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CHARACTERISTIC RELATIONS OF COMPRESSIBLE FLOW WITH THREE INDEPENDENT VARIABLES

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Prepared By

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ABSTRACT

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The theory of compressible, inviscid fluid flow was developed in order to establish a basis to evaluate flow fields. The hyperbolic, second order, non-linear, partial differential characteristics of the gas dynamic equations were solved using a co-ordinate transformation to a characteristic surface and a numerical finite difference technique.

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SUMMARY

The compressible flow equations are derived in detail for the inviscid flow of a homogeneous fluid. The various assumptions utilized in the derivations are listed below, except where they involve a specific circumstance of the derivation. In the latter case, the assumption(s) will be designated at the point where they are utilized.

1. Homogeneous Fluid
2. Inviscid Fluid
3. Uniform Flow State of Approach
4. No External Heat Flux
5. Equilibrium Gas States of Single Phase
6. No External Forces Other Than Those Inherent in the Flow

Where it has been applicable, the final equations have been reduced from the three independent variables to two and then to one independent variable so as to show the influence of the co-ordinate parameters within the flow.

The compressible flow relations have been transformed to characteristic surfaces in a form suitable for solution by numerical procedures. Several problems may be solved by use of the three independent variable method of characteristics; however, the author has attempted to retain the generality of the equations while presenting the solution of an ideal flow field to illustrate the adaptability of this method.

The characteristic relations are derived, the geometrical relationships are resolved and the velocity equations are solved for an ideal flow field utilizing the three independent variable method of characteristics.

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LIST OF SYMBOLS

English Symbols:

A	-	Sound speed
a	-	Co-ordinate in characteristic surface
b	-	Co-ordinate in characteristic surface
C	-	Curve defined
C_p	-	Specific Heat at constant pressure
C_v	-	Specific Heat at constant volume
F	-	Force vector
h	-	Enthalpy
i, j, k	-	Unit vectors in x, y, z directions, respectively
k	-	Heat flux coefficient
ℓ	-	Direction cosine defined by its subscript
n	-	Normal direction or co-ordinate
p	-	Hydrostatic pressure
q	-	Velocity vector
r	-	r direction in spherical co-ordinates
R	-	Region or point as defined by the appropriate figure
R_u	-	Universal gas constant
s	-	Entropy
S	-	Length of line element of flow tetrahedron

English Symbols (Continued):

t	-	Time
T	-	Temperature
$\bar{u}$	-	Internal energy
u	-	Velocity component in x-direction
v	-	Velocity component in y-direction
w	-	Velocity component in z-direction
x	-	Cartesian co-ordinate
y	-	Cartesian co-ordinate
z	-	Cartesian co-ordinate

Greek Symbols:

α	-	As defined
Γ	-	Circulation
δ	-	Angle between two diametral planes of the characteristic cone
ξ	-	x component of the vorticity vector
η	-	y component of the vorticity vector
ζ	-	z component of the vorticity vector
θ	-	Angle between z axis and vector r in spherical co-ordinates
λ	-	Angular displacement of line ε with respect to the velocity vector
ε	-	See Figure 2

Greek Symbols (Continued):

μ	-	Mach angle
ρ	-	Density
σ	-	Surface
ϕ	-	Potential function where $\frac{\partial \phi}{\partial x} = u$
ψ	-	Angle between x-axis and r in the plane xy in spherical co-ordinates
ω	-	Vorticity vector
Ω	-	Vorticity vector

Special Symbols:

Differentiation

$\frac{d}{da}$ - Ordinary derivative with respect to a

$\frac{\partial}{\partial a}$ - Partial derivative with respect to a

$\frac{D}{Da}$ - Substantial derivative with respect to a, defined as

$$\frac{D}{Da} = \frac{\partial}{\partial x}\frac{dx}{da} + \frac{\partial}{\partial y}\frac{dy}{da} + \frac{\partial}{\partial z}\frac{dz}{da} + \frac{\partial}{\partial t}\frac{dt}{da}$$

∇ - $i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

Integration

$\oint_C$ Line integral around curve C

$\iint_\sigma$ Double integral over surface sigma

$\iiint_R$ Triple integral over volume R

INTRODUCTION

Several areas of application have been solved by the method of characteristics; however, relatively little has been published on three independent variable solutions. Axisymmetric studies have been made by various authors and although these are limited in scope, they comprise an important step in the solution of compressible flow problems. Coburn and Dolph[1], Thornhill[2], and Holt[3] have established methods pertaining to the solution of three independent variable problems. Tsung[4] and Ferri (several published papers and specifically the contributions contained in volume VI of the Princeton Series entitled General Theory of High Speed Aerodynamics) have shown certain problem solutions by the three independent variable methods.

The author has attempted to combine these solutions with the basic equations to outline the procedures necessary to solve several areas of flow field problems. Several areas of interest may be attacked by this method where other methods of less scope are inadequate. Some of these areas are nozzle flow fields of clustered jets, arbitrarily oriented re-entry body flow fields, asymmetric flow fields about various body shapes, unsteady nozzle flow fields, unsteady flow about bodies, and many other areas which are too numerous to mention.

The subsequent discussion outlines an approach to a solution of one of these problem areas. The reader may conceivably see the basic steps to the solution of some of the other problem areas with certain modifications of the following analysis.

DERIVATION OF THE METHOD OF CHARACTERISTICS IN THREE INDEPENDENT VARIABLES FOR THE INVISCID FLOW OF A HOMOGENEOUS FLUID

General Equations:

Consider a control volume fixed in space with a space-fixed surface boundary, σ. The control volume may be of arbitrary shape, but it must be a definite shape, i.e., invariant. (Figure 1).

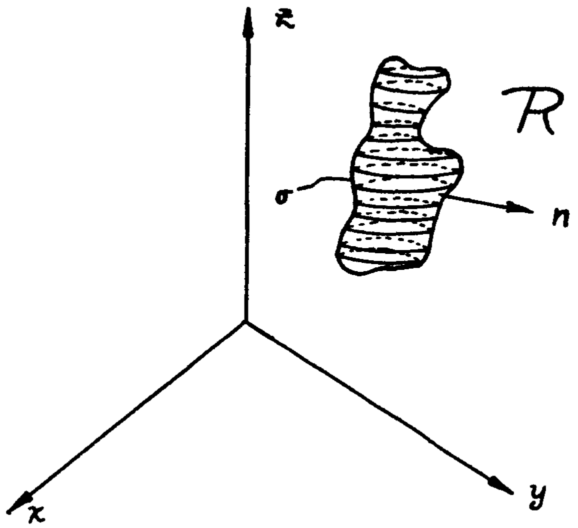

Figure 1: Region R of an arbitrary, but definite shape with surface, σ, and outward drawn normal, $\vec{n}$.

Let $d\sigma$ be a surface element of σ with outward drawn normal, $\vec{n}$. Let $\vec{q}_n$ be the velocity component along the normal $\vec{n}$ which may be defined as,

$$\vec{q}_n = u\cos(n,x) + v\cos(n,y) + w\cos(n,z) \tag{1}$$

where u, v, and w are the velocity components in the x, y, and z directions respectively. The angles (n, x), (n, y), and (n, z) are the angles between the normal and Cartesian axes measured in a counter-clockwise direction.

The mass flux into the spatial region, R, must be equal to the mass flux out of spatial region, R, plus that mass which remains within the region. The mass flux per unit volume is:

$$\rho q_n \, d\sigma$$

where ρ - density

q_n - velocity normal to surface

Since the surface is invariant with time, we may write the continuity equation as:

$$\frac{d}{dt} \iiint_R \rho \, dx \, dy \, dz + \iint_\sigma \rho q_n \, d\sigma \equiv \iiint_R \frac{d\rho}{dt} \, dx \, dy \, dz + \iint_\sigma \rho q_n \, d\sigma = 0 \tag{2}$$

Euler's equations of motion may be derived by use of Newton's law noting that we have momentum conservation which implies three scalar equations in particle mechanics.

Let the force per unit mass be $\vec{F}$ and consider only inviscid flow, then the body force in the x-direction may be defined as:

$$\iiint_R \rho F_x \, dx \, dy \, dz \tag{3}$$

and similarly defined for the y and z directions. The hydrostatic force per unit area, p in the x-direction may be defined as:

$$\iint_\sigma p \cos(n, x) \, d\sigma \tag{4}$$

with corresponding equations in the y and z directions.

Newton's law states that the sum of all forces must be equal to the rate of change in momentum within R. The rate of change of the volume integral of ρu must be included within this change and is indicated in the x-direction by

$$\frac{d}{dt}\iiint_R \rho u \, dx\,dy\,dz \tag{5}$$

and the transport of momentum through the surface, σ, is,

$$\iint_\sigma \rho u q_n \, d\sigma \tag{6}$$

Thus, Euler's equations become (for the x-direction)

$$\frac{d}{dt}\iiint_R \rho u \, dx\,dy\,dz + \iint_\sigma \rho u q_n \, d\sigma = \iiint_R \rho F_x \, dx\,dy\,dz - \iint_\sigma p \cos(n, x) \, d\sigma \tag{7}$$

If we consider that there are no outside body forces, or that they are of such a magnitude that they may be considered negligible, then we may write equation (7) as

$$\frac{d}{dt}\iiint_R \rho u \, dx\,dy\,dz + \iint_\sigma [\rho u q_n + p \cos(n, x)] \, d\sigma = 0 \tag{8}$$

The corresponding y and z directions yield:

$$\frac{d}{dt}\iiint_R \rho v \, dx\,dy\,dz + \iint_\sigma [\rho v q_n + p \cos(n, y)] \, d\sigma = 0 \tag{9}$$

$$\frac{d}{dt}\iiint_R \rho\, w\, dx\,dy\,dz + \iint_\sigma [\rho\, w\, q_n + p \cos(n, z)]\, d\sigma = 0 \tag{10}$$

A third condition of the flow is the energy equation which equates the net effect of body and pressure forces to the rate of change of kinetic energy and internal energy and the energy flux resulting from some outside effect such as heat radiation or conduction.

The total rate of change of kinetic and internal energy within R is

$$\frac{d}{dt}\iiint_R \rho\left(\frac{q^2}{2} + \bar{u}\right) dx\,dy\,dz \tag{11}$$

where $\bar{u}$ is the internal energy of the fluid element.

The total external energy flux, i. e., that through the surface, σ, may be written as

$$\iint_\sigma q_n\, \rho\left(\tfrac{1}{2}q^2 + \bar{u}\right) d\sigma \tag{12}$$

If k is the constant of heat flux, then the total heat flux through σ is:

$$\iint_\sigma k \frac{\partial T}{\partial n}\, d\sigma = \iint_\sigma k \left[\frac{\partial T}{\partial x}\cos(n,x) + \frac{\partial T}{\partial y}\cos(n,y) + \frac{\partial T}{\partial z}\cos(n,z)\right] d\sigma \tag{13}$$

where T is the fluid temperature.

