} = \left\langle E_x(t)^2 \right\rangle - \left\langle E_y(t)^2 \right\rangle \\
s_2 &= i(J_{xy} + J_{yx}) = 2\left\langle E_x(t)E_y(t)\cos(\Delta\phi)\right\rangle \\
s_3 &= i(J_{yx} - J_{yy}) = 2\left\langle E_x(t)E_y(t)\sin(\Delta\phi)\right\rangle,
\end{aligned}
\tag{24}
$$

where the angle brackets denote a time averaging required for nonmonochromatic light. Each Stokes parameter is related to the difference between light intensities of specified orthogonal pairs of polarization states. Thus, the Stokes vector is easily found by measuring the power P_t transmitted through six different polarizers. Specifically,

$$
S = \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} P_{0^\circ} + P_{90^\circ} \\ P_{0^\circ} - P_{90^\circ} \\ P_{+45^\circ} - P_{-45^\circ} \\ P_{rcp} - P_{lcp} \end{bmatrix}, \tag{25}
$$

so that s_0 is the total power or irradiance of the light beam, s_1 is the difference of the powers that pass through horizontal (along $\hat{x}$) and vertical (along $\hat{y}$) linear polarizers, s_2 is the difference between +45° and -45° linearly polarized powers, and s_3 is the difference between right- and left-circularly polarized powers. The values of the Stokes parameters are limited to $s_0^2 \geq s_1^2 + s_2^2 + s_3^2$ and are often normalized so that $s_0 = 1$ and $-1 \leq s_1, s_2, s_3 \leq 1$. Table 1 lists normalized Stokes vectors for several polarization states. The degree of polarization [Eq. (22)] can be written in terms of Stokes parameters as

$$
P = \frac{\sqrt{s_1^2 + s_2^2 + s_2^3}}{s_0^2}. \tag{26}
$$

Additionally, we can define the degree of linear polarization (the fraction of light in a linearly polarized state) by replacing the numerator of Eq. (26) with $\sqrt{s_1^2 + s_2^2}$, or the degree of circular polarization by replacing the numerator with s_3.

An optical component that changes the incident polarization state from **S** to some output state **S'** (through reflection, transmission, or scattering) can be described by a 4 x 4 Mueller matrix **M**. This transformation is given by

$$
S' = \begin{bmatrix} s_0' \\ s_1' \\ s_2' \\ s_3' \end{bmatrix} = MS = \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \tag{27}
$$

where **M** can be a product of n cascaded components $\mathbf{M}_i$ using

$$M = \prod_{i=1}^{n} M_i \tag{28}$$

Matrix multiplication is not commutative and the product must be formed in the order that light reaches each component. For a system of three components in which the light is first incident on component 1 and ultimately exits component 3, **S'** = $\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{S}$, for example.

Examples of Mueller matrices for several homogeneous polarization components are given in Table II. The Mueller matrix for a component can be experimentally obtained by measuring **S'** for at least 16 judiciously selected **S** inputs, and procedures for measurement and data reduction are well developed.

Table I Matrix Representations of Selected Polarization States[a]

State	Coherency matrix	Stokes Vector	Jones Vector
Linear along $\hat{x}$ (α = β = 0º; tan ε = 0)	$I\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$	$\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$
Linear along $\hat{y}$ (α = β = 90º; tan ε = 0)	$I\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$
Linear at 45º (α = β = 45º; tan ε = 0)	$\frac{1}{2}I\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$	$\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$
General linear (-90º; < α < 90º; tan ε = 0)	$\frac{1}{2}I\begin{bmatrix}\cos(\alpha)^2 & \sin\alpha\cos\alpha\\ \sin\alpha\cos\alpha & \sin(\alpha)\end{bmatrix}$	$\begin{bmatrix}1\\ \cos 2\alpha\\ \sin 2\alpha\\ 0\end{bmatrix}$	$\begin{bmatrix}\cos(\alpha)\\ \sin(\alpha)\end{bmatrix}$
Right circular (tan ε = 1; Δϕ = 90º; β = 45º)	$\frac{1}{2}I\begin{bmatrix}1 & i\\ i & 0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$
Left circular (tan ε = -1; Δϕ = -90º; β = 45º)	$\frac{1}{2}I\begin{bmatrix}1 & \\ i & 1\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$
General elliptical		$\begin{bmatrix}1\\ \cos 2\varepsilon\cos 2\beta\\ \cos 2\varepsilon\sin 2\beta\\ \sin 2\varepsilon\end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix}\cos\beta\varepsilon^{-1\Delta\phi/2}\\ \sin\beta\varepsilon^{i\Delta\phi/2}\end{bmatrix}$
Unpolarized	$\frac{1}{2}I\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	None

