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State permutations from manipulation of near level-crossings.

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We discuss some systematic methods for implementing state manipulations in systems formally similar to chains of a few spins with nearest-neighbor interactions, arranged such that there are large and small scales of coupling links. States are permuted by means of bias potentials applied to a few selected sites. Equivalent operations can be performed in some atoms-in-a-cavity models. We note in passing that the establishment of "decoherence-free subspaces" in these models does not require the dissipative processes invoked by other authors.

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Two central concerns of quantum information theory are representation of data through occupancy of quantum states and the manipulation of these data. In this paper we discuss a way of implementing permutations in state occupancies by the application of controlled time-varying potentials to selected sites in the underlying system, beginning with an example of five spins in a line, and continuing with some more complex examples. The dynamics also has two very different scales of inter-site coupling: strong (S) and weak (W).

In the first example we take the five spins to be numbered consecutively from left to right, with pure exchange interactions built of the operators between adjacent spins i and j,

$$h^{(i,j)} = \sigma_+^{(i)}\,\sigma_-^{(j)} + \sigma_-^{(i)}\,\sigma_+^{(j)}. \tag{1}$$

We choose the Hamiltonian

$$H = g_1 h^{(1,2)} + \lambda h^{(2,3)} + \lambda h^{(3,4)} + g_2 h^{(4,5)} + f(t)[\sigma_z^{(3)} - 1], \tag{2}$$

where we have added a single time-dependent term involving the operator of the middle spin $\vec{\sigma}^{(3)}$. We refer to the function $f(t)$ as the bias on site #3. Since this Hamiltonian commutes with the operator $\sum_i \sigma_z^{(i)}$, we can operate within the set of five states with four of the spins up and one spin down. The S-W structure comes from taking $\lambda \ll g_1, g_2$, and $g_1 \neq g_2$. When $f(t) = 0$, this choice has the effect of creating eigenstates of H that are very nearly the following,

$$\begin{aligned}
\psi_1 &= (\uparrow\downarrow + \downarrow\uparrow)(\uparrow\uparrow\uparrow)/\sqrt{2},\\
\psi_2 &= (\uparrow\downarrow - \downarrow\uparrow)(\uparrow\uparrow\uparrow)/\sqrt{2},\\
\psi_3 &= (\uparrow\uparrow\downarrow\uparrow\uparrow),\\
\psi_4 &= (\uparrow\uparrow\uparrow)(\uparrow\downarrow + \downarrow\uparrow)/\sqrt{2},\\
\psi_5 &= (\uparrow\uparrow\uparrow)(\uparrow\downarrow - \downarrow\uparrow)/\sqrt{2}.
\end{aligned} \tag{3}$$

We understand these combinations qualitatively by noting that they would be the exact $f = 0$ eigenstates if the small coupling λ were set to zero, and that, treating λ as a perturbation, the link between the pairs of states (#1,#2) and (#4,#5) is of order λ^2.

Now we ask the question: Of the 120 permutations on this set of five states, how many can we implement by applying a small series of simple pulses in $f(t)$? In forming these pulses we shall tailor $f(t)$ to the permutation that is sought, but we shall insist that $f(t)$ begins at and ends at the value zero. By a permutation we mean just the reshuffling of the states, modulo phase. This demand effectively rules out setting relative phases, which would require fine-tuning in any system that is to be considered over a period of time. The answer to the question is "all 120".

The method uses adiabatic avoided-level-crossing dynamics, with an additional feature that can be embodied in the above Hamiltonian, namely, that the bias on the weakly coupled site can be changed suddenly, without affecting the state of the system appreciably, at all times when the system is far from any (near) level-crossing. As an example, we choose the parameters $g_1 = 30$, $g_2 = 60$, $\lambda = 1$, and begin by defining two bias operations $f(t)$,

$$\begin{aligned}
f_a(t) = (t - t_0)\;\theta[t - t_0]\theta[t_0 + \tau - t] \rightarrow U_a(t_0),\\
f_b(t) = -f_a(t) \rightarrow U_b(t_0).
\end{aligned} \tag{4}$$

