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Elliptic Solution to the Emmons Problem

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The classical Emmons problem provides a well-defined geometry with analytical solutions that is relatively easy to establish experimentally. It has therefore been very useful for flammability assessment of materials. In this paper, the Emmons' problem is formulated in terms of an elliptic equation for the mixture fraction developing in a variable density elliptic flow field. Exact analytical solutions are developed for the mass and mixture fraction conservation equations in parabolic coordinates. The corresponding velocity field incorporates both the Emmons boundary layer result and an elliptic upstream influence that asymptotically satisfies the full Navier-Stokes equations. Thus the solution for the velocity field is exact everywhere outside the boundary layer. In the burning boundary layer, the error is small except in a small region O(20 Stokes lengths ~ 2mm) downstream of the leading edge where the velocity field is only qualitatively correct. However, the singularity at the leading edge is geometrical, and unlike the boundary layer solution, the singularity is confined to a point rather than the whole line x=0. This framework is used to analyze soot transport with generation and destruction. The soot model is also analytically tractable and seems to yield physically plausible results.

1. Introduction

The Emmons' problem is concerned with burning of a stationary flat fuel surface in an oxidizer stream flowing parallel to the surface. The primary objective is to determine the steady burning rate of a vaporizing liquid (or a subliming solid) that supports the flame in the boundary layer. The similarity solution with the boundary layer approximation was obtained by Emmons [1].

Recently, Atreya [2] developed a theory of opposed flow flame spread by using a parabolic coordinate system that allows a realistic model of a charring solid to be solved analytically while retaining the coupled elliptic nature of the classical flame spread problem. An analogous treatment of the gas phase may allow a more realistic description of the fluid mechanics and combustion than is included in [2]. If the solid is assumed to be a two dimensional flat surface, then the gas phase problem must be closely related to the famous "Emmons problem" mentioned above. It became clear that if the solution to the Emmons problem could be extended so that the boundary layer approximation could be removed, the extended solution would be exactly what was needed to complement Atreya's solution and remove the gas phase Oseen approximation.

In this paper, first the Emmons problem is reformulated in terms of an elliptic equation for the mixture fraction developing in a variable density elliptic flow field. It is shown that a suitable choice of coordinate systems allows the mass and mixture fraction conservation equations to be solved exactly. The velocity field incorporates both the Emmons boundary layer result and an elliptic upstream influence that asymptotically satisfies the full Navier-Stokes equations. This framework is used to analyze the de Ris [3] soot model, assuming the mixture fraction and velocity fields are determined by the solution described above. The original form of this model predicts unbounded growth of the soot mass fraction. A modification of the model is introduced whose general form can be justified on physical grounds. This model is also analytically tractable and seems to yield physically plausible results. Since only the general form of the model can be prescribed a priori, the physical constants needed to turn it into a useful tool for combustion and fire research problems have yet to be determined.

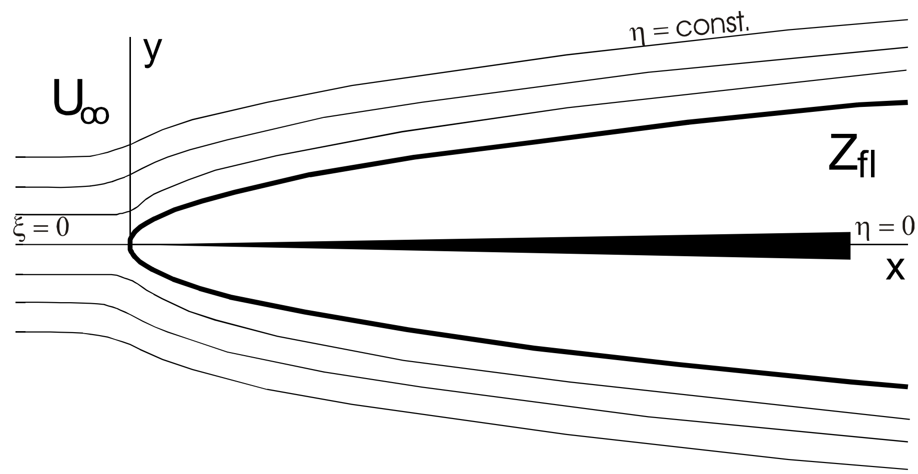

Figure 1: Schematic showing geometry of Emmons problem. The streamlines shown are calculated from the irrotational flow induced by the presence of the flame and boundary layer. The flame sheet denoted by Z_{fl} extends slightly upstream of the plate leading edge.

2. Mathematical Model

The starting point for the analysis is the geometry shown in Fig. 1. The simplest model capable of describing such a problem requires the introduction of a gas density $\rho(x,y)$, a two dimensional flow field $\vec{u}=(u,v)$ and a Mixture Fraction $Z(x,y)$ to account for mass and energy transport induced by the combustion processes. The conservation of mass, gaseous species, and energy can then be reduced to the solution of the following equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0 \qquad (1)$$

$$\rho u\frac{\partial Z}{\partial x}+\rho v\frac{\partial Z}{\partial y}=\frac{\partial}{\partial x}\left(\rho D\frac{\partial Z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\rho D\frac{\partial Z}{\partial y}\right) \qquad (2)$$

The mass conservation equation (1) can be satisfied exactly with the introduction of a stream function $\psi(x,y)$ defined as:

$$\frac{\partial \psi}{\partial y}=\rho u \quad ; \quad \frac{\partial \psi}{\partial x}=-\rho v \qquad (3)$$

If we now assume explicitly that all scalar physical variables can be related uniquely to the Mixture Fraction, then we can define a function *F(Z)* as follows:

$$F(Z) = \int_0^Z \frac{\rho(Z)}{\rho_\infty} \frac{D(Z)}{D_\infty} dZ \tag{4}$$

With a view to the introduction of other coordinate systems, the Mixture Fraction equation (2) can now be written in the form:

$$\frac{\partial(Z,\psi)}{\partial(x,y)} = \rho_\infty D_\infty \nabla^2 F \tag{5}$$