[a] The parameters of α, β, ε, and Δϕ are defined corresponding to elliptical light as discussed in Section I. Extensive lists of Stokes and Jones vectors are available in several texts.

Table II Matrix Representation of Optical Components

Component	Mueller matrix	Jones matrix
Linear diattenuator with maximum (minimum) transmission $p_1^2\,(p_2^2)$ or absorber (if $p = p_1 = p_2$	$\frac{1}{2}\begin{bmatrix} p_1+p_2 & p_1-p_2 & 0 & 0 \\ p_1-p_2 & p_1+p_2 & 0 & 0 \\ 0 & 0 & 2\sqrt{p_1p_2} & 0 \\ 0 & 0 & 0 & 2\sqrt{p_1p_2} \end{bmatrix}$	$\begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$
Linear polarizer at 0º	$\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Linear Polarizer at an angle θ	$\frac{1}{2}\begin{bmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \cos 2\theta \sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta \sin 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$
Half-wave (δ = 180º) linear retarder with the fast axis at 0º	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Quarter-wave (δ = 90º) linear retarder with the fast axis at 0º	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix}$
General linear retarder; retardance δ fast axis at angle β from *x* axis	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 4\beta \sin^2 \delta/2 + \cos^2 \delta/2 & \sin 4\beta \sin^2 \delta/2 & -\sin 2\beta \sin\delta \\ 0 & \sin 4\beta \sin^2 \delta/2 & -\cos 4\beta \sin^2 \delta/2 + \cos^2 \delta/2 & \cos 2\beta \sin\delta \\ 0 & \sin 2\beta \sin\delta & -\cos 2\beta \sin\delta & \cos\delta \end{bmatrix}$	$\begin{bmatrix} e^{i\delta/2}\cos^2\beta + e^{-i\delta/2}\sin^2\beta & i\sin 2\beta \sin\delta/2 \\ i\sin 2\beta\sin\delta/2 & e^{-i\delta/2}\cos^2\beta + e^{i\delta/2}\sin^2\beta \end{bmatrix}$
Right circular retardance δ or rotator with θ = δ/2	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\delta/2 & \sin\delta/2 & 0 \\ 0 & -\sin\delta/2 & \cos\delta/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \cos\delta/2 & \sin\delta/2 \\ -\sin\delta/2 & \cos\delta/2 \end{bmatrix}$
Mirror	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Faraday mirror	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Depolarizer	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	None

C. Jones Calculus

In Jones calculus a two-element vector represents the amplitude and phase of the orthogonal electric field components and the phase information is preserved during calculation. This allows the coherent superposition of waves and is useful for describing the polarization state in systems such as interferometers that combine beams. Since this method is based on coherent waves, however, the Jones vector describes only fully polarized states, and partially or unpolarized states and depolarizing components cannot be represented.