The signal begins at $t = t_0$, grows linearly until $t = t_0 + \tau$ when it is switched off abruptly. We take $\tau = 20$ in the applications that follow. Any bias $f_\alpha(t)$ defines a transformation,

$$U_\alpha(t_0) = T[\exp(-i \int_{t_0}^{t_0+\tau} dt' H(t'))]. \tag{5}$$

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We will show [1] that the operation U_a effects the permutation $\langle 13\rangle$, while U_b effects $\langle 23\rangle$. The turn-off of the bias at $t = t_0 + \tau$ leaves the state at time $t = t_0 + \tau$ evolving with the Hamiltonian of Eq. (2) with $f = 0$, so that we are prepared to apply another signal to get a compound of the permutations. Thus we can write,

$$U_{1\Leftrightarrow 2} \equiv [U_a(2\tau)][U_b(\tau)][U_a(0)], \tag{6}$$

where the time arguments emphasize that the transformations are to be executed serially over a total time of 3τ. Using the composition property of permutations $\langle 13\rangle\langle 23\rangle\langle 13\rangle = \langle 12\rangle$, we see that $U_{1\Leftrightarrow 2}$ just interchanges states #1 and #2, leaving #3 alone. The basis for anticipating the actions of both the individual and compound transformations can be shown in an averted level-crossing diagram based on Eq. (2) with the bias function given by,

$$f(t) = f_a(t) + f_b(t-\tau) + f_a(t-2\tau). \tag{7}$$

Figure 1 shows plots of the energy levels for the first three states as they evolve under the above successive changes of the bias.

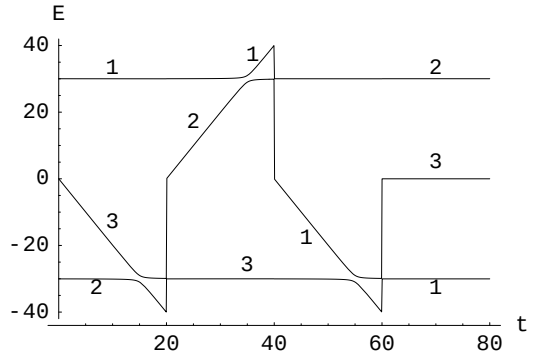

FIG. 1: Level diagrams for the three states that are appreciably affected by the pulse chain of Eq. (7). The numbers on the curves indicate possible initial states, enumerated as in Eq. (3). In the adiabatic limit, the system, beginning in one of these states, follows the path labeled by the state's number. States #4 and #5 remain virtually unmixed and at their original energies.

To show that the states really follow these paths, we construct the matrix $U(t) = T[\exp(-i\int_0^t dt' H(t'))]$ by directly solving the Schrödinger equation with the time-dependent bias of Eq. (7). In Figs. 2 and 3, we plot $|\langle 1,2,3|U(t)|1\rangle|^2$ and $|\langle 1,2,3|U(t)|2\rangle|^2$ against time in this solution, giving the expected behavior, $|U(3\tau)|^2 = |U_{1\Leftrightarrow 2}|^2$ with negligible contamination either from non-adiabaticity in the region of small level separation, or from transitions induced in any of the three sudden changes. The plots show clearly the effects of each of the constituent transitions in turn, giving rise to the factorization indicated in Eq. (6).

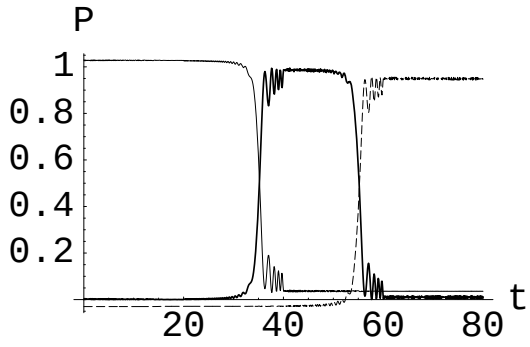

FIG. 2: The evolution of probabilities for the Hamiltonian of Eq. (2) with $f(t)$ from Eq. (7), and the initial state #1. The light solid curve is the probability of the system in state #1 (displaced slightly upward for clarity), the heavy solid curve is for #3, and the dashed curve is for #2 (displaced slightly downward).