We next introduce dimensionless parabolic coordinates ξ , η as follows: Let U_∞ be the ambient wind speed in a coordinate system moving with the flame front, and let ν_∞ be the kinematic viscosity evaluated at ambient temperature. Then:

$$\xi + i\eta = \sqrt{\frac{U_\infty}{\nu_\infty}(x+iy)} \tag{6}$$

The parabolic coordinates can be expressed explicitly as follows:

$$\xi = \sqrt{\frac{U_\infty}{2\nu_\infty}} \frac{y}{|y|} \left(x + \sqrt{x^2+y^2} \right)^{1/2} \qquad \eta = \sqrt{\frac{U_\infty}{2\nu_\infty}} \left(-x + \sqrt{x^2+y^2} \right)^{1/2} \tag{7}$$

Because the mapping from (x, y) to (ξ, η) is conformal, the representation of equation (5) in (ξ, η) coordinates becomes:

$$\frac{\partial(Z,\psi)}{\partial(\xi,\eta)} = \rho_\infty D_\infty \left(\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2} \right) \tag{8}$$

Up to this point the treatment is completely general. We now make the crucial assumption that permits an analytical treatment of the gas phase problem. We *postulate* that the stream function can be written in the following form:

$$\psi = \mu_\infty \xi f(\eta) \tag{9}$$

Given this postulate, we note that equation (8) has solutions of the form $Z = Z(\eta)$. Then, dispensing with the function *F*, equation (8) can be reduced to an ordinary differential equation.

$$f(\eta)\frac{dZ}{d\eta} + \frac{1}{Sc}\frac{d}{d\eta}\left(\frac{\rho(Z)}{\rho_\infty} \frac{D(Z)}{D_\infty} \frac{dZ}{d\eta} \right) = 0 \tag{10}$$

Here, $Sc = \nu_\infty / D_\infty$ is the Schmidt number based on ambient conditions. Note that the assumption that all scalar quantities can be expressed as function of *Z* implies that the Lewis number is one. This means that Sc can be replaced by the Prandtl number $\Pr = \mu C_P / k$ as needed. We will return to this point later. Note that the boundary layer approximation has *not* been made, and that any solutions obtained are thus exact solutions to the full elliptic equations.

The boundary conditions are determined by the fact that all scalar quantities are related to the Mixture Fraction by algebraic "state relations". We require that the temperature $T(x,y)=T_S$ at the burning gas-surface interface $x \geq 0$. We further assume that the gas at the surface is composed of either inert or combustible products of pyrolysis. The state relations will be constructed so that these conditions are consistent with values of the state relations corresponding to $Z = 1$. Similarly, far from the burning fuel surface the gas is assumed to be ambient air moving at a speed U_∞ parallel to the surface. Again, the state relationships will be constructed so that this corresponds to $Z = 0$. Since Z depends only on η, any physical quantity related to Z by state relations will satisfy a zero normal gradient boundary condition for $x \leq 0$. Thus, to summarize, the boundary conditions on Z can be expressed mathematically as:

$$Z(0)=1 \qquad \lim_{\eta\to\infty} Z(\eta)=0 \tag{11}$$

The functional form assumed for ψ is trivially true for a uniform flow for which $f(\eta)=2\eta$. However, by combining the parabolic coordinates with the Howarth transformation a much more interesting result can be obtained. We can anticipate that the velocity field should be roughly similar to that calculated by Emmons in his analysis of a burning flat plate [1]. To see this note that for $x >> y$, the parabolic coordinates can be approximated by:

$$\xi \approx \sqrt{\frac{U_\infty x}{\nu_\infty}} \qquad \eta \approx \frac{y}{2}\sqrt{\frac{U_\infty}{\nu_\infty x}} \tag{12}$$

This "boundary layer" region always exists for a semi-infinite domain. In this region, the x-momentum equation takes the approximate form:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}\right)=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{13}$$

We now choose new coordinates (ξ_1,ς) defined as follows:

$$\xi_1=\sqrt{\frac{U_\infty x}{\nu_\infty}} \qquad \varsigma=\frac{1}{2}\sqrt{\frac{U_\infty}{\nu_\infty x}}\int_0^y \frac{\rho(Z)}{\rho_\infty}dy \tag{14}$$

These coordinates are simply the boundary layer approximation to the parabolic coordinates with the y coordinate replaced by a Howarth transformed variable. However, the second crucial point of the present analysis is the observation that in the boundary layer region, ζ and η are uniquely related as follows:

$$\varsigma=\int_0^\eta \frac{\rho(\eta)}{\rho_\infty}d\eta \tag{15}$$

We now define the variable ζ everywhere by equation (15). With this definition, ζ is a unique function of the parabolic coordinate. We now choose $f(\eta)$ as the solution of the boundary layer equation (13) to get:

$$f(\varsigma)f''(\varsigma)+\frac{d}{d\varsigma}\left(\frac{\rho\mu}{\rho_\infty\mu_\infty}f''(\varsigma)\right)=0 \tag{16}$$

The boundary conditions require that at the burning surface the tangential velocity component must vanish while the normal component of the velocity (the "blowing" velocity) is proportional to *1/ξ*. Far from the burning surface the uniform parallel flow with speed U_∞ is recovered. Mathematically, these conditions take the form:

$$f(0)=-m \qquad f'(0)=0 \qquad \lim_{\varsigma\to\infty} f'(\varsigma)=2 \tag{17}$$

Here, *m* is a dimensionless parameter describing the magnitude of the blowing velocity.

With appropriate assumptions about the temperature dependence of the viscosity, this is essentially the problem solved by Emmons [1]. However, the choice of ζ as independent variable yields additional benefits. Since ζ and η are uniquely related, we have a representation of the velocity field valid almost everywhere. Indeed, Goldstein [4] has shown that for the incompressible flow past a flat plate, using parabolic coordinates ensures that the constant property Navier-Stokes equations are satisfied in the boundary layer with errors $O(1/\xi)$. Moreover, the solution is exact far from the plate. This property arises from the fact that the asymptotic form of the solution for large η is an irrotational flow which exactly satisfies the constant property Navier-Stokes equations.