Recalling Eqs. (1a) and (1b), one can write a vector formulation of complex representation for a fully coherent field

$$E = e^{i\omega t}\begin{vmatrix} E_x e^{i\phi_x} \\ E_y e^{i\phi_y} \end{vmatrix}, \tag{29}$$

where the space-dependent term *kz* has been omitted. When the time dependence is also omitted, this vector is known as the full Jones vector. For generality, the Jones vector **J** is often written in a normalized form

$$J = \begin{vmatrix} \cos\beta \\ \sin\beta e^{i\Delta\phi} \end{vmatrix} = \begin{vmatrix} \cos\beta e^{-i\Delta\phi/2} \\ \sin\beta e^{i\Delta\phi/2} \end{vmatrix}, \tag{30}$$

where $\Delta\phi = \phi_y - \phi_x$ and $\tan(\beta) = E_y / E_x$. The Jones vector can also be found from the polarization azimuthal angle α and ellipticity tan(ε) of the polarization ellipse using

$$\begin{aligned} \Delta\phi &= \tan^{-1}\left[\frac{\tan(2\varepsilon)}{\sin(2\alpha)}\right] \\ \beta &= \frac{1}{2}\cos^{-1}\left[\cos(2\varepsilon)\cos(2\alpha)\right]. \end{aligned} \tag{31}$$

The polarization properties of optical components can be represented as 2 x 2 Jones matrices (Table II). The output polarization state is **J' = MJ**, where the Jones matrix **M** may be constructed from a cascade of components $\mathbf{M}_i$ using Eq. (28). In general the matrices are not commutative and require the same ordering as in Mueller calculus, with the rightmost matrix representing the first element the light is incident upon, and so on.

Jones used this calculus to establish three theorems that describe the minimum number of optical elements needed to describe a cascade of many elements at a given wavelength:

1. A system of any number of linear retarders and rotators (circular retarders) can be reduced to a system composed of only one retarder and one rotator.
2. A system of any number of partial polarizers and rotators can be reduced to a system composed of only one partial polarizer and one rotator.
3. A system of any number of retarders, partial polarizers, and rotators can be reduced to a system composed of only two retarders, one partial polarizer, and, at most, one rotator.

The Jones matrices in Table II assume forward propagation. In some cases, for example, with nonreciprocal components such as Faraday rotators, backward propagation must be explicitly described. Furthermore, since fields are used to represent polarization states, the phase shift arising from normal-incidence reflection may be important. For propagation in reciprocal media, the transformation from the forward Jones matrix to the backward case is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{forward} \rightarrow \begin{bmatrix} a & -c \\ -b & d \end{bmatrix}_{backward} \tag{32}$$

For nonreciprocal behavior, such as the Faraday effect, the transformation is instead

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{forward} \rightarrow \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}_{backward} \tag{33}$$

When **M** is composed of a cascade of $\mathbf{M}_i$ that include both reciprocal and nonreciprocal polarization elements, each matrix must be transformed and a new combined matrix calculated. Upon reflection, the light is now backward propagating and the Jones matrix can be transformed to the forward-propagating form (for direct comparison with the input vector, for example) by changing the sign of the second element; in other words,

$$J_{forward} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} J_{backward} \tag{34}$$

The calculi discussed above are applicable to problems when the polarization properties are lumped, that is, the system consists of simple components such as ideal waveplates, rotators, and polarizers, etc. Because the Jones (or Mueller) matrix from a cascade of matrices depends on the order of multiplication, an optical component with intermixed polarization properties cannot generally be represented by the simple multiplication matrices representing each individual property. For example, a component in which both linear retardance (represented by Jones matrix $\mathbf{M}_L$) and circular retardance ($\mathbf{M}_C$) are both distributed throughout the element is not properly represented by either $\mathbf{M}_L\,\mathbf{M}_C$ or $\mathbf{M}_C\,\mathbf{M}_L$.

A method known as the Jones *N*-matrix formulation can be used to find a single Jones matrix that properly describes the distribution of multiple polarization properties. The *N*-matrix represents the desired property over a vanishingly small optical path. The differential *N*-matrices for each desired property can be summed and the combined properties found by an integration along the optical path. Tables of *N*-matrices and algorithms for calculating corresponding Jones matrices can be found in several references.