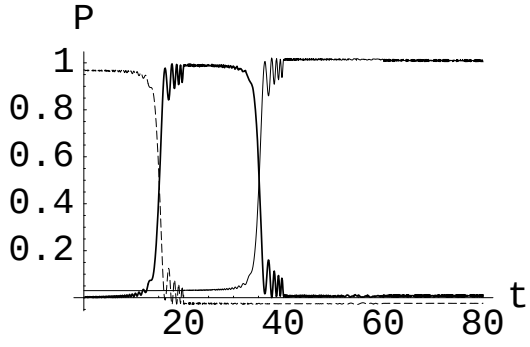

FIG. 3: Same as Fig. 2, but with the initial state taken as #2. Together, Figs. 2 and 3 show the interchange operation effected by the particular pulses of (7).

To implement the transformations involving states #4 and #5, we introduce four more bias functions,

$$\begin{aligned} f_c(t) &= 2(\tau - t)\theta(\tau - t)\theta(t) \rightarrow U_c \rightarrow \langle 352\rangle, \\ f_d(t) &= -f_c(t) \rightarrow U_d \rightarrow \langle 341\rangle, \\ f_e(t) &= -2(\tau - t/2)\theta(\tau - t)\theta(t) \rightarrow U_e \rightarrow \langle 34\rangle, \\ f_f(t) &= -f_e(t) \rightarrow U_f \rightarrow \langle 35\rangle, \end{aligned} \tag{8}$$

where, for simplicity, we have chosen $t_0 = 0$. By straightforward multiplication of the operations in Eqs. (4) and (8), we obtain $U_dU_a \rightarrow \langle 14\rangle$, $U_cU_b \rightarrow \langle 25\rangle$, $U_aU_fU_a \rightarrow \langle 15\rangle$, $U_bU_eU_b \rightarrow \langle 24\rangle$, $U_eU_fU_e \rightarrow \langle 45\rangle$. These five composite operations, together with $U_aU_bU_a \rightarrow \langle 12\rangle$ and the four single pulse operations U_a, U_b, U_e, U_f, give all of the simple interchanges. All permutations can be built from these interchanges, although in most cases it would be more efficient to draw on the cycle-of-three permutation operators U_c and U_d as well, or on further primary pulse

shape variants that directly embody other operators.

Now we consider longer chains with nearest neighbor couplings,

$$H_0 = \sum_{i=j+1} g_{i,j} h^{(i,j)} - E_0, \tag{9}$$

where $g_{i,j}$ are coupling coefficients, and E_0 is the energy that gives $\langle H_0 \rangle = 0$ for the case of all spins up. We again adopt two scales of coupling strength: S and W. As an example, we consider the pattern of couplings $\{g_{1,2},\ g_{2,3},\ g_{3,4}, \ldots\}$ to be {S, W, W, S, W, W, S, W, W, ...}. Under this scheme, the states #3,#6,#9... are only weakly coupled to their neighbors.

As before, we take only states with one spin down and the remainder up. With the W coupling constants turned off, the eigenvalues of the states in which the down spin occupies one of the large blocks, i.e., {(#1,#2), (#4,#5), (#7,#8) ...}, come in pairs of $\pm g_{i,i+1}$. We assume that the S couplings are sufficiently irregular so that the differences in energies between nearest strong blocks, e.g., $g_{1,2} - g_{4,5}$, are still of the strong (S) scale. When we add the W couplings, the eigenvectors will remain almost localized, as can be seen from a perturbation expansion. In general, small energy differences between two nonadjacent blocks are not a concern, since as we move across n additional weak links we can tolerate energy denominators that are smaller by a factor of $(g_W/g_S)^{2n} \ll 1$.

Thus for most cases the eigenvectors can be arranged in a list $\{(\xi_1,\uparrow,\uparrow,\uparrow,...),\ (\eta_1,\uparrow,\uparrow,\uparrow,...),\ (\uparrow,\uparrow,\downarrow,\uparrow,\uparrow,...),\ (\uparrow,\uparrow,\uparrow,\xi_2,\uparrow,...),\ (\uparrow,\uparrow,\uparrow,\eta_2,\uparrow,...),\ (\uparrow,\uparrow,\uparrow,\uparrow,\uparrow,\downarrow,\uparrow,...),\ ...\}$. Here, the ξ's and η's stand for the symmetric and anti-symmetric two-component eigenstates of the strong blocks with one spin up and the other down. If the system begins with all of its amplitude in one block, the amplitude will stay almost within that block under the evolution governed by H_0.