In the present case an analogous situation arises. For large values of ζ, we have $\eta\approx\varsigma-a, f\approx 2\varsigma-b$, where *a* and *b* are constants depending on the dimensionless physical parameters that define the specific material and flow properties of interest. Since the fluid properties become uniform as $\varsigma\to\infty$, the solution again captures the flow induced by the presence of the boundary layer as an exact solution of the Navier-Stokes equations. In fact, for large η, the stream function takes the asymptotic form:

$$\psi = 2\mu_\infty\xi(\eta-c) \tag{18}$$

The term in equation (18) proportional to the positive constant *c* represents the irrotational flow induced by the presence of the boundary layer. Since the irrotational flow of a constant property fluid is also a solution of the Navier Stokes equations, the solution is exact everywhere outside the boundary layer. In the burning boundary layer, the error is still $O(1/\xi)$. For the constant density flat plate problem, Goldstein [4] estimates the solution to be a useful approximation down to values of $\xi^2 \sim 20$. This corresponds to a distance of 20 Stokes lengths downstream from the leading edge. Thus, there is a small region downstream of the leading edge in the boundary layer where the velocity field is only qualitatively correct. The singularity at the leading edge is geometrical, due to the assumption of an infinitely thin plate. However, unlike the boundary layer solution, the singularity is confined to a point rather than the whole line x = 0. The constant *c* in equation (18) can be evaluated in terms of the solutions outlined in the following section. In fact, it is easy to see that:

$$2c = m+\int_0^\infty (2-f'(\eta))\,d\eta = m+\int_0^\infty \left(2\frac{\rho_\infty}{\rho}-f'(\varsigma)\right)d\varsigma \tag{19}$$

Thus, the disturbance to the uniform flow far from the plate is determined by the solution to the combustion problem in the boundary layer, as it should.

Finally, the Mixture Fraction equation (10) is recast with ζ as the independent variable. The result is:

$$f(\varsigma)Z'(\varsigma)+\frac{1}{Sc}\frac{d}{d\varsigma}\left(\frac{\rho^2 D}{\rho_\infty^2 D_\infty}Z'(\varsigma)\right)=0 \tag{20}$$

The boundary conditions remain the same as those presented in equation (11) but with η replaced by ζ. Again, note that solutions to equation (20) satisfy the full elliptic mixture fraction equation. Thus, the "thermal" part of the problem is solved exactly using a velocity field that is asymptotically exact for large ξ and any η, as well as, large η and any ξ.

3. Solutions

3.1 Boundary layer functions

The gas phase problem has been reduced to the solution of two ordinary differential equations. These solutions depend indirectly on all the parameters discussed by Atreya [2]. However, these parameters enter the gas phase analysis only through the blowing parameter m, the Schmidt number Sc, and the state relations. The analysis can be greatly simplified by defining the temperature dependence of the viscosity μ and diffusivity D such that:

$$\rho\mu=\rho_\infty\mu_\infty \qquad \rho^2 D=\rho_\infty^2 D_\infty \tag{21}$$

With these choices the differential equations take the form:

$$f'''(\varsigma)+f(\varsigma)f''(\varsigma)=0 \tag{22}$$

$$Z''(\varsigma)+Sc\, f(\varsigma)Z'(\varsigma)=0 \tag{23}$$

The solution for Z satisfying $Z(0) = 1$ that vanishes as $\zeta\to\infty$ is readily found to be:

$$Z(\varsigma,\, Sc)=1-g(\varsigma,\, Sc)/g(\infty,\, Sc); \qquad g(\varsigma,\, Sc)=\int_0^{\varsigma}[f''(t)]^{Sc}\,dt \tag{24}$$

The problem has now been reduced to the solution of the Blasius equation (22) with boundary conditions given by equation (17). Since the range of physically permissible values of m lies in the range $0 \le m \le 1.239$, a table of such solutions (or more realistically a program for generating them efficiently) completes the analysis of the gas phase equations. Since these solutions are easy to generate and the analytical properties of the solutions are well known, representations in terms of this function can be considered analytical solutions with as much justification as one obtained in terms of any of the higher transcendental functions of mathematical physics.

There are two further issues that warrant discussion, the dependence on the Schmidt number and the development of the state relations. While it is tempting to simply put $Sc = 1$ and reduce the whole gas phase solution to the Blasius function, that seems to be an unwise simplification. Putting $Sc = Pr$, where Pr is the Prandtl number is more consistent with the assumptions underlying the analysis and more realistic as well. The additional quadrature needed to obtain Z for other values of Sc is a small price to pay. Second, the conversion of the results into the physical coordinates requires an unscrambling of the relation between ζ and η given by equation (15). The easiest way to proceed is to postulate a piecewise linear relationship between the dimensionless specific volume ρ_∞/ρ and Z (see Baum et al. [5]).

The starting point is the definition of the Mixture Fraction in terms of the combustible pyrolyzate mass fraction Y_f and the oxygen mass fraction Y_O.

$$Z = \frac{Y_f S - \left(Y_o - Y_o^\infty\right)}{\chi S + Y_o^\infty} \tag{25}$$

Here, χ is the fraction of the pyrolyzate leaving the surface that is combustible, and $S = (\nu_o m_o)/(\nu_f m_f)$ is the stoichiometrically weighted molecular weight ratio. Since the fuel and oxidizer must vanish at the flame sheet, its location $\zeta = \zeta_f$ is found from the condition:

$$Z(\varsigma_{fl}) = \frac{Y_o^\infty}{\chi S + Y_o^\infty} = Z_{fl} \tag{26}$$

The postulated state relation for the specific volume then can be written in terms of the pyrolyzate density at the surface ρ_p and the flame density ρ_{fl} in the form:

$$\frac{\rho_\infty}{\rho} = 1 + \left(\frac{\rho_\infty}{\rho_{fl}} - 1\right)\frac{Z}{Z_{fl}} \qquad 0 \le \frac{Z}{Z_{fl}} \le 1 \tag{27}$$

$$\frac{\rho_\infty}{\rho} = \frac{\rho_\infty}{\rho_p} + \left(\frac{\rho_\infty}{\rho_{fl}} - \frac{\rho_\infty}{\rho_p}\right)\frac{1-Z}{1-Z_{fl}} \qquad \frac{Z}{Z_{fl}} \ge 1 \tag{28}$$

