Survey of new continuum numerical multiscale approaches and limitations

Małgorzata Peszyńska [1]

Department of Mathematics, Oregon State University

DOE Summer School, Corvallis, OR, June 29-July 3, 2007

[1] Supported by DOE 98089 "Modeling, Analysis, and Simulation of Multiscale Preferential Flow"' and NSF DMS 0511190 "Model adaptivity in porous media"

Outline

Model problem

Second order elliptic PDE on an open bounded $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, $\mathbf{K} = \mathbf{K}^T$, $\mathbf{K} = \mathbf{K}(x)$ bounded, strictly elliptic: $\lambda_{min}(x) \geq \lambda_0 > 0$

$$\begin{cases} -\nabla \cdot \mathbf{K}\nabla p = f, & x \in \Omega \\ p = 0, & x \in \partial\Omega \end{cases} \iff \begin{array}{c} Ap = f \\ X, Y \text{ are Banach (Hilbert) spaces} \\ A : X \mapsto Y \text{ or } A : X \mapsto Y' \end{array}$$

- Sobolev spaces $H^m(\Omega)$, $m \geq 0$, ($\mathbf{K} \in C^1(\bar{\Omega})$, $\partial\Omega$ is C^2 smooth)

$$\underbrace{-\nabla \cdot \mathbf{K}\nabla}_{A} : \quad H^{m+2}(\Omega) \cap H_0^1(\Omega) \mapsto H^m(\Omega), m \geq 0$$

Model problem

Second order elliptic PDE on an open bounded $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, $\mathbf{K} = \mathbf{K}^T$, $\mathbf{K} = \mathbf{K}(x)$ bounded, strictly elliptic: $\lambda_{min}(x) \geq \lambda_0 > 0$

$$\begin{cases} -\nabla \cdot \mathbf{K}\nabla p = f, & x \in \Omega \\ p = 0, & x \in \partial\Omega \end{cases} \iff \begin{array}{c} Ap = f \\ X, Y \text{ are Banach (Hilbert) spaces} \\ A : X \mapsto Y \text{ or } A : X \mapsto Y' \end{array}$$

- Sobolev spaces $H^m(\Omega)$, $m \geq 0$, ($\mathbf{K} \in C^1(\bar{\Omega})$, $\partial\Omega$ is C^2 smooth)

$$\underbrace{-\nabla \cdot \mathbf{K}\nabla}_{A} : \quad H^{m+2}(\Omega) \cap H^1_0(\Omega) \mapsto H^m(\Omega), m \geq 0$$

- a weaker notion .... ($\mathbf{K} = \mathbf{K}(x)$ can be discontinuous, $\partial\Omega$ is polygonal)

$$A : H^1_0(\Omega) \mapsto H^{-1}(\Omega)$$

- Sobolev spaces $W^{m,p}$, $1 \leq p < \infty$, when $\mathbf{K} = \mathbf{K}(x, \Theta)$ or is degenerate (also for transient problems)

Weak (variational) formulation and FE formulation

Use smooth $q : q|_{\partial\Omega} = 0$, integrate by parts $-\nabla \cdot \mathbf{K}\nabla p = f$, use $p|_{\partial\Omega} = 0$ to get

$$\int_\Omega \mathbf{K}(x)\nabla p(x)\nabla q(x)dx = \int_\Omega f(x)q(x)dx,$$

Abstract setting

Define $V := H_0^1(\Omega), a(p,q) := \int_\Omega \mathbf{K}\nabla p \nabla q dx,\ (f,q) := \int_\Omega fq dx$.

$$\text{Find } p \in V : a(p,q) = (f,q), \forall q \in V$$

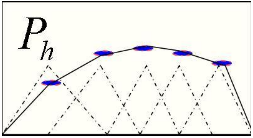
Theory: $\exists! p$: continuously depending on the data $f, \mathbf{K}, \Omega$.