Since we are assuming that the heat flux into region R is positive, the sense of (13) is necessarily positive. This implies that $\frac{\partial T}{\partial n}$ is a negative quantity if heat is added to region R.

Utilizing the pressure term of the momentum equations (8, 9, and 10), we may write the energy equation as:

$$\frac{d}{dt}\iiint_R \rho\left(\tfrac{1}{2}q^2 + \bar{u}\right)dx\,dy\,dz + \iint_\sigma \left(\tfrac{1}{2}q^2 + \bar{u} + \frac{P}{\rho}\right)\rho q_n\,d\sigma$$

$$= \iint_\sigma k\left[\frac{\partial T}{\partial x}\cos(n,x) + \frac{\partial T}{\partial y}\cos(n,y) + \frac{\partial T}{\partial z}\cos(n,z)\right]d\sigma \tag{14}$$

The utilization of integral relations for the derivation of the gas dynamic equations allows the treatment of compression shocks or other finite discontinuities in the fluid stream in the regions where the differential equations break down. Secondly, the integral relations are invariant during a rotation of the co-ordinate system and in Galilei transformations.

The transition from the integral relations to the differential relations requires that the functions be smooth. If we transform the surface integrals to volume integrals by means of the Gauss integral theorem which states that for every function, G,

$$\iint_\sigma G q_n\,d\sigma \equiv \iint_\sigma \left[Gu\cos(n,x) + Gv\cos(n,y) + Gw\cos(n,z)\right]d\sigma$$

$$= \iiint_R \left[\frac{\partial(Gu)}{\partial x} + \frac{\partial(Gv)}{\partial y} + \frac{\partial(Gw)}{\partial z}\right]dx\,dy\,dz$$

$$= \iiint_R (G\vec{q})\,dx\,dy\,dz \tag{15}$$

If $G = \rho$, then applying Gauss's theorem, 15, to the continuity equation, 2, results in the following.

$$\iiint_R \frac{\partial \rho}{\partial t}\, dx\, dy\, dz + \iiint_R \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right. dx\, dy\, dz = 0 \tag{16}$$

As R becomes small, in the limit, the functions under the integral sign vary less and less with R, so that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = 0 \tag{17}$$

By a similar transformation equations 8, 9, and 10 become

$$\rho \frac{D\vec{q}}{Dt} = -\nabla P \tag{18}$$

The energy equation, 14, may be transformed to yield

$$\frac{\partial}{\partial t}\left[\rho \left(\tfrac{1}{2}q^2 + \bar{u}\right) \right] + \nabla \left[\rho \vec{q} \left(\tfrac{1}{2}q^2 + \bar{u} + \frac{P}{\rho}\right) \right] = \nabla \cdot \nabla (k T) \tag{19}$$

SPECIAL CASES

General Discussion:

At this point in the derivation the equations are general except for the assumptions listed on page ii which primarily restrict the flow to one which is inviscid and uniform. Several other assumptions may be made about the flow which will reduce the complexity of the general equations and give rise to a solution. These may be listed as follows.

1. No heat flux across the boundaries of the flow. This says that $\nabla(\bar{k}\,T) = 0$, i.e. that there exists an adiabatic flow field.

2. The fluid is isoenergetic. This further restricts the problem to one where the stagnation enthalpy of the flow field remains constant, or in equational form.

$$T\frac{Ds}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial t} \tag{20}$$

3. The fluid is an ideal fluid. This assumption achieves an equation of state related by

$$P = \rho R_u T \tag{21}$$

where Ru is the universal gas constant and gives the user several relationships between sound speed, velocity, pressure, enthalpy, entropy, density, and temperature which may be used to further simplify the general compressible flow relations.

4. The fluid is calorically perfect. This states that the isobaric specific heat constant, Cp, and the isovolumic specific heat constant, Cv, remain constant in the flow field. This also states that gamma, the ratio of Cp to Cv, remains constant.

Steady Isoenergetic Flow

The utilization of assumption 2 above results in an important relation in gas dynamics entitled Bjerknes' theorem which may be obtained as follows.

Enthalpy, h, may be defined as

$$h \equiv \bar{u} + P/\rho \tag{22}$$

The stagnation enthalpy is that enthalpy which a fluid would possess if the fluid were brought to rest reversibly and adiabatically and it is defined by

$$h_s \equiv h + \tfrac{1}{2}q^2 \tag{23}$$

Using equations 22 and 23 equation 19 becomes

$$\frac{\partial}{\partial t}\left[\rho\left(\tfrac{1}{2}q^2 + \bar{u}\right)\right] + \nabla\left[\rho\vec{q}h_s\right] = \nabla \cdot \nabla (k T) \tag{24}$$

or

$$\frac{\partial}{\partial t}\left[\rho\left(h_s - \frac{P}{\rho}\right)\right] + \nabla[\rho\vec{q}h_s] = \nabla \cdot \nabla (kT) \tag{24a}$$

Since the fluid may be rotational we define this as the curl of $\vec{q}$,*
$\nabla x \vec{q}$, where

$$\nabla x \vec{q} = 2\omega_x i + 2\omega_y j + 2\omega_z k \tag{25}$$

$$\nabla x \vec{q} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) i + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) k \tag{26}$$

utilizing Stokes theorem** we arrive at (along the normal direction)

$$\iint_\sigma \nabla_n x \vec{q} \, d\sigma = 0 \tag{27}$$

Utilizing the Helmholtz' first vortex theorem and defining Γ as

$$\Gamma = \iint_\sigma \nabla_n x \vec{q} \, d\sigma \tag{28}$$

then within an inviscid fluid with no body forces, the material derivative of Γ may be expressed as

$$\frac{d\Gamma}{dt} = -\oint \left[\frac{1}{\rho} \frac{\partial P}{\partial x} dx + \frac{1}{\rho} \frac{\partial P}{\partial y} dy + \frac{1}{\rho} \frac{\partial P}{\partial z} dz \right] \tag{29}$$

* See For Instance Reference 5.

** Refer to any text of hydrodynamics or vector analysis, for instance Reference 6.

or

$$\frac{d\Gamma}{dt} = -\oint_C \frac{dP}{\rho} \tag{30}$$

where C is the boundary curve of the surface σ .

Combining the first and second laws of thermodynamics, the following may be written:

$$-\frac{dp}{\rho} \doteq Tds - dh \tag{31}$$

equation (30) becomes, having noted that $\oint dh = 0$ since enthalpy is a property and not a path function,

$$\frac{d\Gamma}{dt} = -\oint_C Tds = -\oint_C T\frac{ds}{dr}\,dr = -\iint_\sigma \left(\nabla \times T\frac{ds}{dr}\right) d\sigma \tag{32}$$

this may be written as Bjerknes' theorem* which is of the form

$$\frac{d\Gamma}{dt} = \iint_\sigma (\nabla T \times \nabla s)_n \, d\sigma \tag{33}$$

Transforming Euler's equations and using the theorem of Bjerknes, we obtain for steady isoenergetic flow,

$$-\vec{q} \times (\nabla \times \vec{q}) + \nabla \frac{q^2}{2} = -\frac{1}{\rho}\nabla P = -\nabla h + T\nabla s \tag{34}$$

which for steady flow reduces to Crocco's equation.

*See Appendix A

Steady Inviscid Flow

For steady flow equations 17, 18, and 24a are as follows:

$$\nabla \cdot (\rho \vec{q}) = 0 \tag{35}$$

$$\rho \frac{D\vec{q}}{Dt} + \nabla p = 0 \tag{36}$$

$$\nabla(\rho \vec{q} h_s) = \nabla \cdot \nabla(k T) \tag{37}$$

At this point it is necessary to require that the approaching fluid is uniform and further, assuming that the heat flux into the boundary σ is zero or negligible, the energy equation may be neglected since the implication of a steady inviscid flow with no heat flux is that the flow is isoenergetic. Therefore,

$$\nabla h_s = \nabla h + \nabla \frac{q^2}{2} = 0 \tag{38}$$

noting that

$$\frac{D\vec{q}}{Dt} = \frac{\partial \vec{q}}{\partial t} + \nabla \frac{q^2}{2} - \vec{q} \times \operatorname{curl} \vec{q} \tag{39}$$

where $\frac{\partial \vec{q}}{\partial t} = 0$

then equation (36) becomes

$$\rho \left[\nabla \frac{q^2}{2} - \vec{q} \times \operatorname{curl} \vec{q} \right] + \nabla P = 0 \tag{40}$$

Using equation (31) and combining equation (38) with (40) gives Crocco's equation

$$\vec{q} \times \operatorname{curl} \vec{q} = -T \nabla s = \frac{\nabla P}{\rho} + \frac{\nabla q^2}{2} \tag{41}$$

Let us examine equation (41) to determine the properties of the isoenergetic, steady, inviscid flow. If the flow is anisentropic, then the curl $\vec{q}$ cannot vanish and the flow field cannot be irrotational. Therefore, a gradient of entropy which is non-zero impliesthat the fluid is rotational and conversely.

If grad s is zero, then two conditions may exist for the left-hand side of equation (41). Either the velocity vector and the vorticity vector are parallel which is the case in Beltrami flow, or the curl $\vec{q}$ is zero which results in irrotational, potential flow.

Performing the scalar product of the velocity vector, $\vec{q}$, and equation (41) results in

$$\vec{q} \cdot \operatorname{grad} s = 0 \tag{42}$$

Examining this equation leads to the obvious conclusion that either $\vec{q}$ or grad s is zero, which are trivial cases, or that the gradient of the entropy is in a direction normal to the velocity vector. Further, the streamlines traced by the vector, $\vec{q}$, for steady, isoenergetic, and inviscid flow exhibit no change of entropy. Therefore, the flow must be isentropic along streamlines provided that no discontinuities in the flow exist (e.g., shock waves). If the flow field upstream is everywhere uniform, then we may conclude that the entire flow field is isentropic provided that there are no discontinuities in the fluid stream.

Since the flow along streamlines is isentropic, we may utilize the definition of the sound speed along the streamline.

$$A^2 = \left.\frac{\partial P}{\partial \rho}\right|_s \tag{43}$$

where A is the local speed of sound.