Note that ρ_p can be easily related to the surface temperature T_s using the equation of state since by hypothesis only pyrolyzate is present in the gas adjacent to the surface. Similarly, the flame density can be found by observing that the state relations for the nitrogen mass fraction Y_n and the non-combustible pyrolyzate mass fraction Y_p take the form:

$$Y_n = Y_n^\infty\left(1 - Z\right) \qquad Y_p = \left(1 - \chi\right)Z \tag{29}$$

Since the only other species present at the flame is the "combustion product", knowledge of Z_{fl} and the flame temperature yields the flame density. Let Y_c be the mass fraction of the combustion products. Then, evaluating Y_c at the flame yields:

$$Y_c(\varsigma_{fl}) = \frac{\chi Y_o^\infty (1+S)}{\chi S + Y_o^\infty} \tag{30}$$

We defer any detailed prescription for the state relations of the reactive species entering the problem. However, we continue to assume that suitable relations of the form $Y_f(Z)$, $Y_o(Z)$ and $Y_c(Z)$ are available. Note that the temperature state relation can then be found from the equation of state as:

$$\frac{T}{T_\infty} = \frac{\rho}{\rho_\infty}(Z)\left(\frac{Y_o^\infty}{m_o} + \frac{Y_n^\infty}{m_n}\right)\left(\sum_\alpha \frac{Y_\alpha(Z)}{m_\alpha}\right)^{-1} \tag{31}$$

Since the flame density is to be determined from the equation of state and knowledge of the flame temperature, the flame temperature can be adjusted if desired to account for radiative losses. Thus, although radiative transport is not included explicitly in this model, some of its consequences can be accounted for.

The parabolic coordinate $\eta(\zeta)$ can now be readily found. The definition of ζ, equation (15) can be rewritten as:

$$\eta = \int_0^\varsigma \frac{\rho_\infty}{\rho}(t)\,dt \tag{32}$$

The integral can be expressed in terms of the parameters introduced above and a new function $h(\zeta)$ defined as:

$$h(\varsigma) = \int_0^\varsigma t[f''(t)]^{Sc}\,dt \tag{33}$$

The expression for $\eta(\zeta)$ then takes the following form for $0 \le \zeta \le \zeta_{fl}$:

$$\eta = \frac{\rho_\infty}{\rho_p}\varsigma + \left(\frac{\rho_\infty}{\rho_{fl}} - \frac{\rho_\infty}{\rho_p}\right)\frac{1}{1-Z_{fl}}\left[\varsigma(1-Z(\varsigma)) - \frac{h(\varsigma)}{g(\infty)}\right] \tag{34}$$

Thus, the location of the flame in physical coordinates is given by:

$$\eta_{fl} = \frac{\rho_\infty}{\rho_{fl}}\varsigma_{fl} - \left(\frac{\rho_\infty}{\rho_{fl}} - \frac{\rho_\infty}{\rho_p}\right)\left(\frac{1}{1-Z_{fl}}\right)\left[\frac{h(\varsigma_{fl})}{g(\infty)}\right] \tag{35}$$

Finally, when $\zeta \ge \zeta_{fl}$:

$$\eta = \eta_{fl} + (\varsigma - \varsigma_{fl}) + \left(\frac{\rho_\infty}{\rho_{fl}} - 1\right)\left[\varsigma\frac{Z(\varsigma)}{Z_{fl}} - \varsigma_{fl} + \frac{1}{Z_{fl}}\frac{1}{g(\infty)}(h(\varsigma) - h(\varsigma_{fl}))\right] \tag{36}$$

At this stage of the analysis all the information required to obtain the velocity, density, and mixture fraction everywhere in the flow field has been presented. In particular, the location of the flame sheet can be determined without further information. The remaining unknown hydrodynamic variable is the pressure perturbation, which cannot be determined from the boundary layer analysis. However, the asymptotic flow field induced by the presence of the boundary layer permits an estimate of the pressure field using Bernoulli's equation. We now examine this asymptotic flow in more detail.

3.2 **Asymptotic Flow Field**

The starting point is the asymptotic form of the stream function given by equation (18). Examination of this result shows that the $\psi = 0$ streamline has two branches; determined by the curves $\xi = 0$ and $\eta = c$ respectively. The first branch corresponds to the negative x axis, while the second is a parabola whose equation is:

$$\left(\frac{U_\infty}{2c\nu_\infty}\right)^2 y^2 - c^2 = \left(\frac{U_\infty}{2c\nu_\infty}\right)x \tag{37}$$

Thus, the asymptotic field corresponds to the irrotational flow past a fictitious bluff body with a stagnation point located at $x = -(\nu_\infty / U_\infty)\, c^2$. As noted earlier, since an irrotational flow is an exact solution of the constant property Navier-Stokes equations, the solution in the far field is both exact and fully elliptic with an upstream influence manifested in part by the location of the inviscid stagnation point *upstream* of the plate.

The velocity components far from the body can be readily found by differentiating the asymptotic form of the stream function. Thus:

$$\frac{\partial \psi}{\partial x} = -2\,\mu_\infty\, c\,\frac{\partial \xi}{\partial x} \qquad \frac{\partial \psi}{\partial y} = \rho_\infty\,\mu_\infty - 2\,\mu_\infty\, c\,\frac{\partial \xi}{\partial y} \tag{38}$$

Using the Cauchy-Riemann equations and the definition of the parabolic coordinates given in equation (6):

$$\frac{\partial \xi}{\partial x} - i\frac{\partial \xi}{\partial y} = \frac{\xi - i\eta}{2\sqrt{x^2 + y^2}} \tag{39}$$

The asymptotic velocity components then take the form:

$$u = U_\infty - \frac{\nu_\infty c\,\eta}{\sqrt{x^2+y^2}} \qquad v = \frac{\nu_\infty c\,\xi}{\sqrt{x^2+y^2}} \tag{40}$$

Finally, the pressure distribution is determined from Bernoulli's equation.

$$\frac{p - p_\infty}{\rho_\infty U_\infty^2} = \frac{\nu_\infty c}{2U_\infty}\frac{(2\eta - c)}{\sqrt{x^2+y^2}} = \frac{c}{2}\frac{(2\eta - c)}{(\eta^2 + \xi^2)} \tag{41}$$

Note that the boundary layer solution for this problem yields $p = p_\infty$. Thus, the solution given in equation (41) is the first approximation to the pressure everywhere. It is too small a perturbation to affect the solution inside the boundary layer, while outside the boundary layer it is exact.