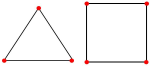
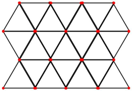
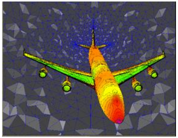
FE for the model elliptic problem

(D) PDE and b.c $-\nabla \cdot \mathbf{K}\nabla p = f, p|_{\partial\Omega} = 0$

(V): weak form: $a(p,q) = (f,q), \forall q \in V$

- Partition $\overline{\Omega_h} = \bigcup_{T \in \mathcal{T}_\langle} \bar{T} \approx \overline{\Omega} \subset \mathbb{R}^d$ into elements T: segments (1D) or triangles/quadrilaterals (2D), tetrahedra/prisms/bricks (3D)



- Define local polynomial basis (globally $C^0(\Omega)$ only) of degree $k = 1, 2, \ldots$ for space $V_h^k(\Omega)$

(FE) Finite element solution:

find $p_h \in V_h^k$: $a(p_h, q_h) = (f, v_h), \forall q_h \in V_h^k$

Error analysis

- $a(\cdot,\cdot)$ continuous: $a(p,q) \leq C \parallel p \parallel_V \parallel q \parallel_V$
- $a(\cdot,\cdot)$ elliptic: $a(u,u) \geq \alpha_0 \parallel u \parallel_V^2$
- Conforming FE: $V_h \subsetneq V$ (nonconforming $\approx$ variational crimes)
- Galerkin orthogonality $a(p - p_h, v_h) = 0$
- Céa's lemma $\parallel p - p_h \parallel_V \leq \inf_{q_h \in V_h} \parallel p - q_h \parallel_V$
- Interpolation estimate(s) $\parallel u - I_h u \parallel_V \leq Ch \parallel u \parallel_{H^2(\Omega)}$

lead to ...

$$\parallel p - p_h \parallel_{H^1(\Omega)} \leq Ch \parallel p \parallel_{H^2(\Omega)}$$

Error analysis

- $a(\cdot,\cdot)$ continuous: $a(p,q) \leq C \parallel p \parallel_V \parallel q \parallel_V$
- $a(\cdot,\cdot)$ elliptic: $a(u,u) \geq \alpha_0 \parallel u \parallel_V^2$
- Conforming FE: $V_h \subsetneq V$ (nonconforming $\approx$ variational crimes)
- Galerkin orthogonality $a(p - p_h, v_h) = 0$
- Céa's lemma $\parallel p - p_h \parallel_V \leq \inf_{q_h \in V_h} \parallel p - q_h \parallel_V$
- Interpolation estimate(s) $\parallel u - I_h u \parallel_V \leq Ch \parallel u \parallel_{H^2(\Omega)}$

lead to ...

$$\parallel p - p_h \parallel_{H^1(\Omega)} \leq Ch \parallel p \parallel_{H^2(\Omega)}$$

- use of Aubin-Nitzche duality "trick" based on $\parallel p \parallel_{H^2(\Omega)} \leq \parallel f \parallel_{L^2(\Omega)}$:

$$\parallel p - p_h \parallel_{L^2(\Omega)} \leq Ch^2 \parallel p \parallel_{H^2(\Omega)}$$

Finding p_h: implementation

Use $\{\phi_1(\mathbf{x}), \dots \phi_{N_h}(\mathbf{x})\}$ as the (piecewise polynomial) basis for V_h^k
Write $p_h = \sum_i p_i \phi_i(\mathbf{x})$ and $p_h \equiv \mathbf{P} = (p_i)_{i=1}^{N_h}$

(FE) Finite element solution $p_h \equiv \mathbf{P}$

$$a(p_h, q_h) = (f, q_h), \forall q_h \in V_h^k \leftrightarrow \sum_i p_i a(\phi_i, \phi_j) = (f, \phi_j), \forall j$$

Note: the system $\mathbf{AP} = \mathcal{F}$ has dimension N_h.

Finding p_h: implementation

Use $\{\phi_1(\mathbf{x}), \dots \phi_{N_h}(\mathbf{x})\}$ as the (piecewise polynomial) basis for V_h^k
Write $p_h = \sum_i p_i \phi_i(\mathbf{x})$ and $p_h \equiv \mathbf{P} = (p_i)_{i=1}^{N_h}$

(FE) Finite element solution $p_h \equiv \mathbf{P}$

$$a(p_h, q_h) = (f, q_h), \forall q_h \in V_h^k \leftrightarrow \sum_i p_i a(\phi_i, \phi_j) = (f, \phi_j), \forall j$$

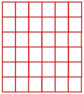
Note: the system $\mathbf{AP} = \mathcal{F}$ has dimension N_h.