Now, generalizing the earlier five-site example, we can ask whether biases placed at every third site, #3, #6, #9, ..., which are weakly coupled both to the right and to the left, have the capability of moving information around the whole system. We have looked at an eight-site example, with independently manipulable biases at sites #3 and #6. In complete analogy with the five-site case, we find it is easy, for example, to move one of the (#1,#2) eigenstates successively to the blocks (#3), (#4,#5), (#6), (#7,#8), again by numerically solving the Schrödinger equation for the system.

A summary and a generalization of these outcomes:

(1) If we have a chain of blocks of states connected, one to another, through an intermediary state that is weakly coupled to both blocks, the dynamics within the blocks will be nearly self-contained. That is, if the probability is localized within a particular block at a particular time, in the form of any superposition of the eigenstates belonging to that block, then the probability will remain in that block. Likewise, a block of several states in which the mutual couplings and the diagonal elements are weak will, in general, not admix appreciably with strongly coupled states to its left or right.

(2) By putting controllable biases on a weak connection site to bring the energy of an associated level to (nearly) coincide with a level in an adjacent strong block, shifts of probability from block to block can be implemented in an orderly and complete fashion.

We turn to another application of these general conclusions, appropriating the model of Ref. [2], a paper entitled "Quantum computing using dissipation to remain in a decoherence-free subspace". These authors consider a hypothetical system in which two identical three-level atoms, with levels $\{0,1,2\}$, are placed in a cavity tuned exactly to the $1-2$ level spacing, with the atoms separately addressible by weak laser fields that drive Rabi oscillations. The cavity mode, if excited, is also allowed to escape through conversion to photons at a partially transmitting wall. This is the only decohering process. The connection scheme is depicted in Fig. 4.

$$\begin{array}{ccccccccc} & & & & & & |2,2\rangle_0 & \Longleftrightarrow & |s\rangle_1 \rightarrow D \\ & & & & & & \updownarrow & & \\ D \leftarrow |s\rangle_1 & \Longleftrightarrow & |2,2\rangle_0 & \leftrightarrow & |a\rangle_0 & \leftrightarrow & |1,0\rangle_0 \leftrightarrow |2,0\rangle_0 & \Longleftrightarrow & |1,0\rangle_1 \rightarrow D \\ & & & & \updownarrow & & \updownarrow & & \\ & & & & |1,1\rangle_0 & & |s\rangle_0 & \Longleftrightarrow & |1,1\rangle_1 \rightarrow D \end{array}$$

FIG. 4: The state connection scheme in the model of Ref. [2]. In a notation close to that of Ref. [2], the states are labeled as $|\text{atom}\,\#1, \text{atom}\,\#2\rangle_{\text{cav}}$ where "cav" is the excitation of the cavity mode. The states denoted "s" and "a" (for the atomic part) are $|\text{s,a}\rangle = (|1,2\rangle \pm |2,1\rangle)/\sqrt{2}$. The connections are those variously induced by the three laser couplings and the cavity mode, as explained in the text.

The excitation of the cavity mode from the states in the chain that are coupled thereto are the S links, denoted by fat double arrows. The thin double arrows are W links, and the single arrows leading to D represent the cavity mode leakage to the outside. In Ref. [2] this leakage is stated to be the dissipation that creates a decoherence-free subspace (DFS). All W links come from the laser interactions, which are three-fold: on atom #2, an exactly tuned $0 \leftrightarrow 2$ signal and an exactly tuned $1 \leftrightarrow 2$ signal; on atom #1, just an exactly tuned $1 \leftrightarrow 2$ signal, 180° out of phase with the corresponding signal on the other atom.

The structure shown in Fig. 4 has a general property in common with the models discussed above; there is an S block within which all connections are weak,

$$|1,1\rangle_0 \leftrightarrow |a\rangle_0 \leftrightarrow |1,0\rangle_0, \tag{10}$$

characterized in Ref. [2] as a DFS. Every other state in the picture has at least one strong connection. Our observation is that if the system begins in this subspace,

then the dynamics remains almost entirely within the subspace, just from the S-W configuration of the couplings. The cavity leakage, the only dissipation in this system, is irrelevant to this conclusion. We have verified this assertion with a direct calculation using the full model that includes all of the states shown in Fig. 4.