If we restate the fundamental equations in cartesian form,

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \tag{44}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{1}{\rho}\frac{\partial P}{\partial y} = 0 \tag{45}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial P}{\partial z} = 0$$

and write the scalar product of the entropy gradient and the velocity vector along a streamline,

$$u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial y} + w \frac{\partial s}{\partial z} = 0 \tag{46}$$

and combine the definition of sound speed and Euler's equations by first multiplying them by -u, -v, and -w in x, y, and z directions respectively,

$$u^2 \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} + uw \frac{\partial u}{\partial z} + \frac{1}{\rho} A^2 \frac{\partial \rho}{\partial x} = 0$$

$$uv \frac{\partial v}{\partial x} + v^2 \frac{\partial v}{\partial y} + vw \frac{\partial v}{\partial z} + \frac{1}{\rho} A^2 \frac{\partial \rho}{\partial y} = 0$$

$$uw \frac{\partial w}{\partial x} + vw \frac{\partial w}{\partial y} + w^2 \frac{\partial w}{\partial z} + \frac{1}{\rho} A^2 \frac{\partial \rho}{\partial z} = 0 \tag{47}$$

Combining these with continuity equation

$$\rho \frac{\partial u}{\partial x} + \frac{\partial \rho}{\partial x} u + \rho \frac{\partial v}{\partial y} + \frac{\partial \rho}{\partial y} v + \rho \frac{\partial w}{\partial z} + \frac{\partial \rho}{\partial z} w = 0 \tag{48}$$

we get,

$$(A^2 - u^2)\frac{\partial u}{\partial x} + (A^2 - v^2)\frac{\partial v}{\partial y} + (A^2 - w^2)\frac{\partial w}{\partial z} - uv\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$- uw\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - vw\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0 \tag{49}$$

which is the gas dynamic equation for steady, inviscid, isoenergetic flow of a homogeneous fluid along a streamline.

Defining the vorticity vector by its components ξ , η , and ζ as

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 2\omega_x$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 2\omega_y$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\omega_z \qquad (50)$$

Since the steady, inviscid flow of the fluid is isoenergetic, the entropy may be related to the vorticity vector components by the following:

$$v\zeta - w\eta = -T\frac{\partial s}{\partial x}$$

$$w\xi - u\zeta = -T\frac{\partial s}{\partial y}$$

$$u\eta - v\xi = -T\frac{\partial s}{\partial z} \qquad (51)$$

Unsteady Inviscid Flow

The fundamental relations may be restated as follows:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = 0 \tag{17}$$

$$\rho \frac{D\vec{q}}{Dt} + \nabla p = 0 \tag{18}$$

Assuming that it is possible to express the speed of sound as a function of p, ρ, and T,

$$A = A(p, \rho, T) \tag{52}$$

It is possible to obtain a solution to the energy equation provided that the thermal conductivity, k , is identically zero. Equation (31) may be differentiated with respect to time to yield:

$$T \frac{Ds}{Dt} = \frac{Dh_s}{Dt} - \frac{1}{\rho}\frac{\partial p}{\partial t} \tag{53}$$

which is equation (20) for non-isoenergetic flow. Thus, equation (53) implies that in an isentropic unsteady process, the stagnation enthalpy is a function of the local pressure variation and the density of the flow field. In addition, the gradients of enthalpy and entropy may be seen to be directly related with the pressure variation in the flow; therefore, any solution of an unsteady nature must contain gradients of entropy and enthalpy and local-variation of pressure-density to achieve any true representation of the flow field.

CHARACTERISTIC RELATIONS

General Derivation of the Characteristic Relations n-Dimensional Euclidean Space:

Assume that there exists a system of equations quasi-linear in their first derivatives consisting of k independent variables:

$$u_1, u_2, u_3, \ldots\ldots, u_k$$

In this section of this report, u_i is the velocity component in the i^{th} direction, and x^j refers to the co-ordinate direction. The general tensor summation convention will be used and it will be understood that all i will vary from one to k and all j from one to n.

The general second order equation which is quasi-linear in its first derivatives represents the inner product of the a_{ij} tensor with the first derivative of the u n-tuple,

$$b_i = a_{ij} \frac{\partial u_i}{\partial x^j} \tag{54}$$

Since each i^{th} equation, that is each b n-tuple, is a scalar, they may be summed over all i to give a final sum, B, where B and a_{ij} are functions only of u_i and not of $\dfrac{\partial u_i}{\partial x^j}$.

The system of equations contains $(k)^{n-1}$ coefficients, a_{ij}, and nk derivatives of u_i which establishes the mathematical problem that must be solved. That is, given (n - 1) derivatives of u_j at a point $(x^1, x^2, x^3, \ldots x^n)$, calculate the remaining k derivatives from the system of equations. The (n - 1)k derivatives may be thought of as being in some hyperspace, S, with the remaining k derivatives normal to that space, S. The hyperspace, S, exists only if it can be shown that the tensor, a_{ij}, is of such a character that the remaining k derivatives are normal to S. If this system does exist, it will be designated as a characteristic space and the system of equations are hyperbolic in nature (that is, the space is considered as positive definite).

Clearly if these equations are linearly dependent, then there exists such a characteristic space. Thus multiplying each i^{th} equation by an invariant α^i and summing over all i,

$$\alpha^i a_{ij} \frac{\partial ui}{\partial x^j} \delta^s{}_i = \alpha^i b_i = B_\alpha \tag{55}$$

Define:

$$A^i{}_j = \alpha^i a_{ij}$$

and

$$u_{i/j} = \partial u_i / \partial x^j$$

Then equation (55) becomes

$$A^i_j \, u_i/j = B_\alpha$$

If equations (54) are linearly dependent, then the tensors $A^i{}_j$ will (for some set $\alpha^1, \alpha^2, \alpha^3, \ldots\ldots, \alpha^k$) be parallel to the characteristic space, S. The characteristic space may be found by first determining some arbitrary tensor, $d^s{}_r$ which is normal to S, that is, that tensor, $d^s{}_r$, for which the inner product of $d^s{}_r$ and $A^i{}_j$ is zero;

$$d^s{}_r \; A^i{}_j \; \delta^j{}_s \; \delta^r{}_i = 0 \tag{56}$$

where $\delta^i{}_j$ is the Kronecker delta. The solution may be made by solving for each of its components in n-dimensional space. Since these equations have a finite set of solutions (α^1, α^2, α^3, ... , α^k) if and only if the denominator determinant is zero (that is, only the indeterminacies of the derivatives are of interest, since these produce the special characteristic space), then the solution for $d^s{}_r$ is unique and of the form of equation (56).

Particular Solution for Three-Dimensional, Rotational, Isoenergetic, Inviscid, Steady Flow of a Homogeneous Gas

In this space (n=3) and (k=5) the co-ordinate system is defined as follows:

a) Let x^1 lie along the direction of the velocity vector and

b) x^2 be normal to x^1 and

c) x^3 to be normal to both x^1 and x^2.

The five independent variables are u_1, u_2, u_3, p, and ρ. The five equations are equations (17) and (18) plus the following:

$$\frac{\partial p}{\partial x^1} - A^2 \frac{\partial \rho}{\partial x^1} = 0$$

The determinant appears as:

$$\begin{vmatrix} \rho d_1 & \rho d_2 & \rho d_3 & 0 & u_1 d_1 \\ \rho u_1 d_1 & 0 & 0 & d_1 & 0 \\ 0 & \rho u_1 d_2 & 0 & d_2 & 0 \\ 0 & 0 & \rho u_1 d_3 & d_3 & 0 \\ 0 & 0 & 0 & d_1 & -A^2 d_1 \end{vmatrix} = 0$$

Expanding the determinant yields:

$$(\rho)^3 \ (u_1)^2 \ (d_1)^3 \ \{ \ (u_1)^2 \ (d_1)^2 \ - A^2 \ [\ (d_1)^2 + (d_2)^2 + (d_3)^2] \ \} \quad = 0$$

For a non-trivial solution, the density, ρ, and the velocity, u_1, cannot be identically equal to zero; therefore, two solutions are possible as follows:

$$(d_1)^3 = 0$$

Since d_1 must be real for the solution to be hyperbolic in nature, d_1 must be identically zero. This leads to the interesting conclusion that the streamlines of the flow, regardless of the velocity, are characteristics of the flow; however, this does not lead to the general solution of the problem. The second possible solution is:

$$\left[\left(\frac{u_1}{A}\right)^2 - 1\right] (d_1)^2 - [\ (d_2)^2 + (d_3)^2 \ ] = 0 \qquad (57)$$

The second solution shows that if $(u_1 / A < 1)$, there are no real values for the d_1's, that is, there exists no real characteristic space. If $(u_1 / A = 1)$, there is no real solution unless both d_2 and d_3 are identically zero and even then, the solution for d_1 is indeterminate. Since the transonic region is of no interest in this report, this solution will not be explored further. If $(u_1 / A > 1)$, the d's are real and the solutions speed range is defined, that of supersonics.

Equation (57) is, upon close inspection, the equation of a real cone of revolution about the x^1 axis; therefore, every line normal to the d's also forms a cone of revolution about the x' axis each of which possesses a slope of its genearatrices defined by the Mach angle, μ, as:

$$\mu = \arcsin \frac{A}{u_1} \tag{58}$$

and

$$\tan \mu = \pm \sqrt{\frac{1}{\left(\frac{u_1}{A}\right)^2 - 1}} = \frac{dx^2}{dx^1} \tag{59}$$

The vorticity vector has components:

$$\Omega_1 = \frac{\partial u_3}{\partial x^2} - \frac{\partial u_2}{\partial x^3} = 0 \tag{60}$$

$$\Omega_2 = \frac{\partial u_1}{\partial x^3} - \frac{\partial u_3}{\partial x^1} = - \frac{\partial s}{\partial x^3} T \tag{61}$$

$$\Omega_3 = \frac{\partial u_2}{\partial x^1} - \frac{\partial u_1}{\partial x^2} = \frac{\partial s}{\partial x^2} T \tag{62}$$

Transforming the relationships to the plane of the velocity vector and bi-characteristic* yields:

$$\begin{aligned} \frac{\partial u_1}{\partial x^1} &= \frac{du_1}{dx^1} - \frac{\partial u_1}{\partial x^2}\frac{dx^2}{dx^1} \\ \frac{\partial u_2}{\partial x^2}\frac{dx^2}{dx^1} &= \frac{du_2}{dx^1} - \frac{\partial u_2}{\partial x^1} \end{aligned} \tag{63}$$

Substitute equations (64) and (59) into equation (57) using the definition of d_1,

$$\left[1 - \left(\frac{u_1}{A}\right)^2\right] \frac{du_1}{dx^1} \mp \left[1 - \left(\frac{u_1}{A}\right)^2\right] \tan \mu \frac{\partial u_1}{\partial x^2} \pm \cot \mu \frac{du_2}{dx^1}$$

$$\mp \cot \mu \frac{\partial u_2}{\partial x^1} + \frac{\partial u_3}{\partial x^3} = 0 \tag{64}$$

*See Figure 4, page 37.