In practice the matrix entries $A_{ij} := a(\phi_i, \phi_j) = \int_\Omega \mathbf{K}(\mathbf{x}) \nabla \phi_i(\mathbf{x}) \nabla \phi_j(\mathbf{x}) dA = \sum_T \int_T \mathbf{K}(\mathbf{x}) \nabla \phi_i(\mathbf{x}) \nabla \phi_j(\mathbf{x}) dA$

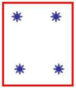
are computed approximately using numerical integration (quadrature): $A_{ij} \approx \sum_T (\int_T \mathbf{K}(\mathbf{x}) \nabla \phi_i(\mathbf{x}) \nabla \phi_j(\mathbf{x}) dA)_h$

Numerical implementation and solving a linear system

- Discretize: define Ω_h, choose V_h^k

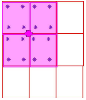


- *Assembly* process with quadrature: compute for each T the approximation

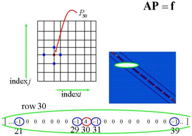


$$\int_T (\mathbf{K}(\mathbf{x})\nabla\phi_i(\mathbf{x})\nabla\phi_j(\mathbf{x})dA)_h := \sum_m w_m \mathbf{K}(\mathbf{x}_m)\nabla\phi_i(\mathbf{x}_m)\nabla\phi_j(\mathbf{x}_m)$$

Add over all elements T adjacent to node j for each j (cost is $O(N_h)$)



- Solve linear system ($\mathbf{A}$ is sparse spd)



$$\mathbf{AP} = \mathcal{F}$$

...this requires $O(N_h^r)$ computational time r=3 for full GE ... to ... r=1 for Full Multigrid solvers,

Mixed formulation: $\mathbf{u} = -\mathbf{K}\nabla p$, $\nabla \cdot \mathbf{u} = f$

$W := L^2(\Omega)$ and $\mathbf{V} := \mathbf{H}(\mathbf{div};\Omega)$. Find $\Theta := (\mathbf{u}, p) \in \mathbf{V} \times W$

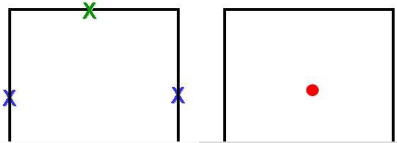
$$\begin{aligned}(K^{-1}\mathbf{u}, \mathbf{v}) &= (p, \nabla \cdot \mathbf{v}) - < g, \mathbf{v} \cdot \mathbf{n} >, \quad \forall \mathbf{v} \in \mathbf{V} \\ (\nabla \cdot \mathbf{u}, w) &= (f, w), \quad \forall w \in W\end{aligned}$$

Discrete mixed formulation

$W_h \subset W$, $\mathbf{V}_h \subset V$ are $\mathbf{RT}_{[0]}$ spaces. Find $\Theta_h := (\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$:

$$\begin{aligned}(K^{-1}\mathbf{u}_h, \mathbf{v}_h) &= (p_h, \nabla \cdot \mathbf{v}_h) - < g_h, \mathbf{v}_h \cdot \mathbf{n} >_{\partial\Omega}, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla \cdot \mathbf{u}_h, w_h) &= (f, w_h), \quad \forall w_h \in W_h.\end{aligned}$$

Error estimates: $\| p - p_h \|_{L^2(\Omega)} = O(h)$, $\| \mathbf{u} - \mathbf{u}_h \|_K = O(h)$



Discrete mixed $\mathbf{RT}_{[0]}$ spaces $\equiv$ cell-centered FD

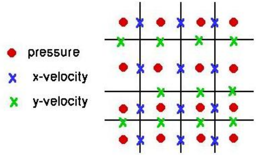
Basis functions for $\mathbf{RT}_{[0]}$ spaces *[RavTho77]*

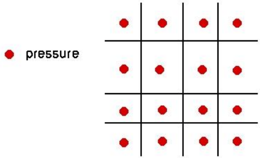
$\mathbf{u}_h \in \mathbf{V}_h \subset V$ are piecewise linears $\times$ piecewise constants
$p_h \in W_h \subset W$ are piecewise constants