Thus, the D's can be deleted from the ends of the chain in Fig. 4 without producing an appreciable effect on the evolution of the subspace given in Eq. (10). Note, however, that even though the cavity mode is not excited in the operative subspace, we cannot turn off the coupling to the cavity (the double arrows in Fig. 4) without destroying the isolation of the subspace. Without the strong cavity coupling, the laser pulses would excite the states $|2,2\rangle_0$ and $|2,0\rangle_0$; in the presence of the cavity coupling these states become isolated in the respective left and right S blocks in Fig. 4.[3]

We now write an effective Hamiltonian operating within the block of Eq. (10), utilizing the laser coupling strengths and phases of Ref. [2], but allowing the laser frequencies to be detuned by a small amount,

$$H_{\rm eff}=\frac{\Omega}{2}\left[e^{i\Delta_A t}|1,0\rangle_0\langle a|_0+e^{i\Delta_B t}|a\rangle_0\langle 1,1|_0\right]+\text{h.c.}, \quad (11)$$

where $\Delta_{A,B}$ are the respective detunings of the $0\leftrightarrow 2$ and $1\leftrightarrow 2$ lasers. Transition probabilities in the indicated basis will be unaffected by the transformation (acting in the atomic space only), $\Psi'(t)=\exp(-i\Lambda t)\Psi(t)$, where

$$\begin{aligned}2\Lambda \;=\; &(\Delta_A+\Delta_B)|1,0\rangle\langle 1,0| \;+\; (\Delta_B-\Delta_A)|a\rangle\langle a| \\ &-(\Delta_A+\Delta_B)|1,1\rangle\langle 1,1|, \end{aligned} \quad (12)$$

giving the new effective Hamiltonian,

$$H'_{\rm eff}=\Lambda+\frac{\Omega}{2}[(|1,0\rangle_0\langle a|_0)+(|a\rangle_0\langle 1,1|_0)]+\text{h.c.} \quad (13)$$

This is exactly the first example of the present paper, with the five states reduced to three by taking only the symmetric states in place of the (#1,#2) and (#4,#5) complexes. Thus by adiabatically changing the combination $\Delta_A-\Delta_B$, while keeping $\Delta_A+\Delta_B$ constant, we can effect the interchange of the states $|1,1\rangle_0$ and $|1,0\rangle_0$ exactly as before. We begin with a steady situation in which the detuning parameters $\Delta_{A,B}$ are substantially greater than Ω, and the initial state is (very stably) either $|1,1\rangle_0$ or $|1,0\rangle_0$. Then the detuning is manipulated, using simultaneous slow changes of both laser frequencies in order to interchange these two states, in the three-step process parallel to that described in Eq. (6) and shown graphically in Fig. 3. By contrast, Ref. [2] uses the perfectly tuned case $\Delta_{A,B}=0$, and takes $\Omega=0$ until a pulse turn-on time. The pulse is then turned off at exactly the time for the interchange $|1,0\rangle_0\leftrightarrow|1,1\rangle_0$ to have occurred under the influence of precession alone. In either method, the transformation represents a c-not gate, the states $|0,1\rangle_0$ and $|0,0\rangle_0$ being frozen due to the S-W effect. Note that our implementation does not require accurate timing of applied fields.

A number of models similar to that of Ref. [2] can be found in the recent literature [4, 5, 6, 7], with dissipation or continuous measurement cited as the key element in creating decoherence-free subspaces. However, since the S-W paradigm provides both the intuitive and the computational basis for the conclusions, it is unnecessary to add either theoretical interpretations in terms of measurement or gratuitous couplings to the model like the D links of Fig. 4. In particular, we note that the outcomes of the first three of the four models described in Ref. [5] are directly explicable by S-W considerations.

The S-W classification, really an elementary observation about eigenvalues, is not the only way in which subspaces can become mutually isolated. Indeed Ref. [8] and many subsequent works have discussed cases in which rapid incoherent scattering or frequently interrupted external field interactions [9, 10] can freeze a system in a single state or subspace of states. However, none of these considerations are relevant to the models presented in the present paper.

To summarize, in systems with appropriate arrangements of strong and weak couplings, variable potentials applied to a relatively small number of sites can efficiently effect state permutations for spin chains with pure-exchange coupling. Using the same approach, we found a new way to implement a c-not gate in an atoms-in-a-cavity model discussed by previous authors.

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[1] Permutations are denoted by $\langle\cdots\rangle$, where, e.g., $\langle 123\rangle\langle 4\rangle\langle 5\rangle$ implies $1\to 2$, $2\to 3$, $3\to 1$.
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