Combining equations (59), (62), and (64) yields:

$$\cot\mu \frac{du_1}{dx^1} \mp \frac{du_2}{dx^1} - \tan\mu \frac{\partial u_3}{\partial x^3} \mp \Omega_3 = 0 \tag{65}$$

In order to achieve a second relation, it is necessary to explore:

$$\frac{du_3}{dx^1} = \frac{\partial u_3}{\partial x^1} + \frac{\partial u_3}{\partial x^2}\frac{dx^2}{dx^1} + \frac{\partial u_3}{\partial x^3}\frac{dx^3}{dx^1} \tag{66}$$

However, along the bi-characteristic,

$$\frac{dx^3}{dx^1} = 0 = \frac{dx^3}{dx^2}$$

Combining equations (60), (61), and (66) yields:

$$\frac{du_3}{dx^1} = -\Omega_2 + \frac{du_1}{dx^3} \pm \tan\mu\,\Omega_1 \pm \tan\mu \frac{du_2}{dx^3} \tag{67}$$

The streamline co-ordinates are related to the Cartesian system by an angle, δ , which is defined as the angle between the diametral plane of the characteristic cone passing through the bi-characteristic and the diametral plane parallel to the z-axis. Figure 2 shows the relationship between the polar spherical co-ordinates and the Cartesian system.

If the velocity is expressed in spherical polar co-ordinates (q, θ, ψ) for the streamline solution, the u vector becomes:

$$du_1 = dq$$

$$\frac{du_2}{q} = \cos\delta\, d\theta + \sin\delta \sin\theta\, d\psi$$

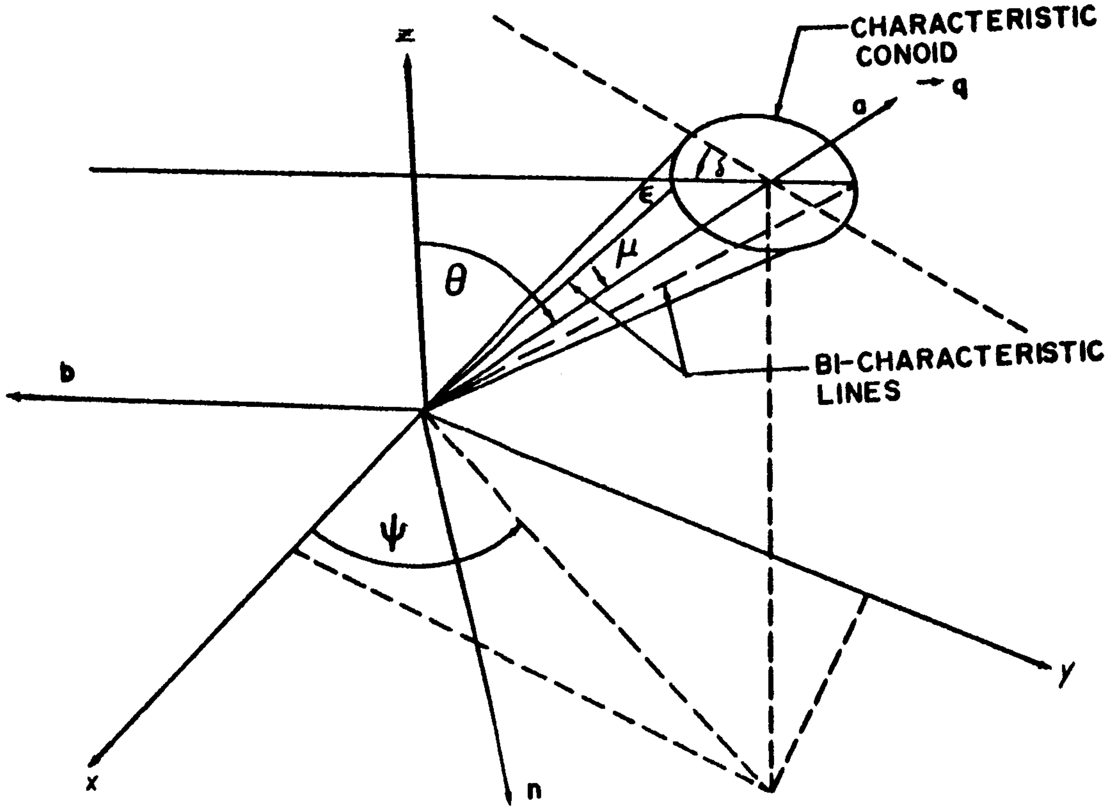

FIGURE 2

CHARACTERISTIC, VELOCITY, AND CARTESIAN CO-ORDINATE RELATIONS

$$\frac{du_3}{q} = -\sin \delta \, d\theta + \cos \delta \sin \theta \, d\psi$$

Also, the x^1 co-ordinate may (for convenience) be transformed to the bi-characteristic line, ε, and redefining,

$$x^1 = a$$

$$x^2 = b$$

$$x^3 = n$$

Then, noting that

$$\frac{d\varepsilon}{da} = \sec \mu$$

Equations (65) and (67) become:

$$\frac{1}{q}\frac{dq}{d\varepsilon} \mp \tan \mu \cos \delta \frac{d\theta}{d\varepsilon} \mp \tan \mu \sin \delta \sin \theta \frac{d\psi}{d\varepsilon} + \tan^2 \mu \sin \delta \cos \mu \frac{d\theta}{dn}$$

$$- \frac{\sin^2 \mu}{\cos \mu} \cos \delta \sin \theta \frac{d\psi}{dn} \pm \frac{\Omega_3 \sin \mu}{q} = 0 \tag{68}$$

and

$$\frac{1}{q}\frac{dq}{dn} + \frac{\sin \delta}{\cos \mu}\frac{d\theta}{d\varepsilon} - \frac{\sin \theta \cos \delta}{\cos \mu}\frac{d\psi}{d\varepsilon} \pm \tan \mu \cos \delta \frac{d\theta}{dn}$$

$$\pm \tan \mu \sin \theta \sin \delta \frac{d\psi}{dn} - \frac{\Omega_2}{q} \pm \tan \mu \frac{\Omega_1}{q} = 0 \tag{69}$$

The characteristic equations for this particular solution are in a form suitable for numerical analysis. This will be explored further later in this report.

Particular Solution for Two-Dimensional, Irrotational, Inviscid, Unsteady Flow of a Homogeneous Gas

The following derivation will utilize the preceding development and assumptions, except that it will be necessary to further restrict the fluid to one which satisfies the following equation of state

$$P = \rho R_u T$$

that is, an ideal fluid.

Consider equations 17, 18, and 24a.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = 0 \tag{17}$$

$$\rho \frac{D\vec{q}}{Dt} + \nabla P = 0 \tag{18}$$

$$\frac{\partial}{\partial t}\left[\rho\left(h_s - \frac{P}{\rho}\right)\right] + \nabla(\rho \vec{q} h_s) = \nabla \cdot \nabla(kT) \tag{24a}$$

The density, ρ, may be considered to be a single valued function of P along some curve C. Therefore, Kelvin's theorem for an incompressible fluid will apply to ρ as follows.

$$\frac{d\Gamma}{dt} \equiv \frac{d}{dt}\oint q_t \, dr = \oint - \frac{dP}{\rho} = - \oint T \, ds \tag{70}$$

It is feasible to describe homentropic flow at this point. Homentropic* implies that there exists spatial invariability of the entropy in a cloud of

*See Modern Developments in Fluid Dynamics, High Speed Flow. Ed. L. Howarth. Oxford, 1953. p.3

particles engaged in isentropic flow. Further, that within this homentropic cloud vorticity can neither be created, nor annihilated within such a flow field; however, the surroundings of this cloud need not be isentropic.

Therefore, requiring the fluid to be homentropic and assuming an irrotational state of approach of the flow validates the use of Kelvin's theorem in equation (70), although Kelvin's theorem was originally derived for an incompressible fluid. Further, with these assumptions, Euler's equations may be written as follows.

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[\frac{u^2 + v^2 + w^2}{2} + \int_{\rho_1}^{\rho} \frac{A^2}{\rho}\, d\rho\right] = 0 \tag{71}$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial y}\left[\frac{u^2 + v^2 + w^2}{2} + \int_{\rho_1}^{\rho} \frac{A^2}{\rho}\, d\rho\right] = 0 \tag{72}$$

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial z}\left[\frac{u^2 + v^2 + w^2}{2} + \int_{\rho_1}^{\rho} \frac{A^2}{\rho}\, d\rho\right] = 0 \tag{73}$$

Representing the velocity vector by a potential field:

$$\begin{aligned} u &= \phi_x = \frac{\partial \phi}{\partial x} \\ v &= \phi_y = \frac{\partial \phi}{\partial y} \\ w &= \phi_z = \frac{\partial \phi}{\partial z} \end{aligned} \tag{74}$$

$$\phi_t = \frac{\partial \phi}{\partial t} = \tfrac{1}{2}q^2 + \int_{\rho_1}^{\rho} \frac{A^2}{\rho} d\rho$$

and if the fluid is calorically ideal, the integral relations become:

$$\int_{\rho_1}^{\rho} \frac{A^2}{\rho} d\rho = h - h_1 \tag{75}$$

$$\int_{\rho_1}^{\rho} \frac{A^2}{\rho} d\rho = \frac{1}{\gamma - 1} (A^2 - A_1^2) \tag{76}$$

The integration over definite limits erases the integral sign and the derivatives of ϕ_t are

$$-\phi_{tt} = \frac{\partial^2 \phi}{\partial t^2} = u\phi_{tx} + v\phi_{ty} + w\phi_{tz} + \frac{A^2}{\rho}\frac{\partial \rho}{\partial t}$$

$$-\phi_{tx} = \frac{\partial^2 \phi}{\partial t \partial x} = u\phi_{xx} + v\phi_{xy} + w\phi_{xz} + \frac{A^2}{\rho}\frac{\partial \rho}{\partial x} \tag{77}$$

$$-\phi_{ty} = \frac{\partial^2 \phi}{\partial t \partial y} = u\phi_{yx} + v\phi_{yy} + w\phi_{yz} + \frac{A^2}{\rho}\frac{\partial \rho}{\partial y}$$

$$-\phi_{tz} = \frac{\partial^2 \phi}{\partial t \partial z} = u\phi_{zx} + v\phi_{zy} + w\phi_{zz} + \frac{A^2}{\rho}\frac{\partial \rho}{\partial z}$$

Combining equation 2 with equation 77 eliminates the density from the continuity equation and yields upon rearranging

$$(A^2 - u^2)\phi_{xx} + (A^2 - v^2)\phi_{yy} + (A^2 - w^2)\phi_{zz} - \phi_{tt} - 2uv\phi_{xy} - 2vw\phi_{yz}$$

$$- 2wu\phi_{zx} - 2u\phi_{xt} - 2v\phi_{yt} - 2w\phi_{zt} = 0 \tag{78}$$

Equation 78 represents the gas dynamic equation for unsteady, inviscid flow of an ideal homogeneous fluid in three dimensions. The linearized form of the above equation for parallel flow of velocity u may be shown to be:

$$(1 - M_\infty^2)\phi_{xx} + \phi_{yy} + \phi_{zz} - \phi_{t't'} - 2M_\infty\phi_{xt'} = 0 \tag{79}$$

where M_∞ = Mach number of the parallel flow and

$t' = A_\infty t$

∞ = represents the free stream conditions

This equation, 79, is the usual point of beginning the solution of oscillatory processes and has been transformed to a wave equation of the form

$$\phi_{tt} = A_0^2 (\phi_{xx} + \phi_{yy}) \tag{80}$$

where A_0 is the stagnation value of sonic speed.