Jones and Mueller matrices can be related to each other under certain conditions. Jones matrices differing only in absolute phase (in other words, a phase common to both orthogonal eigenpolarizations) can be transformed into a unique Mueller matrix that will have up to seven independent elements, though the phase information will be lost. Thus Mueller matrices for distributed polarization properties can be derived from Jones matrices calculated using *N*-matrices. Conversely, nondepolarizing Mueller matrices [which satisfy the condition $\mathrm{Tr}(\mathbf{M}\mathbf{M}^T) = 4m_{00}$, where $\mathbf{M}^T$ is the transpose of **M**] can be transformed into a Jones matrix.

D. Poincaré Sphere

The Poincaré sphere provides a visual method for representing polarization states and calculating the effects of polarizing components. Each state of polarization is represented by a unique point on the sphere defined by its azimuthal angle α, the ellipticity tan |ε|, and the handedness. Orthogonal polarizations occupy points at opposite ends of a sphere diameter. Propagation through retarders is represented by a sphere rotation that translates the polarization state from an initial point to a final polarization.

Figure 10 shows a Poincaré sphere with several polarizations labeled. Point *x* represents linear polarization along the *x* axis and point *y* represents *y*-polarized light. Right-circular polarization (tan ε = 1) lies at the north pole, and all polarizations above the equator are right-elliptical. Similarly, the south pole represents left-circular polarization (tan ε = -1), and states below the equator are left-elliptically polarized. (In many texts the locations of the circular states are reversed; while a source of confusion, this change is valid so long as other conventions are observed.)

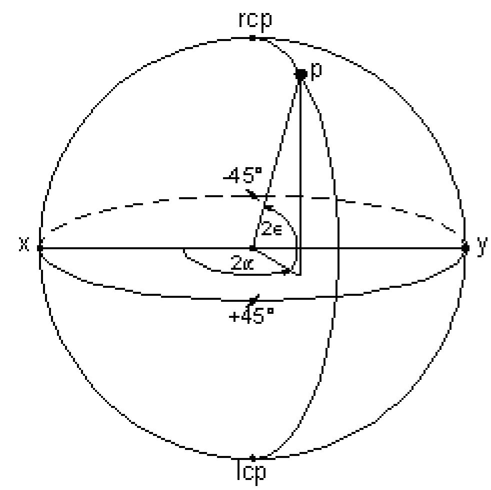

Figure 10 The Poincaré sphere. The polarization represented by point *p* is located using the azimuthal angle α (in the equatorial plane measured from point *x*) and the ellipticity angle ε (a meridional angle measured from the equator toward the north pole). Linear polarization along the *x* axis is located at point *x*, linear polarization along the *y* axis is represented by point *y*, and rcp and lcp denote right- and left-circularly polarized states, repesctively. The origin represents unpolarized light.

In Fig. 10, a general polarization state with azimuthal angle α and ellipticity angle ε is represented by the point *p* with longitude 2α and latitude 2ε. Linear polarizations have zero ellipticity (tan |ε| = 0) and are located along the equator. A linear polarization with azimuthal angle α from the *x* axis is located at a longitudinal angle 2α along the equator from point *x*. Polarization states that lie upon a circle parallel to the equator have the same ellipticity but different orientations. Polarizations at opposite diameters have the same ellipticity, perpendicular azimuthal angles, and opposite handedness.

The Poincaré sphere can also be used to show the effect of a retarder on an incident polarization state. A retarder oriented with a fast axis at α and an ellipticity and handedness given by tan ε can be represented by a point *R* on the sphere located at angles 2α and 2ε. For a given input polarization represented by point *p*, a circle centered at point *R* that includes point *p* is the locus of the output polarization states possible for all retardance magnitudes. A specific retardance magnitude δ is represented by a clockwise arc of angle δ along the circle from the point *p*. The endpoint of this arc represents the polarization state output from the retarder.