4. Cross Flow

The solution described above assumes that the flow field contains two components of velocity, directed parallel and normal to the plate and lying in a plane containing the line that generates the plate. However, it is possible to extend the solution to contain a third component of the velocity w, which is normal to the plane. This velocity component must depend only on the two coordinates (x, y). However, there is no a priori restriction on the magnitude of the velocity. The solution obtained is exact in the sense that the full elliptic momentum equation in the direction normal to the plane (the "cross-flow") is satisfied without approximation.

The starting point for the analysis is the cross-flow momentum equation, which takes the following form for a flow depending on only the in-plane coordinates (x, y):

$$\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}\right)=\frac{\partial}{\partial x}\left(\mu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right) \tag{42}$$

Using the stream function, this can also be written in the Jacobian form:

$$\frac{\partial(w,\psi)}{\partial(x,y)}=\frac{\partial}{\partial x}\left(\mu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right) \tag{43}$$

The cross-flow velocity component *w* is now assumed to have the form:

$$w=\alpha W_\infty W(\eta) \tag{44}$$

Here, W_∞ is the prescribed value of the cross-flow velocity component far from the plate, and α is a constant to be determined. Then, using equations (9) and (44), the left hand side of equation (43) can be written as:

$$\frac{\partial(w,\psi)}{\partial(x,y)}=-\mu_\infty f(\eta)\,\alpha\,W_\infty\frac{dW}{d\eta}\frac{\partial(\xi,\eta)}{\partial(x,y)} \tag{45}$$

Similarly, using equation (44) again and the fact that $\eta(x, y)$ is a solution of Laplace's equation, the cross-flow equation takes the form:

$$-\mu_\infty f(\eta)\,\alpha\,\frac{dW}{d\eta}\frac{\partial(\xi,\eta)}{\partial(x,y)}=\left(\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2\right)\frac{d}{d\eta}\left(\mu\frac{dW}{d\eta}\right) \tag{46}$$

Finally, using the Cauchy-Riemann equations once more, the following equation for $W(\eta)$ is obtained:

$$f(\eta)\frac{dW}{d\eta}+\frac{d}{d\eta}\left(\frac{\mu}{\mu_\infty}\frac{dW}{d\eta}\right)=0 \tag{47}$$

Rewriting this equation in terms of the variable ζ defined in equation (15), we obtain:

$$f(\varsigma)\frac{dW}{d\varsigma}+\frac{d}{d\varsigma}\left(\frac{\rho\mu}{\rho_\infty\mu_\infty}\frac{dW}{d\varsigma}\right)=0 \tag{48}$$

Comparing this equation with equation (16), it is easy to see that the choice $\alpha = 1/2$ implies that $W(\zeta)$ must follow the same boundary conditions as the function $f'(\varsigma)$ and that equations (48) and (16) are identical. Hence, the exact solution for the cross-flow velocity component is:

$$w=\frac{W_\infty}{2}\frac{df}{d\varsigma} \tag{49}$$

5. Soot Transport

A suitable soot model in the present context is taken to mean one that properly accounts for the transport of soot in a combustion environment, with simple empirically prescribed sources and sinks of particulate matter. Such a model is entirely compatible with the analysis of the Emmons problem described above. In particular, we assume that the dominant transport processes are advection by the bulk velocity and thermophoretic motion down the combustion induced temperature gradients. The soot generation rate $\dot{m}_g$ and the destruction or oxidation rate $\dot{m}_d$ are empirically related to the mixture fraction. The result is a single *linear* equation for the soot mass fraction Y_s.

$$\rho\left(u\frac{\partial Y_s}{\partial x}+v\frac{\partial Y_s}{\partial y}\right)-\frac{\partial}{\partial x}\left(\rho Y_s\frac{K\nu}{T}\frac{\partial T}{\partial x}\right)-\frac{\partial}{\partial y}\left(\rho Y_s\frac{K\nu}{T}\frac{\partial T}{\partial y}\right)=\dot{m}_g-\dot{m}_d \tag{50}$$

Here, $K = 0.55$ is the coefficient in the expression for the thermophetic velocity $\vec{v}_T$ which has the general vector form:

$$\vec{v}_T=-\frac{K\nu}{T}\nabla T \tag{51}$$

Since the analysis will ultimately be performed in the parabolic coordinate system, we note that equation (50) can be written in the following form:

$$\frac{\partial(Y_s,\psi)}{\partial(\xi,\eta)}-Y_s\frac{d^2H}{d\eta^2}-\frac{dH}{d\eta}\frac{\partial Y_s}{\partial\eta}=\frac{\partial(x,y)}{\partial(\xi,\eta)}(\dot{m}_g-\dot{m}_d) \tag{52}$$

$$Where,\quad H=\int_{T_\infty}^{T}K\frac{\mu(T)}{T}dT \tag{53}$$

Note that the function H depends only on $Z(\eta)$. Equation (52) is completely general in the sense that Y_s can depend on both ξ and η, and holds for any form of the soot formation and oxidation terms. The only assumption is that the flow and thermal environment is prescribed by the Emmons problem.

We now turn to an analysis of the de Ris [3] model. The model can be written for the present purposes in the following general form:

$$\dot{m}_g=F_g(Z)\qquad\qquad \dot{m}_d=F_d(Z)Y_s \tag{54}$$

The most important feature of the model outside its general form is the fact that both F_g and F_d are sharply peaked in the mixture fraction space, with the oxidation term peaking at lower values of Z than the generation term. Since the mixture fraction decreases with increasing η as one moves away from the burning surface, the soot oxidation region lies outside the soot generation region. In the de Ris model, both these regions are between the flame and the burning surface. Since the resultant convective and thermophoretic velocities carry the soot from the oxidation zone towards the soot generation zone, the only way the oxidation term can play a role in this geometry is if there is soot in the ambient flow. Given the linearity of the soot transport equation, this possibility can be considered separately without loss of generality.