Reducing the problem from 4 variables in equation 78 to three variables requires only that z = 0 and w = 0. Equation 78 then becomes:

$$(A^2 - u^2)\phi_{xx} + (A^2 - v^2)\phi_{yy} - \phi_{tt} - 2uv\phi_{xy} - 2u\phi_{xt} - 2v\phi_{yt} = 0 \tag{81}$$

which is equation 78 in two dimensions with three independent variables.

Equation 81 becomes upon rearranging and noting that an ideal fluid has been assumed**

$$A^2 (\phi_{xx} + \phi_{yy}) = -\left(\phi_x \frac{\partial A}{\partial x} + \phi_y \frac{\partial A}{\partial y} + \frac{\partial A}{\partial t}\right)\frac{2A}{\gamma - 1} \tag{82}$$

Equation 82 is similar to the steady supersonic inviscid flow gas dynamic equation 49. Since there exists this similarity between the equations, a characteristic solution of $t = f(x, y)$ may possibly exist.

Assume equation 82 has characteristic surfaces

$$t = f(x, y) \tag{83}$$

then they must be defined by

$$(A^2 - u^2)(f_{xx}) + (A^2 - v^2)(f_{yy}) - 1 + 2vf_y + 2uf_x - 2uvf_xf_y = 0 \tag{84}$$

where f_x and f_y are the partial derivatives of $f(x, y)$ with respect to x and y defined as

$$f_x = \frac{\partial f(x,y)}{\partial x}$$

$$f_{xx} = \frac{\partial^2 f(x,y)}{\partial x^2} \tag{85}$$

$$f_y = \frac{\partial f(x,y)}{\partial y}$$

$$f_{yy} = \frac{\partial^2 f(x,y)}{\partial y^2}$$

$$f_{xy} = f_{yx} = \frac{\partial^2 f(x,y)}{\partial x \partial y}$$

**See Appendix B.

The direction cosines of the normal to the characteristic surfaces are proportional to f_x, f_y, and -1.

$$A^2 \; (f_{xx} + f_{yy}) \; = \; (uf_x + vf_y - 1)^2 \tag{86}$$

or parametrically as,

$$\frac{f_x}{\cos \delta} = \frac{f_y}{\sin \delta} = \frac{1}{u \cos \delta + v \sin \delta + A} \tag{87}$$

where δ is defined as per Figure 3.

Since the unsteady flow process yields a hyperbolic equation for every speed range, the characteristic surfaces are real for all velocities. Consider a co-ordinate system involving x, y and t as in Figure 3. If at time t_0 the fluid particle is located at $P(x_0, y_0)$ with velocity q and components u and v, after some time dt, the particle, will move to another point $P_1(x_1, y_1)$, where

$$\begin{aligned} x_1 &= x_0 + udt \\ y_1 &= y_0 + ydt \end{aligned} \tag{88}$$

Thus, the fluid particle has propagated itself after some disturbance from $P(x_0, y_0)$ to $P_1(x_1, y_1)$ which lies within a circle defined by its center $x_0 + udt$, $y_0 + vdt$ and its radius Adt.*

The characteristic conoid (that is this circle) is defined by:

$$(x - x_0 - udt)^2 + (y - y_0 - vdt)^2 = (Adt)^2 \tag{89}$$

*Note: Sonic waves propagate themselves at sonic speed; therefore, after some time, dt, and with velocity, A, the particle will have moved through a distance Adt.

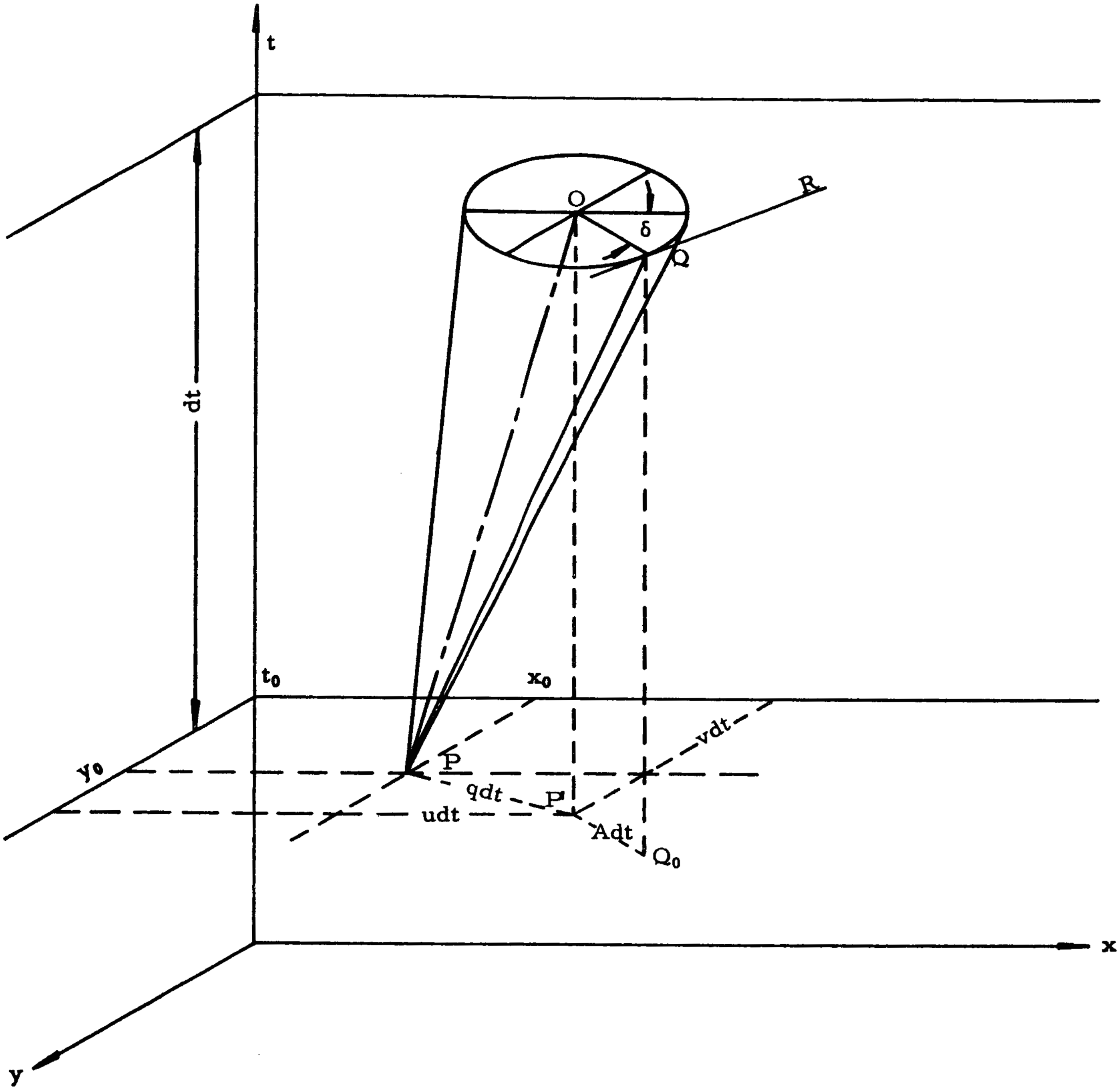

Figure 3

Co-ordinate System for the Mach Conoid in Unsteady, Inviscid, Two-Dimensional Flow

The characteristic cone from $P(x_o, y_o)$ may be defined as,

$$(x - x_0)^2 + (y - y_0)^2 + (t - t_0)^2 (u^2 + v^2 - A^2) - 2u(t - t_0)(x - x_0)$$

$$- 2v(t - t_0)(y - y_0) = 0 \tag{90}$$

The bi-characteristic line, $\overline{PQ}$, has

$$\left(\frac{dx}{Adt}\right)_{PQ} = \frac{u}{A} + \cos \delta$$

$$\left(\frac{dy}{Adt}\right)_{PQ} = \frac{v}{A} + \sin \delta$$

The line $\overline{QR}$ normal to line $\overline{OQ}$ in the plane t = constant locates line $\overline{PQ}$ with respect to the Mach cone. If line $\overline{OQ}$ is inclined at an angle δ to the x axis, then

$$\left(\frac{dy}{dx}\right)_{OQ} = \tan \delta \tag{91}$$

$$\left(\frac{dt}{dy}\right)_{OQ} = \left(\frac{dt}{dx}\right)_{OQ} = 0$$

Since $\overline{PQ}$ is the bi-characteristic curve of the characteristic cone which emanates from $P(x_o, y_o)$, then along $\overline{PQ}$, the bi-characteristic curve, since $\overline{PQ}$ has been defined in the characteristic plane, equation 87 applies; therefore, if $\frac{\partial}{\partial a}$ indicates the variation along $\overline{PQ}$ and $\frac{\partial}{\partial n}$ the variation along $\overline{QR}$, the following may be obtained from equations 78 and 86.

$$(A + v \sin \delta)\left(\frac{\partial u}{\partial a} - \frac{\partial v}{\partial n} \right) - v \cos \delta \left(\frac{\partial v}{\partial a} + \frac{\partial u}{\partial n} \right) + \frac{2A}{\gamma - 1} \cos \delta \frac{\partial A}{\partial a}$$

$$+ \frac{2}{\gamma - 1} (v + A \sin \delta) \frac{\partial A}{\partial n} = 0 \qquad (92)$$

and

$$\sin \delta \left(\frac{\partial u}{\partial a} - \frac{\partial v}{\partial n} \right) - \cos \delta \left(\frac{\partial v}{\partial a} + \frac{\partial u}{\partial n} \right) + \frac{2}{\gamma - 1} \frac{\partial A}{\partial n} = 0 \qquad (93)$$

which upon rearranging produce

$$\left(\frac{\partial u}{\partial a} - \frac{\partial v}{\partial n} \right) + \frac{2}{\gamma - 1} \left(\cos \delta \frac{\partial A}{\partial a} + \sin \delta \frac{\partial A}{\partial n} \right) = 0 \qquad (94)$$

$$\left(\frac{\partial u}{\partial n} + \frac{\partial v}{\partial a} \right) + \frac{2}{\gamma - 1} \left(\sin \delta \frac{\partial A}{\partial a} - \cos \delta \frac{\partial A}{\partial n} \right) = 0 \qquad (95)$$

Equations 94 and 95 are the velocity equations to be used in the analysis of unsteady flow fields of all speed regimes. The solution of these by a numerical technique is quite similar to the one now utilized in the solution of the steady flow equations. A portion of the Numerical Solutions section will be devoted to this development.