Consider x-polarized light incident on a quarter-wave linear retarder oriented with its fast axis at +45° from horizontal; using Jones calculus, we find that right circular polarization should exit the waveplate. To show this graphically using the Poincaré sphere, we locate the point +45°, which represents the retarder orientation. The initial polarization is at point *x*; for a retardance δ = 90°, we trace a clockwise arc centered at the point +45° that subtends 90° from point *x*. This arc ends at the north pole, so the resulting output is right-circular polarization. If the retardance was δ = 180°, the arc would subtend 180°, and the output light would be *y*-polarized. Similarly, left-circular polarization results if δ = 270° (or if δ = 90° and the fast axis is oriented at -45°). The evolution of the polarization through additional components can be traced by locating each retarder's representation on the sphere, defining a circle centered by this point and the polarization output from the previous retarder, and tracing a new arc through an angle equal to the retardance.

Comparing the Poincaré sphere definitions to Eq. (25) shows that for normalized Stokes vectors ($s_0 = 1$), each vector element corresponds to a point along Cartesian axes centered at the sphere's origin. Stokes element s_1 (= cos2ε cos2α) falls along the axis between *x*- and *y*-polarized; $s_1 = 1$ corresponds to point *x* and $s_1 = -1$ corresponds to point *y*. Values of s_2 (= cos2ε cos2α) correspond to points along the diameter connecting the ±45° linear polarization points; $s_2 = -1$ corresponds to the -45° point. Element s_3 (= sin2ε) is along the axis between the north and south poles. These projections on the Poincaré sphere can be equivalently represented by rewriting Eq. (25) and normalizing to obtain

$$\begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \cos(2\varepsilon)\cos(2\alpha) \\ \cos(2\varepsilon)\sin(2\alpha) \\ \sin(2\varepsilon) \end{bmatrix}. \tag{35}$$

Any fully polarized state on the surface of the sphere can be found using these Cartesian coordinates. Partially polarized states will map to a point within the sphere, and unpolarized light is represented by the origin.

V. POLARIMETRY

Polarimetry is the measurement of a light wave's polarization state, or the characterization of an optical component's or material's polarization properties. Complete polarimeters measure the full Stokes vector of an optical beam or measure the full Mueller matrix of a sample. In many cases, however, some characteristics can be neglected and the measurement of all Stokes or Mueller elements is not necessary. Incomplete polarimeters measure a subset of characteristics and may be used when simplifying assumptions about the light wave (for example, that the degree of polarization is 1) or sample (for example, a retarder exhibits negligible diattenuation or depolarization) are appropriate. In this section, a few techniques are briefly described for illustration.

A. Light Measurement

A *polarization analyzer*, or light-measuring polarimeter, characterizes the polarization properties of an optical beam. An optical beam's Stokes vector can be completely characterized by measuring the six optical powers listed in Eq. (25) using ideal polarizers. When the optical beam's properties are time invariant, the measurements can be performed sequentially by measuring the power transmitted through four orientations of a linear polarizer and two additional measurements with a quarter-wave retarder (oriented ±45° with respect to the polarizers axis) placed before the polarizer. In practice, as few as four measurements are required since $s_2 = 2P_{+45°} - s_0$ and $s_3 = 2P_{\mathrm{rcp}} - s_0$.

The Stokes vector can alternatively be measured with a single circular polarizer made by combining a quarterwave plate (with the fast axis at 45°) with a linear polarizer. P_{rcp} is measured when the retarder side faces the-source. Flipping so that the retarder faces the detector allows measurement of $P_{0°}$, $P_{90°}$, and $P_{\pm 45°}$.

The Stokes vector elements can be measured simultaneously with multiple detector configurations. *Division of amplitude* polarimeters use beamsplitters to direct fractions of the power to appropriate polarization analyzers. Using *division of wavefront* polarization analyzers, we assume that the polarization is uniform over the optical beam and subdivisions of the beam's cross section are directed to appropriate analyzers.