Now, we replace η with ζ as one of the independent variables, dispense with the function H, and evaluate the Jacobian of the transformation as follows:

$$\frac{\partial(x,y)}{\partial(\xi,\eta)}=\left(\frac{2\nu_\infty}{U_\infty}\right)^2\left(\xi^2+\eta^2\right) \tag{55}$$

The soot transport equation now takes the form:

$$\xi \frac{df}{d\varsigma}\frac{\partial Y_s}{\partial \xi} - f(\varsigma)\frac{\partial Y_s}{\partial \varsigma} - \frac{\partial}{\partial \varsigma}\left(Y_s \frac{d\Theta}{d\varsigma} \right) = \frac{4\nu_\infty}{\rho_\infty U_\infty^2}\frac{\rho_\infty}{\rho}(\xi^2+\eta^2)\; F_g(Z) \quad (56)$$

$$\Theta = K \log\left(\frac{T}{T_\infty} \right) \quad (57)$$

The only boundary condition that can be applied to this first order partial differential equation is that $Y_s \to 0$ as $\eta \to \infty$. This enforces the condition that there is no soot in the ambient environment. All the soot in the domain is generated in the soot formation region. The particulate reaching the surface $\eta = 0$ is assumed to be deposited there. The de Ris model represents the function F_g as a product $\dot{m}_{g,\max}$ times a dimensionless quantity labeled here as $\tilde{F}_g$. The soot mass fraction can then be written without approximation as:

$$Y_s = \left(4\nu_\infty \dot{m}_{g,\max} \right)\left(\rho_\infty U_\infty^2 \right)^{-1}\left(\tilde{Y}_s^{(0)}(\varsigma) + \xi^2 \tilde{Y}_s^{(1)}(\varsigma) \right) \quad (58)$$

The functions $\tilde{Y}_s^{(0)}$ and $\tilde{Y}_s^{(1)}$ are solutions to the following equations, subject to the boundary condition that they both vanish as $\zeta \to \infty$.

$$f(\varsigma)\frac{d\tilde{Y}_s^{(0)}}{d\varsigma} + \frac{d}{d\varsigma}\left(\tilde{Y}_s^{(0)} \frac{d\Theta}{d\varsigma} \right) = -\eta^2 \tilde{F}_g(Z) \quad (59)$$

$$f(\varsigma)\frac{d\tilde{Y}_s^{(1)}}{d\varsigma} + \frac{d}{d\varsigma}\left(\tilde{Y}_s^{(1)} \frac{d\Theta}{d\varsigma} \right) - 2\frac{df}{d\varsigma}\tilde{Y}_s^{(1)} = -\tilde{F}_g(Z) \quad (60)$$

At this point the difficulties posed by this model become clear. While the functions $\tilde{Y}_s^{(0)}$ and $\tilde{Y}_s^{(1)}$ are well behaved, the functional dependence of the solutions on ξ shown in equation (58) means that the soot concentrations in the boundary layer region of the flow become unbounded as $\xi \to \infty$. The problem arises from the fact that the maximum value of the soot formation rate is not connected to the fuel consumption rate. Since the soot formation rate is typically a small fraction of the fuel consumption rate, and can certainly never exceed it, the absence of a coupling between these two quantities is disastrous when applied to the Emmons problem. On the other hand, the de Ris model possesses about the right level of complexity for use in fire and other complex combustion simulations. A viable model should not require an elaborate set of new equations to solve, and it should be amenable to analysis in simple situations. The next section contains a proposal for a modified version of the de Ris model that could be the basis for future work.

6. Modified Soot Model

The idea behind the model presented below is very simple. The soot formation process requires fuel to be consumed, so the soot formation rate should be directly tied to the fuel consumption rate. Moreover, since the gaseous constituents are functions of the mixture fraction, the local fuel consumption rate can be represented in terms of the mixture fraction. Similarly, the oxidation of soot imposes two requirements. First, there must be soot present to be oxidized. Second, the oxidation of soot requires a depletion of the oxygen supply. Again, the existence of a state relation for oxygen allows the oxygen consumption rate to be related to the mixture fraction. The result is recipes for the soot generation rate $\dot{m}_g$ and oxidation rate $\dot{m}_d$ that take the following form:

$$\dot{m}_g = \beta \dot{m}_f \qquad\qquad \dot{m}_d = \lambda Y_s \dot{m}_o \tag{61}$$

Here, $\dot{m}_f$ and $\dot{m}_o$ are the fuel and oxygen consumption rates respectively. The quantities β and λ are empirical parameters which may depend on Z. β can be interpreted physically as the fraction of the fuel converted to soot, while λ is a measure of the efficiency the soot oxidation mechanism. Both of these parameters are assumed to be fuel dependent.

Clearly, this model is designed to be similar to the de Ris model, and has about the same physical content. The close resemblance is maintained by choice, since (as noted above) that model represents a good balance between the conflicting goals of physical realism and mathematical complexity. When the soot rate processes are coupled to the fuel and oxygen consumption rates, the unphysical behavior noted in the previous section disappears. In fact, the solution for Y_s becomes much simpler with the modified model. To see this, it is necessary to express the fuel and oxidizer consumption rates in terms of the mixture fraction. Since this model will eventually be applied to three dimensional transient problems, the relationship will be derived for the more general case.