At this point, the solutions of the three dimensional gas dynamic equations by characteristic techniques are complete. Modifications may be made to each to alleviate some of the stronger assumptions necessary to the final forms presented in this section. The numerical techniques used in the solution of these equations for a physical application are discussed in the following section.

NUMERICAL SOLUTION

The velocity equations in the preceding section are in general not amenable to solution by separation of variables, superposition or any of the other well-defined techniques of solving partial differential equations. These equations are hyperbolic non-linear, second order partial differential equations and either a series solution or a finite difference technique must be utilized. The author has chosen the use of finite differences to achieve a solution since this adapts quite well to high speed digital computer techniques.

Thornhill[2] has established a method to accomplish this numerical procedure which may be summarized as follows.

Noting that there exist characteristic conoids in Euclidean three-space which are generated by bi-characteristic curves that lie on the characteristic cones, the solution of the flow field may be made by numerical integration along a hexahedral network of characteristic surfaces. These surfaces may (in sufficiently small units) be approximated by plane sections which are normal to the characteristic cones at the bi-characteristics. Thus, the already determined physical and dynamic values at the known three points (that is, those at P_1, P_2, and P_3) may be used, coupled with the six relations (three physical and three dynamic) of the plane sections, to calculate the velocity and space co-ordinates of the fourth point, P_4. Thus, the geometrical significance of the problem allows systematic numerical integration along a grid of hexahedral surfaces to completely describe a given hyperbolic flow field of three independent variables.

General Solution of the Supersonic, Isoenergetic, Irrotational, Inviscid Flow of a Homogeneous Gas

Steady Flow in Three Dimensions:

The numerical integration of the hexahedral network must be related to the intersection of the characteristic conoids with the characteristic cones and their subsequent intersection to determine the location of the space co-ordinates of the fourth point, P_4. Figure 4 shows the intersection of the three Mach cones while Figure 5 shows the relationship between the characteristic cones, the characteristic conoids, and the bi-characteristic curves. Figures 4 and 6 show the characteristic tetrahedron which is the primary calculating network that will be used for the solution of the flow field.

Consider three non-collinear, arbitrarily located points in three-space (see Figure 6) defined by their three-space co-ordinates (x_j, y_j, z_j) and their dynamic co-ordinates (q_j, θ_j, ψ_j) where the velocity is expressed in spherical polar notation and j takes on values of 1, 2, and 3 depending upon the point under consideration. Figure 2 shows the relationship of the co-ordinate system and the characteristic cone.

The characteristic conoids (general characteristic surfaces) intersect the characteristic cones at the bi-characteristics. These general characteristic surfaces may, in a sufficiently small unit, be approximated by