Incomplete light-measuring polarimeters are useful when the light is fully polarized (degree of polarization approaches 1). For example, the ellipticity magnitude and azimuth can be found by analyzing the light with a rotating linear polarizer and measuring the minimum and maximum transmitted powers. Linear polarization yields a detected signal with maximum modulation, while minimum modulation occurs for circular polarization. The handedness of the ellipticity can be found using a right(or left-) circular polarizer.

These methods are photometric, and accurate optical power measurements are required to determine the light characteristics. Before the availability of photodetectors, null methods that rely on adjusting system settings until light transmission is minimized were developed, and these are still useful today. For example, an incomplete polarimetric null system for analyzing polarized light uses a calibrated Babinet-Soleil compensator followed by a linear polarizer. Adjusting both the retardance δ and angle θ between the fast axis and polarizes axis until the transmitted power is zero yields the ellipticity angle ω (using $\sin 2\omega = \sin 2\theta \sin\delta$) and azimuthal angle α (using $\tan \alpha = \tan 2\theta \cos\delta$). When unpolarized light is present, the minimum transmission is not zero, and photometric measurement of this power can be used to obtain the degree of polarization.

B. Sample Measurement

A *polarization generator* is used to illuminate the sample with known states of polarization to measure the sample's polarization properties. The reflected or transmitted light is then characterized by a polarization analyzer, and the properties of the sample are inferred from changes between the input and output states.

A common configuration for determining the Mueller matrix combines a fixed linear polarizer and a rotating quarter-wave retarder for polarization generation with a rotating quarter-wave retarder followed by a fixed linear polarizer for analysis. Power is measured as the two retarders are rotated at different rates (one rotates five times faster than the other) and the Mueller matrix elements are found from Fourier analysis of the resulting time series. Alternatively, measurements can be taken at 16 (or more) specific combinations of generator and analyzer states, typically with the polarizers fixed and at specified retarder orientations. Data reduction techniques have been developed for efficiently determining the Mueller matrix from such measurements. Several methods include measurements at additional generator/analyzer combinations to overdetermine the matrix; least-squares techniques are then applied to reduce the influence of nonideal system components and decrease measurement error.

Because of the simplicity and reduction of variables, incomplete polarizers can often provide a more accurate measurement of a single polarization property when other characteristics are negligible. For example, there are many methods for measuring linear retardance in samples with negligible circular retardance, diattenuation, and depolarization, and these are often applicable to measurements of high-quality waveplates.

In a rotating analyzer system, the retarder is placed between two linear polarizers so that the input polarization bisects the retarder's birefringence axes. Linear retardance is calculated from measurements of the transmitted power when the analyzer is parallel ($P_{0°}$) and perpendicular ($P_{90°}$) to the input polarizer using $|\delta| = \cos^{-1} [(P_{0°} - P_{90°})/(P_{0°} + P_{90°})]$. In this measurement, retardance is limited to two quadrants (for example, measurements of 90° and 270° = -90° retarders will both yield δ = 90°). If a biasing quarter-wave retarder is placed between the input polarizer and retarder and both retarders are aligned with the fast axis at 45°, retardance in quadrants 1 and 4 ($|\delta| \leq 90°$) can be measured from $\delta = \sin^{-1}[(P_{90°} - P_{0°})/(P_{90°} + P_{0°})]$. There are several null methods, including those that use a variable compensator aligned with the retarder at 45° between crossed polarizers (retardance is measured by adjusting a calibrated compensator until no light is detected) or that use a fixed quarter-wave-biasing retarder and rotate the polarizer and/or analyzer until a null is obtained.

Ellipsometry is a related technique that allows the measurement of isotropic optical properties of surfaces and thin films from the polarization change induced upon reflection. Linearly polarized light is directed toward the sample at known incidence angles, and the reflected light is analyzed to determine its polarization ellipse.

Application of electromagnetic models to the configuration (for example, via Fresnel equations) allows one to calculate the refractive index, extinction coefficient, and film thickness from the measured ellipticities. Ellipsometry can be extended to other configurations using various incident polarizations and polarization analyzers to measure polarimetric quantities, blurring any distinction between ellipsometry and polarimetry.

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