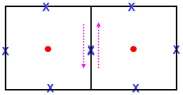
$\mathbf{u}_h \cdot \mathbf{n}$ are continuous

Equivalent to CCFD (up to quadrature)

$\mathbf{u}_h \cdot \mathbf{t}$ are NOT continuous







Reality of FE computations vs theory

For optimal convergence of numerical methods

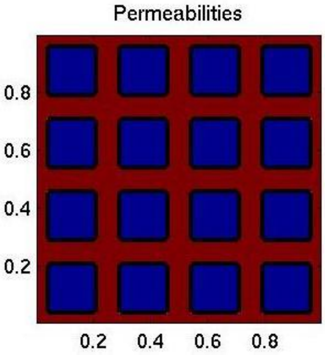
$$\| p - p_h \|_{L^2(\Omega)} \leq Ch^2 \| p \|_{H^2(\Omega)}$$

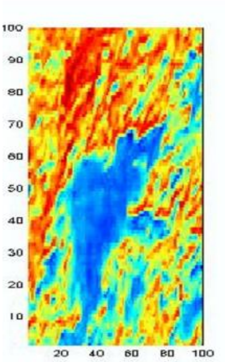
one needs (at least local) smoothness of the true solution p ... but in practice

- f, Ω possibly not smooth
- **K** not smooth: multiscale character

What is $\mathbf{K} = \mathbf{K}(x)$ like ?

$\mathbf{K} = \mathbf{K}^T$ (permeability, conductivity, mobility,...) is in general anisotropic.Here we focus on two sources of difficulties:



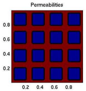


two scales K_{fast} / $K_{slow} = 10^{\beta}, \beta \geq 1$
periodic character of $\mathbf{K}$

strong heterogeneity
no separation of scales ?

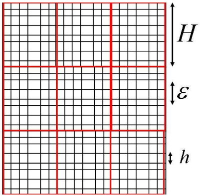
Numerical simulation: periodic multiscale case

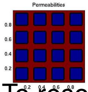
Assume $\mathbf{K}(\mathbf{x}) = \mathbf{K}^{\epsilon}(\mathbf{x}) = \mathbf{K}(\mathbf{x}, \frac{\mathbf{x}}{\epsilon})$



Numerical simulation: periodic multiscale case

Assume $\mathbf{K}(\mathbf{x}) = \mathbf{K}^{\epsilon}(\mathbf{x}) = \mathbf{K}(\mathbf{x}, \frac{\mathbf{x}}{\epsilon})$

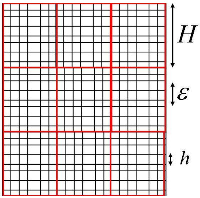




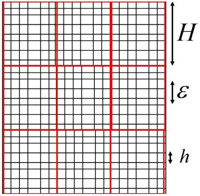
To resolve the scales in $\mathbf{K}(\mathbf{x})$, we need a grid with $h << \epsilon$ .. this means solving $\mathbf{AP} = \mathcal{F}$ with $O(N_h^r)$ complexity and may be prohibitively complex in $\mathbb{R}^2, \mathbb{R}^3$. [a]

[a] Keep in mind the big picture and solving nonlinear transient problems $\mathbf{K} = \mathbf{K}(\mathbf{x}, p, \nabla p)$

How not to solve with *h*



How not to solve with *h*



Some solutions and ideas when using $\epsilon < H$

- use a special linear solver technique (multigrid ?)
- find effective $\mathbf{K}^*{}_H$ and solve for $p^*{}_H \approx p^0$
- solve for $p^*{}_H \approx p^0$ using *multiscale FE*
- if needed, recover (reconstruct) next order effects (correctors, downscaling)

Multigrid/multilevel methods

Well suited to handle large systems

$$\mathbf{AP} = \mathcal{F}$$

- Standard multigrid not useful for problems with highly varying coefficients:
- Must use a special grid transfer operator (not bilinear)
- Idea: construct an effective **K** using a special grid transfer operator *[Knapek, Moutlon, Dendy]*