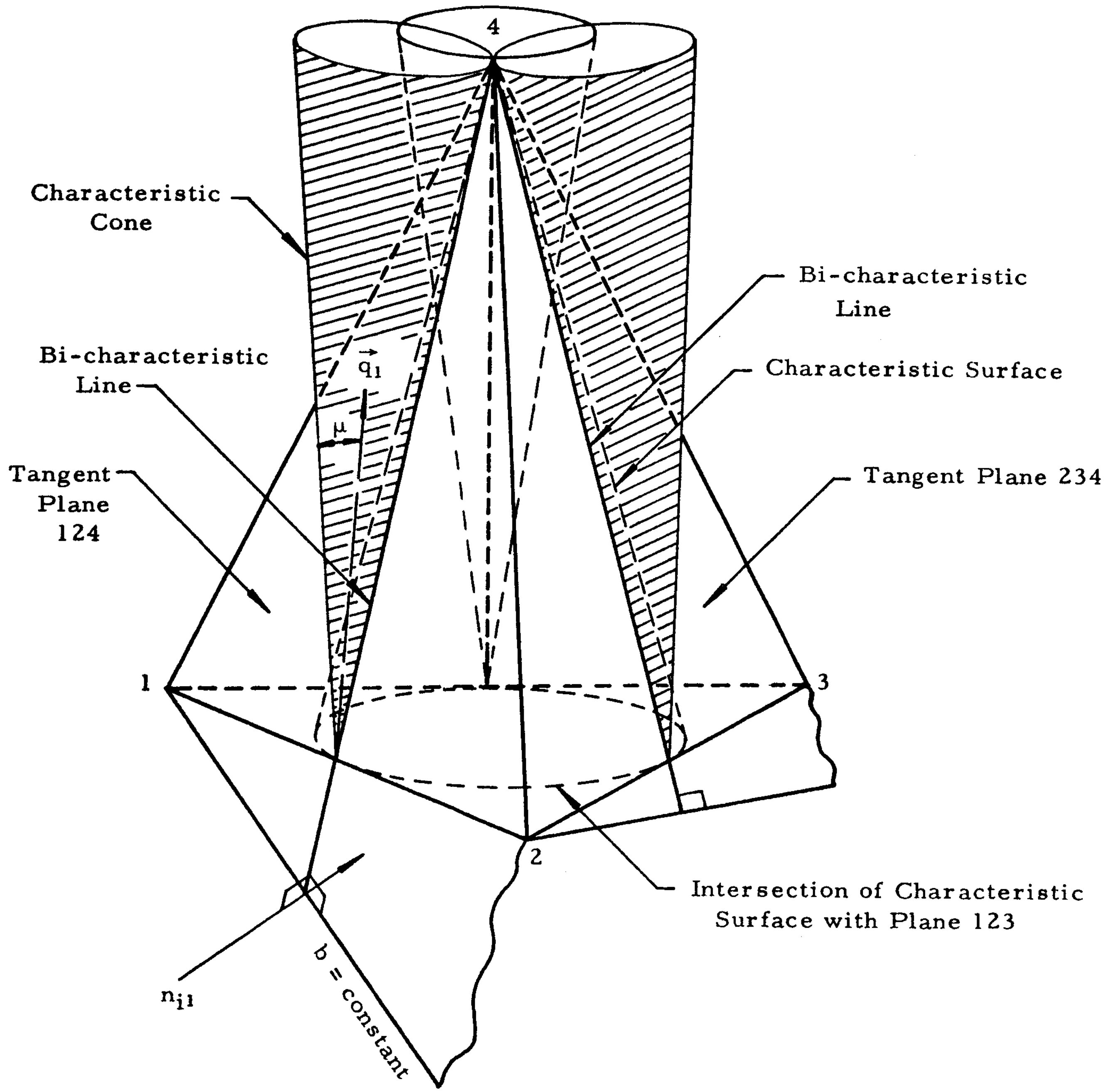

Figure 4

Typical Hexahedral Network

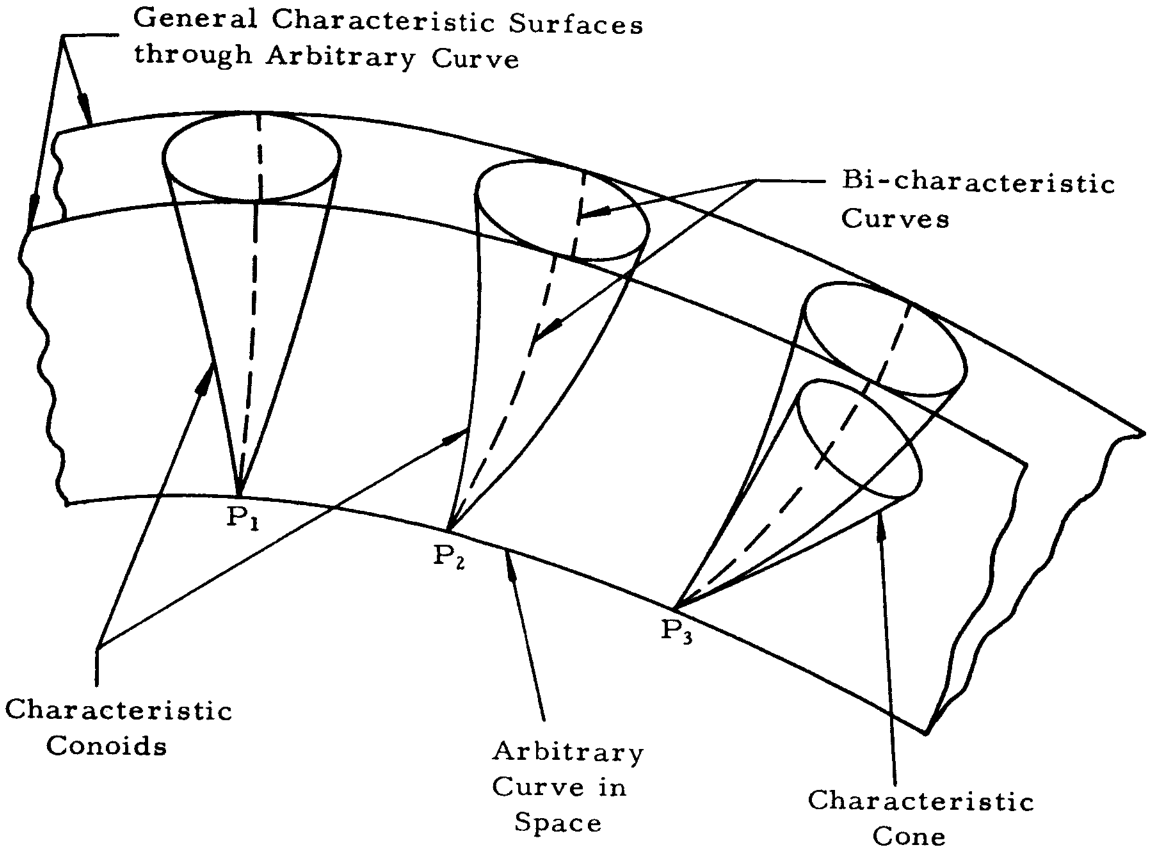

Figure 5

Characteristic Surfaces, Conoids, and Curves

Characteristic Tetrahedron

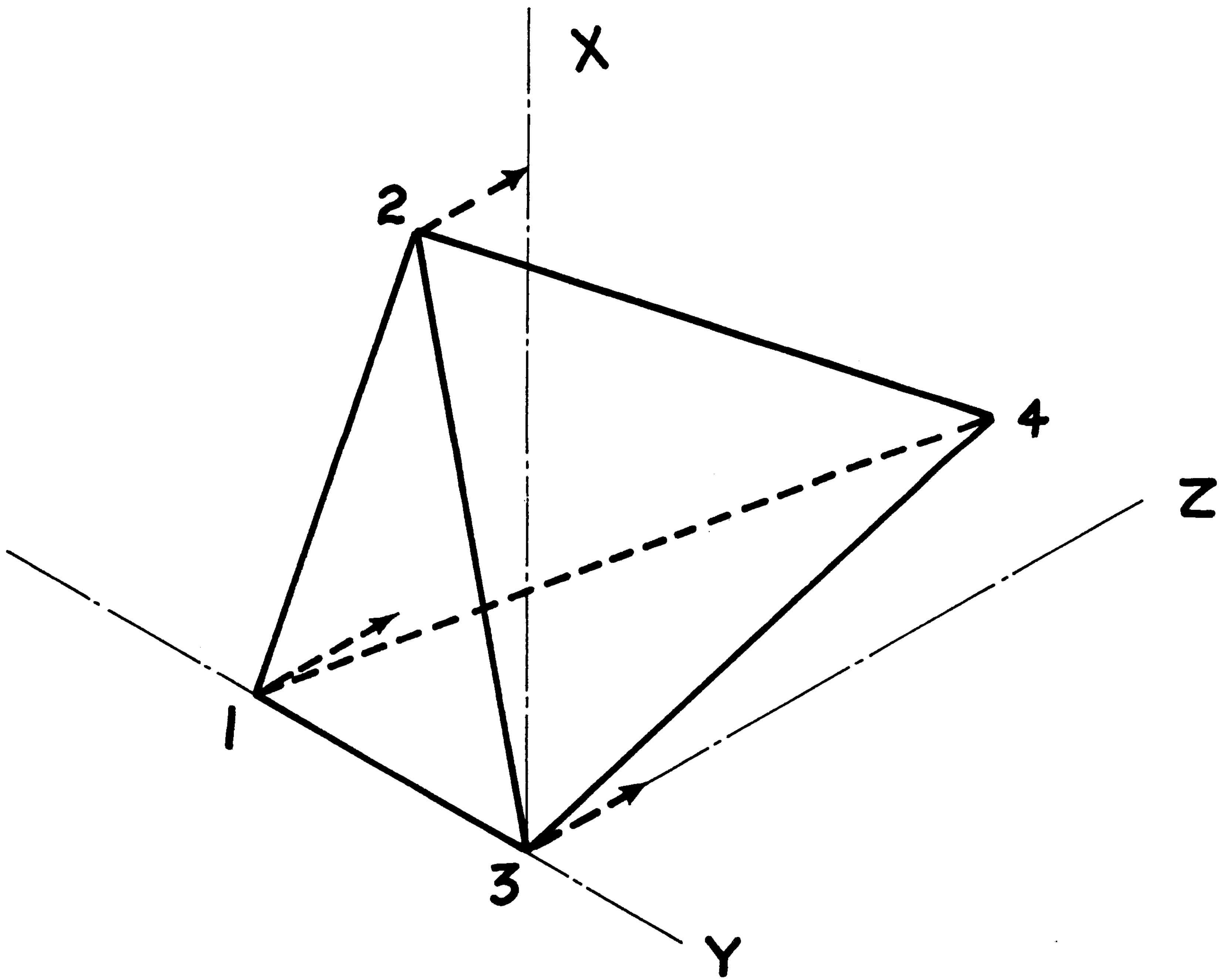

Figure 6

plane sections, hereafter referred to as the tangent planes to the characteristic cone. Thus, the tangent planes contain the bi-characteristics and are everywhere inclined at an angle $\pm\,\mu$ to the velocity vector.

The outward drawn normals to the tangent planes are inclined at an angle $\pm(\pi/2 + \mu)$ to the velocity vector. Expressing the outward drawn normals to the planes, $\vec{n}_j$, in terms of their direction cosines, n_{ij}, where i refers to the co-ordinate direction and j to the number of the plane under consideration as follows:

$j = 1$ defines plane P_1, P_2, and P_4

$j = 2$ defines plane P_2, P_3, and P_4

$j = 3$ defines plane P_3, P_1, and P_4

The inner product of the outward drawn normal, $\vec{n}_j$, with any line in the j^{th} tangent plane may be expressed as:

$$\vec{n}_j \cdot \vec{L}_j = \cos(\pi/2) = 0 \tag{96}$$

The lines, $\vec{L}_j$, can be defined by their direction cosines, ℓ_{ij}. Choosing the lines, $\vec{L}_j$, to be the base lines of the tetrahedron (line 1-2, line 2-3, and line 3-1), ℓ_{ij} may be defined as follows:

$$\ell_{ij} = \frac{x_{j+1} - x_j}{\sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2 + (z_{j+1} - z_j)^2}} \tag{97}$$

and similarly for ℓ_{2j} and ℓ_{3j} until all base lines have been determined.

The inner product may now be rewritten,

$$n_{ij} \cdot \ell_{ij} = 0 \tag{98}$$

A second relation can be obtained by performing the inner product of the normal, n_{ij}, with the velocity vector,

$$n_{ij} \cdot \vec{q} = \cos(\pi/2 + \mu) = -\sin\mu \tag{99}$$

where the velocity vector, $\vec{q}$, has been normalized. A third and final relation comes from a property of the direction cosines.

$$\sum_i n_{ij}^2 = 0 \tag{100}$$

Since equations (98), (99), and (100) result in a quadratic system of equations containing nine unknowns and nine equations, the solution, in general, results in two roots for each system of three equations, i.e., for each plane. However, only one of these roots is a valid solution in each plane. In order to determine the proper root to use, the inner product of the normal, n_{ij}, and any line in the (P_1, P_2, P_3) base plane drawn from the point directly opposite the base line of the j^{th} tangent plane is performed with each of the roots of the above system. If both inner products are positive, or if they are of opposite sense, the largest algebraic value will determine the proper root to choose. If both inner products are negative, the largest absolute value determines the proper root.

The tangents are defined by:

$$n_{1j}(x_4 - x_j) + n_{2j}(y_4 - y_j) + n_{3j}(z_4 - z_j) = 0 \tag{101}$$

If these are solved simultaneously in a linear system of equations, the space co-ordinates (x_4, y_4, z_4) may be determined.

The solution for the space co-ordinates provides the basis for determining the characteristic co-ordinates which enables the solution of the velocity equations. Equation (68) was chosen for the velocity relationship which coupled with the following relationship (which may be found in any compressible flow text):

$$\frac{dM^*}{M^*} = \frac{dq}{q} \tag{102}$$

where M^* is the dimensionless Mach number defined as the ratio of the flow velocity to that of the critical velocity, can be used to solve the dynamic portion of the numerical integration. If equation (68) is set up in the following manner,

$$C_{1j}\,\theta_4 + C_{2j}\,\psi_4 - \frac{M_4^*}{C_{3j}} = C_{4j} \tag{103}$$

where C_{1j}, C_{2j}, C_{3j}, and C_{4j} are defined as follows:

$$C_{1j} = \tan\mu_j \cos\delta + \tan\mu_j \sin\delta \sin\mu_j\, C_{5j}$$

$$C_{2j} = \tan\mu_j \sin\delta \sin\theta_j - \tan\mu_j \cos\delta \sin\mu_j \sin\theta_j\, C_{5j}$$

$$C_{3_j} = M^*_{j,4}$$

$$C_{4_j} = \tan\mu_j \sin\mu_j \sin\delta\,(\theta_{j+1} - \theta_j)\,C_{6j}$$
$$- \tan\mu_j \cos\delta \sin\mu_j \sin\theta_j\,(\psi_{j+1} - \psi_j)\,C_{6j}$$
$$+ C_{1j}\theta_j + C_{2j}\psi_j - M^*_j/C_{3j}$$

and

$$C_{5_j} = \left(\frac{\Delta a}{\Delta b}\right)_j$$

$$C_{6_j} = \left(\frac{\Delta a}{\Delta b}\right)_{j,4}$$

A linear system of equations results, i.e., a three by four matrix. The subscript j designates the plane, while (j, 4) designates the value along the bi-characteristic through point 4.

The preceding derivation represents the solution of a point away from all surfaces. The addition of a general surface for the flow to follow simplifies the problem. The surface may be described by a general quadric equation as follows.

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + qz + s = 0 \tag{106}$$

After determination of the normal direction cosines of the three tangent planes, the solution of the solid boundary flow must be restricted to the boundaries. Solving the line intersections of two of these planes

$$n_{1j}(x_4 - x_j) + n_{2j}(y_4 - y_j) + n_{3j}(z_4 - z_j) = 0 \quad j = 1,3 \tag{107}$$

yields

$$\frac{x_4 - x_2}{K_1} = \frac{y_4 - y_2}{K_2} = \frac{z_4 - z_2}{K_3} \tag{108}$$

where

$$\begin{aligned} K_1 &= n_{21}\, n_{32} - n_{31}\, n_{22} \\ K_2 &= n_{31}\, n_{12} - n_{11}\, n_{32} \\ K_3 &= n_{11}\, n_{22} - n_{21}\, n_{12} \end{aligned} \tag{109}$$

The solution of the line intersection of the two tangent planes with the surface equation yields the co-ordinates x_4, y_4, z_4. The normal direction cosines to the surface equation may then be obtained.

With this latter data, the following equation will apply since the velocity vector must be in the plane of the flow, (see Figure 7).

$$n_{i3} \cdot \vec{q} = \cos(\pi/2) = 0 \tag{110}$$

Using this equation and equations (104), the velocity vector of the fourth point may be calculated. An iteration procedure similar to that of the preceding yields the proper solution.