First consider the fuel consumption rate. The equations for the fuel mass fraction and mixture fraction can be written in the following form:

$$\rho\left(\frac{\partial Y_f}{\partial t} + \vec{u}\cdot\nabla Y_f\right) - \nabla\cdot\left(\rho D \nabla Y_f\right) = -\dot{m}_f \tag{62}$$

$$\rho\left(\frac{\partial Z}{\partial t} + \vec{u}\cdot\nabla Z\right) - \nabla\cdot\left(\rho D \nabla Z\right) = 0 \tag{63}$$

Now, assuming that $Y_f = Y_f(Z)$ and using the mixture fraction equation, the following expression for the fuel consumption rate is obtained.

$$\dot{m}_f = \frac{d^2 Y_f}{dZ^2}\rho D\left(\nabla Z\right)^2 \tag{64}$$

A similar analysis using the oxygen mass fraction equation yields the corresponding expression for the oxygen consumption rate.

$$\dot{m}_o = \frac{d^2 Y_o}{dZ^2}\rho D\left(\nabla Z\right)^2 \tag{65}$$

Note that since the idealized state relation is a piecewise linear function, the second derivatives appearing in equations (64) and (65) would be Dirac delta functions if theses forms were used. In fact, the piecewise linear state relations yield:

$$\frac{d^2 Y_f}{dZ^2} = \frac{Y_o^\infty}{S\, Z_{fl}}\,\delta\left(Z - Z_{fl}\right) \qquad\qquad \frac{d^2 Y_o}{dZ^2} = \frac{Y_o^\infty}{Z_{fl}}\,\delta\left(Z - Z_{fl}\right) \tag{66}$$

However, if the delta functions are replaced by more general functions that are sharply peaked in mixture fraction space (Gaussian distributions for example) with peaks shifted slightly with respect to the flame sheet, then functional forms very similar to those appearing in the de Ris model are obtained. For the present, we will defer any specific choice of functional form and return to the solution for the Emmons problem.

In order to completely specify the soot transport equation (52) in parabolic coordinates, it is necessary to evaluate $(\nabla Z)^2$. Again using the Cauchy-Riemann equations it is easy to show that:

$$(\nabla Z)^2 = \left(\frac{dZ}{d\eta}\right)^2 \frac{\partial(\xi, \eta)}{\partial(x, y)} \tag{67}$$

Finally, replacing η with the Howarth transformed variable ζ and using the Emmons problem stream function together with the assumptions $\rho\,\mu = \rho_\infty\,\mu_\infty$ and $\rho^2 D = \rho^2 D_\infty$, equation (52) takes the form:

$$\left(f + \frac{d\Theta}{d\varsigma}\right)\frac{dY_s}{d\varsigma} + Y_s \frac{d^2\Theta}{d\varsigma^2} + \frac{1}{Sc}\left(\frac{dZ}{d\varsigma}\right)^2 \left(\beta \frac{d^2 Y_f}{dZ^2} - \lambda \frac{d^2 Y_o}{dZ^2} Y_s\right) = 0 \tag{68}$$

Note that equation (68) is a first order linear ordinary differential equation. The only boundary condition required is that the ambient soot mass fraction is specified. Let this value be Y_s^∞. The boundary condition then becomes:

$$\lim Y_s(\varsigma \to \infty) = Y_s^\infty \tag{69}$$

The solution to equation (68) can be written as a sum of two terms. The homogenous contribution only involves the oxidation of the ambient soot. Denoting this component of the solution as Y_s^d, the oxidation process can be described as a product of two factors. A transport function Y_{tr} describes the evolution of the soot mass fraction under the influence of convective and thermophoretic transport. An oxidation function Y_{ox} accounts for the destruction of the soot as it passes through the flame region. Thus:

$$Y_s^d = Y_s^\infty Y_{tr}(\varsigma) Y_{ox}(\varsigma) \tag{70}$$

$$Y_{tr} = \exp\left(\int_\varsigma^\infty \left[\frac{d^2\Theta}{d\varsigma^2}\left(f + \frac{d\Theta}{d\varsigma}\right)^{-1}\right] d\varsigma\right) \tag{71}$$

$$Y_{ox} = \exp\left(-\frac{\lambda}{Sc}\int_\varsigma^\infty \left(\frac{dZ}{d\varsigma}\right)^2 \frac{d^2 Y_o}{dZ^2}\left(f + \frac{d\Theta}{d\varsigma}\right)^{-1} d\varsigma\right) \tag{72}$$

Note that the function Y_{tr} depends only on the solution to the Emmons problem and is independent of any model for soot oxidation. Thus, the transport role played by the flame is to control the thermal environment that determines the convective and thermophoretic velocities acting on the particulate matter. If the flame was chemically inert with respect to the soot, this would completely determine the soot distribution everywhere, including the rate at which soot was deposited on the surface. This type of problem has been studied in detail by Batchelor and

Shen [6], who proposed a general relation between the soot deposition rate and the convective heat transfer to any surface. Unfortunately, their analysis does not apply to a system with combustion and large temperature differences.

The soot oxidation function Y_{ox} is exponentially dependent on both the oxidative efficiency parameter λ and the oxygen state relation. The physical content of this function can be understood more clearly if we take advantage of the simplification offered by assuming that the second derivative of this state relation is sharply peaked about a point ζ_{ox} which corresponds to point Z_{ox} in mixture fraction space. This assumption is entirely consistent with the Gaussian form used in the de Ris model to represent soot formation. It is not necessary to assume a particular form for this function. However, we are implicitly assuming there is a small dimensionless parameter analogous to the parameter σ in equation (5.3) of [3]. Under these circumstances, the argument that leads to Watson's Lemma may be applied here. Since the mixture fraction and the net particulate convection velocity $f+\Theta'(\varsigma)$ are smoothly varying functions of ζ, the integral in the expression for Y_{ox} can be approximated by evaluating all terms except $Y_o''(Z)$ at ζ_{ox} and carrying out the remaining integration. The result is:

$$Y_{ox} = \exp\left(-\frac{\lambda}{Sc}\frac{dZ}{d\varsigma}(\varsigma_{ox}) \left(f + \frac{d\Theta}{d\varsigma} \right)^{-1} (\varsigma_{ox}) \left(\frac{dY_o}{dZ}(Z=0) - \frac{dY_o}{dZ}(Z) \right) \right) \tag{73}$$

Finally, note that whatever smoothed form is ultimately used for the state relation $Y_o(Z)$, its slope at $Z = 0$ must take the classical value $-Y_o^{\infty}/Z_{fl}$. If we also use the fact that both $Z'(\varsigma)$ and $Y_o'(Z)$ are negative for the Emmons problem, the expression for Y_{ox} can be written in the somewhat more transparent form:

$$Y_{ox} = \exp\left(-\frac{\lambda}{Sc}\left|\frac{dZ}{d\varsigma}(\varsigma_{ox})\right| \left(f + \frac{d\Theta}{d\varsigma} \right)^{-1} (\varsigma_{ox}) \left(\frac{Y_o^{\infty}}{Z_{fl}} - \left|\frac{dY_o}{dZ}(Z)\right| \right) \right) \tag{74}$$