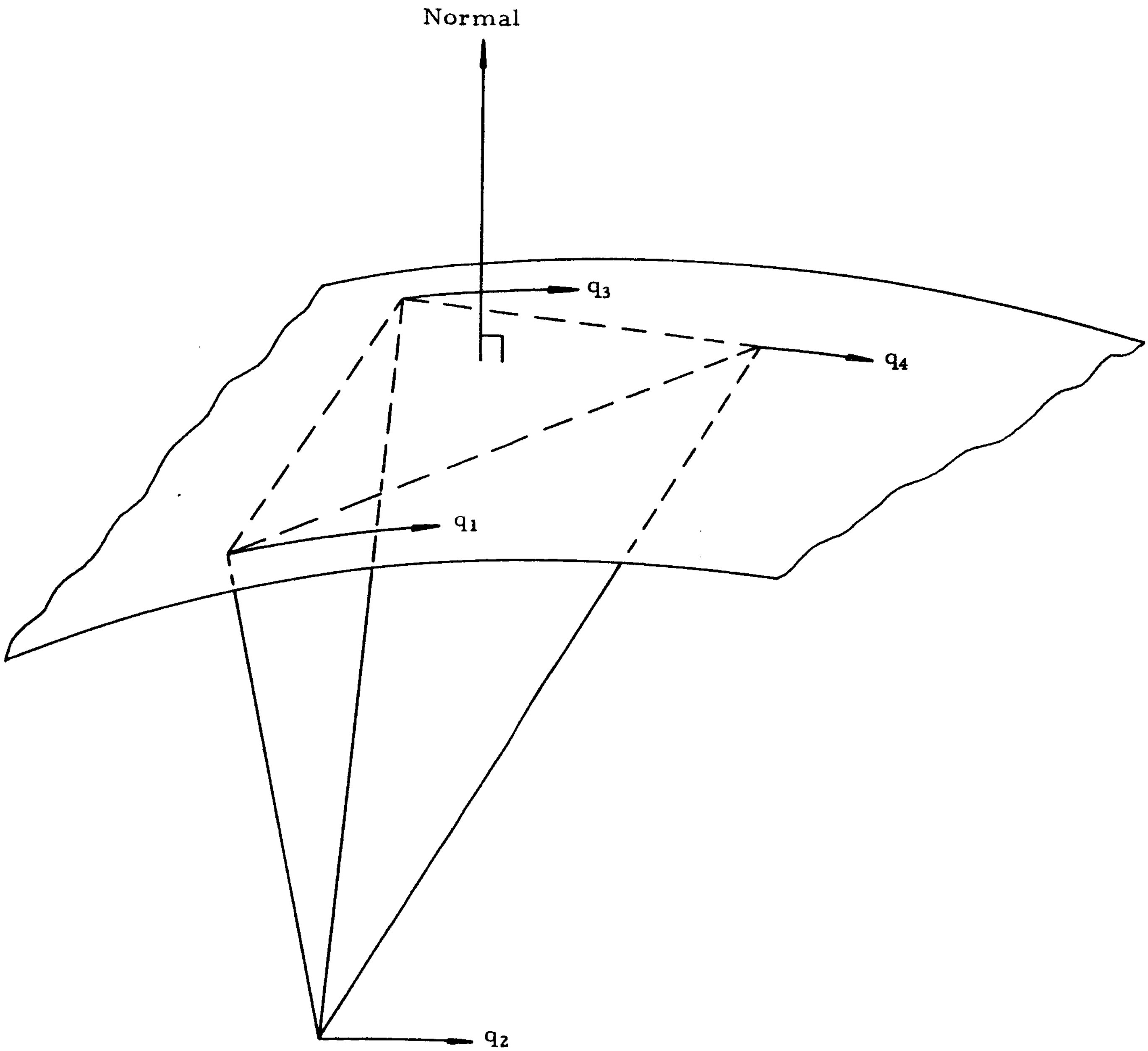

Figure 7

Solution of Velocity Vector along Solid Boundary

Unsteady Flow in Two Dimensions

The equations (94) and (95) can be prepared for finite difference solution by considering the following relations:

$$\frac{du}{da} = \frac{\partial u}{\partial a} + \frac{\partial u}{\partial t}\frac{dt}{da} \tag{111}$$

$$\frac{du}{dn} = \frac{\partial u}{\partial n} \tag{112}$$

$$\frac{dv}{da} = \frac{\partial v}{\partial a} + \frac{\partial v}{\partial t}\frac{dt}{da} \tag{113}$$

$$\frac{dv}{dn} = \frac{\partial v}{\partial n} \tag{114}$$

$$\frac{dA}{da} = \frac{\partial A}{\partial a} + \frac{\partial A}{\partial t}\frac{dt}{da} \tag{115}$$

$$\frac{dA}{dn} = \frac{\partial A}{\partial n} \tag{116}$$

and

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a}\frac{da}{dt} \tag{117}$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial a}\frac{da}{dt} \tag{118}$$

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} \tag{119}$$

From Figure 3:

$$\tan \lambda = \frac{dt}{da} = \pm \sqrt{(a \cos \delta + u)^2 + (a \sin \delta + v)^2} \tag{120}$$

Using equations (111), (114), (115), (116), and (120), equation (94) becomes:

$$\left(\frac{du}{da} \mp \frac{\partial u}{\partial t} \tan \lambda\right) - \frac{dv}{dn} + \frac{2}{\gamma - 1}\left[\cos \delta \left(\frac{dA}{da} \mp \frac{\partial A}{\partial t} \tan \lambda\right) + \sin \delta \left(\frac{dA}{dn}\right)\right] = 0 \tag{121}$$

Using equations (112), (113), (115), (116), and (120), equation (95) becomes:

$$\frac{du}{dn} + \frac{dv}{da} \mp \frac{\partial v}{\partial t} \tan \lambda + \frac{2}{\gamma - 1}\left[\sin \delta \left(\frac{dA}{da} \mp \frac{\partial A}{\partial t} \tan \lambda\right) - \cos \delta \, \frac{dA}{dn}\right] = 0 \tag{122}$$

The vorticity relationships are:

$$\Omega_a = \frac{\partial v}{\partial t} - \frac{\partial A}{\partial n} = - T \frac{\partial s}{\partial a} \tag{123}$$

$$\Omega_n = \frac{\partial A}{\partial a} - \frac{\partial u}{\partial t} = - T \frac{\partial s}{\partial n} \tag{124}$$

$$\Omega_t = \frac{\partial v}{\partial a} - \frac{\partial u}{\partial n} = - T \frac{\partial s}{\partial t} \tag{125}$$

Using these with equation (121) yields:

$$\left(\frac{du}{da} \mp \tan\lambda \frac{\partial u}{\partial t}\right) - \frac{dv}{dn} + \frac{2}{\gamma - 1}\left[\cos\delta\left(\frac{dA}{da} \mp \frac{dA}{dt}\tan\lambda\right)\right.$$

$$\left. + \sin\delta\frac{dA}{dn}\right] = 0 \tag{126}$$

and equation (122) is:

$$\frac{du}{dn} + \frac{1}{\gamma - 1}\left[\sin\delta\left(\frac{dA}{da} \mp \tan\lambda\frac{dA}{dt} \pm \cos\delta\frac{dA}{dn}\right]\right. = T\frac{\partial s}{\partial t} \tag{127}$$

Equations (126) and (127) are in a form suitable for solution by finite difference techniques. The relationships may be further expanded to a form similar to those in the steady flow analysis and solved by the velocity matrix,

$$C_{1j}u_4 + C_{2j}v_4 + C_{3j}A_4 = C_{4j}$$

However, it may be more convenient to transform the velocity relationships to some other co-ordinate system (such as cylindrical or polar) to achieve unit velocity vectors.

The geometrical solution is quite similar to that of the steady with one major exception which is, λ must be used instead of μ.

DISCUSSION

The three independent variable method of characteristics has been converted to a digital computer solution of the flow field. The general method was programmed for use on the IBM 7090 computer. Preliminary results establish the solution as being valid for all values of velocity vectors above a Mach of 1.2. At this time, an intensive check-out program is underway to determine any possible singularities in the computer solution.

This report must be considered as completing development of the theory and the preliminary results of the check out. After the intensive check-out phase has been completed, the computer solution and results will be presented.

Further development of the theory and numerical solution will be presented at a later date. This will include the treatment of discontinuities, mixing phenomena, jet wakes, and exhaust plumes.

CONCLUSIONS

The three-independent variable method of characteristics may be utilized to solve problems of steady and unsteady flow regimes. The technique of finite differences may be utilized to effectively solve the potential equation in the characteristic surface.

The applications of this method may be extended to solve problems related to asymmetric flow fields, unsteady flow fields, and mixed flow fields for either inside flow or outside flow conditions. Thus, it may be possible to determine the flow characteristics of nozzles, wakes, re-entry bodies, and many other types of problems encountered in the field of gas dynamics.

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APPENDICES

APPENDIX A
PROOF OF THE THEOREM BY BJERKNES

Beginning with equation 31 which states that

$$\frac{d\Gamma}{dt} = - \iint_{\sigma} \left(\nabla \times T \frac{ds}{dr} \right) d\sigma \qquad \text{A-1}$$

we may transform the values of the curl of $T\nabla s$ by the following

$$\nabla \times T \frac{ds}{dr} = \nabla \times T \nabla s$$

The x-direction yields

$$i \left[\frac{\partial}{\partial y} \left(T \frac{ds}{dz} \right) - \frac{\partial}{\partial z} \left(T \frac{ds}{dy} \right) \right] = i \left[\frac{\partial T}{\partial z} \frac{ds}{dy} - \frac{\partial T}{\partial y} \frac{ds}{dz} \right] \qquad \text{A-2}$$

The y-direction gives

$$j \left[\frac{\partial}{\partial z} \left(T \frac{ds}{dx} \right) - \frac{\partial}{\partial x} \left(T \frac{ds}{dz} \right) \right] = j \left[\frac{\partial T}{\partial x} \frac{ds}{dz} - \frac{\partial T}{\partial z} \frac{ds}{dx} \right] \qquad \text{A-3}$$

The z-direction becomes

$$k \left[\frac{\partial}{\partial x} \left(T \frac{ds}{dy} \right) - \frac{\partial}{\partial y} \left(T \frac{ds}{dx} \right) \right] = k \left[\frac{\partial T}{\partial y} \frac{ds}{dx} - \frac{\partial T}{\partial x} \frac{ds}{dy} \right] \qquad \text{A-4}$$

By inspection, the equations become

$$\nabla \times T \frac{ds}{dr} = \nabla \times T \nabla s = \nabla T \times \nabla s \qquad \text{A-5}$$

and V. Bjerknes' theorem follows:

$$\frac{d\Gamma}{dt} = - \iint_{\sigma} (\nabla T \times \nabla s)_n \, d\sigma \tag{A-6}$$

APPENDIX B

Equation 78 states

$$(A^2 - u^2)\phi_{xx} + (A^2 - v^2)\phi_{yy} + (A^2 - w^2)\phi_{zz} - \phi_{tt} - 2uv\phi_{xy} - 2vw\phi_{yz}$$

$$- 2wu\phi_{zx} - 2u\phi_{xt} - 2v\phi_{yt} - 2w\phi_{zt} = 0 \tag{78}$$

Using equation 77 and substituting into 78 yields:

$$A^2 (\phi_{xx} + \phi_{yy} + \phi_{zz}) + \frac{A^2}{\rho}\frac{\partial \rho}{\partial t} + \frac{uA^2}{\rho}\frac{\partial \rho}{\partial x} + \frac{vA^2}{\rho}\frac{\partial \rho}{\partial y} + \frac{wA^2}{\rho}\frac{\partial \rho}{\partial z} = 0 \tag{B-1}$$

Along streamlines

$$A^2 = \left.\frac{\partial p}{\partial \rho}\right|_s$$

$$A^2 (\phi_{xx} + \phi_{yy} + \phi_{zz}) + \frac{1}{\rho}\left(\frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} + v\frac{\partial P}{\partial y} + w\frac{\partial P}{\partial z}\right) = 0 \tag{B-2}$$

Assuming ideal fluid and using Bernoulli's relation yields

$$\frac{dP}{\rho} = d\left(\frac{A^2}{\gamma - 1}\right) \tag{B-3}$$

therefore,

$$A^2 (\phi_{xx} + \phi_{yy} + \phi_{zz}) + \left(\frac{\partial A}{\partial t} + u\frac{\partial A}{\partial x} + v\frac{\partial A}{\partial y} + w\frac{\partial A}{\partial z}\right)\frac{2A}{\gamma - 1} = 0 \tag{B-4}$$

which is equation 82 in three dimensions and four independent variables.