Equation (74) can be interpreted as follows: The oxidation of the ambient soot takes place in a narrow but finite width region centered about the curve $\zeta = \zeta_{,ox}$. The function Y_{ox} makes a transition from the value $Y_{ox} = 1$ outside the oxidation zone where only soot transport occurs, to a limiting value inside the zone where the residual fraction of the ambient soot Y_{res} can be expressed as:

$$Y_{res} = \exp\left(-\frac{\lambda}{Sc}\left|\frac{dZ}{d\varsigma}(\varsigma_{ox})\right| \left(f + \frac{d\Theta}{d\varsigma} \right)^{-1} (\varsigma_{ox}) \frac{Y_o^{\infty}}{Z_{fl}} \right) \tag{75}$$

The exponential describing the residual soot mass fraction is basically the product of the oxygen consumption rate at or near the flame, the oxidation efficiency factor, and the particulate transit time through the oxidation zone. The location of the oxidation zone is determined by the precise form of the oxygen state relation, which must be determined on empirical grounds.

We now turn to the soot generation term Y_s^g, which mathematically is the inhomogeneous contribution to the solution of equation (68). The solution consistent with the boundary condition of no ambient soot can be readily expressed without approximation in the form:

$$Y_s^g = Y_{tr}(\varsigma) Y_{ox}(\varsigma) Y_{gen}(\varsigma) \tag{76}$$

$$Y_{gen} = \frac{\beta}{Sc} \int_{\varsigma}^{\infty} \left(\frac{dZ}{d\varsigma}\right)^2 \frac{d^2 Y_f}{dZ^2} \left[\left(f + \frac{d\Theta}{d\varsigma}\right) Y_{tr}\, Y_{ox}\right]^{-1} d\varsigma \tag{77}$$

Again, the solution is more readily interpreted if we take advantage of the sharply peaked nature of the second derivative of the fuel state relation $Y_f''(Z)$. It is also necessary to assume that the peak is shifted with respect to the flame in a way that does not coincide with the peak in $Y_o''(Z)$. For consistency with the de Ris model, we assume that the soot generation zone lies inside the oxidation zone; i.e., at a location ζ_{gen} that corresponds to a value $Z_{gen} > Z_{ox}$. Under these circumstances, the same approximation used to evaluate Y_{ox} can be used to obtain the following result:

$$Y_{gen} = \frac{\beta}{Sc} \left|\frac{dZ}{d\varsigma}(\varsigma_{gen})\right| \left[\left(f + \frac{d\Theta}{d\varsigma}\right) Y_{tr}\, Y_{ox}\right]^{-1} (\varsigma_{gen})\, \frac{dY_f}{dZ}(Z) \tag{78}$$

The function Y_{gen} is the product of the soot generation per unit mass of fuel consumed parameter β, the local fuel consumption rate which is proportional to $Y_f'(Z)$, and the particle transport time through the soot formation zone $\left(f + \Theta'(\varsigma)\right)$ evaluated in the soot formation zone. The remaining terms $\left(Y_{tr}\, Y_{ox}\right)^{-1}(\varsigma_{gen})$ are present to cancel all or part of the corresponding factors in the full solution for the soot generation term given by equation (76). Since the soot oxidation factor Y_{ox} has reached its residual value given by equation (75) before it enters the soot formation zone, the Y_{ox} factors cancel completely as they should. The remaining factor $(Y_{tr}(\zeta_{gen}))^{-1}$ serves to cancel that portion of the soot transport function that accounts for motion outside the soot formation zone. That leaves only the soot transport from the soot formation zone inward towards the solid surface in this solution.

Thus, the full soot generation term Y_s^g can be written in final form as a product of a modified transport function $\tilde{Y}_{tr}$ and a modified generation function $\tilde{Y}_{gen}$.

$$Y_s^g = \tilde{Y}_{tr}(\varsigma)\, \tilde{Y}_{gen}(\varsigma) \tag{79}$$

$$\tilde{Y}_{tr} = \exp\left(\int_{\varsigma}^{\varsigma_{gen}} \left[\frac{d^2\Theta}{d\varsigma^2} \left(f + \frac{d\Theta}{d\varsigma} \right)^{-1} \right] d\varsigma \right) \tag{80}$$

$$\tilde{Y}_{gen} = \frac{\beta}{Sc} \left|\frac{dZ}{d\varsigma}(\varsigma_{gen})\right| \left(f + \frac{d\Theta}{d\varsigma}\right)^{-1} (\varsigma_{gen})\, \frac{dY_f}{dZ}(Z) \tag{81}$$

This solution could have been obtained by assuming that there is no soot in the region $\zeta > \zeta_{gen}$, and there is no possibility of oxidation. The first statement is true in the sense that the presence of ambient soot is accounted for by the homogeneous term Y_s^d. There is no possibility of oxidation because the soot transport is directed inwards from the flame towards the surface, and the oxidation region lies between the soot generation region and the flame.

7. Concluding Remarks

In this work, an elliptic solution to the Emmons' problem is obtained in parabolic coordinates. The analytical solutions are exact for the mass and mixture fraction conservation equations. The corresponding velocity field incorporates both the Emmons boundary layer result and an elliptic upstream influence that asymptotically satisfies the full Navier-Stokes equations. Thus the solution for the velocity field is exact everywhere outside the boundary layer. In the burning boundary layer, the error is small except in a small region *O*(20 Stokes lengths ~ 2mm) downstream of the leading edge where the velocity field is only qualitatively correct. The solution for mixture fraction and velocity fields is then used to analyze soot transport with generation and destruction. The soot model is also analytically tractable and seems to yield physically plausible results. Since only the general form of the model can be prescribed a priori, the physical constants of the soot model will have to be determined experimentally. However, given the exact transport and mixture fraction fields, this model aids the data interpretation.

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