Find effective $\mathbf{K}^*{}_H$ by homogenization

- Homogenization formulas for $\mathbf{K}^*$

$$\mathbf{K}^*{}_{jk} = \frac{1}{|\Omega_0|}\int_{\Omega_0} K_{jk}(\mathbf{y})(\delta_{jk} + \partial_k \omega_j(\mathbf{y}))dA$$

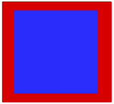
$$\begin{cases} -\nabla \cdot \mathbf{K}\nabla\omega_j(\mathbf{y}) & = \nabla \cdot (\mathbf{K}\mathbf{e}_j), \quad \mathbf{y} \in \Omega_0 \\ \omega_j & \Omega_0 - \text{ periodic} \end{cases}$$

- Analytical formulas and bounds[2] available for special geometries only
- Finding $\mathbf{K}^*$ numerically: for every T_H, solve a local problem with grid h for $\mathbf{K}^*{}_H(\mathbf{x})$
- exploit two-scale numerical FE approaches *[Matache/Schwab]*

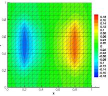
[2] Wiener, Matheron, Cardwell, Parsons, Torquato, Rubinstein, Hashin, Shtrikman, Dagan

Find effective $\mathbf{K}^*{}_H$ by homogenization

- Homogenization formulas for $\mathbf{K}^*$



$$\mathbf{K}^*{}_{jk} = \frac{1}{|\Omega_0|} \int_{\Omega_0} K_{jk}(\mathbf{y})(\delta_{jk} + \partial_k \omega_j(\mathbf{y}))dA$$



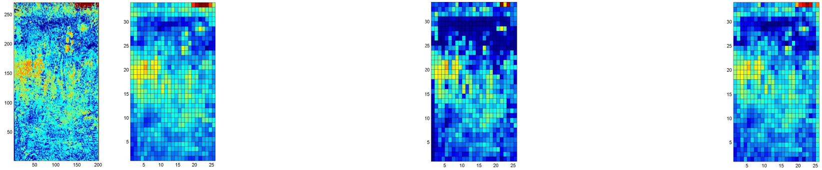
$$\begin{cases} -\nabla \cdot \mathbf{K}\nabla\omega_j(\mathbf{y}) & = \nabla \cdot (\mathbf{K}\mathbf{e}_j), \quad \mathbf{y} \in \Omega_0 \\ \omega_j & \Omega_0 - \text{ periodic} \end{cases}$$

- Analytical formulas and bounds[2] available for special geometries only
- Finding $\mathbf{K}^*$ numerically: for every T_H, solve a local problem with grid h for $\mathbf{K}^*{}_H(\mathbf{x})$
- exploit two-scale numerical FE approaches *[Matache/Schwab]*

[2] Wiener, Matheron, Cardwell, Parsons, Torquato, Rubinstein, Hashin, Shtrikman, Dagan

Find an effective $\mathbf{K}^*{}_H$ by upscaling

Examples of upscaled $\mathbf{K}^*{}_H$ using simple averaging:

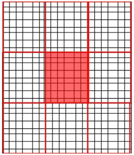


Original field $\mathbf{K}_h$ with $h \approx$ 217x201, upscaled $\mathbf{K}_H$ with $H \approx$ 34x26 by arithmetic, harmonic averaging, and renormalization *[King]*

(P)ressure based upscaling: solve

$$-\nabla \cdot \mathbf{K}^{\epsilon}(\mathbf{x})\nabla w = 0, \quad \mathbf{y} \in T_H$$

$$w|_{\partial T_H} = ?$$



Then get $\mathbf{K}^*{}_H|_{T_H}$ by matching fluxes with $\langle \nabla w \rangle$.
Boundary conditions: Dirichlet (prevents crossflow) or periodic.

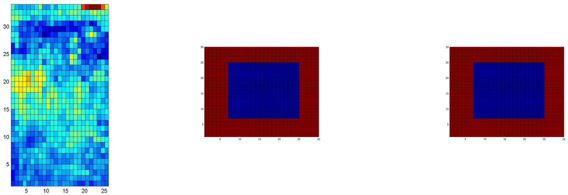
Comparison of upscaling methods

Given $\mathbf{K}_h$, $h \approx 34x26$...

Heterogeneous case
$\mathbf{K}_h \in [1.9, 242.2]$

Fracture cell
$\mathbf{K}_h(\mathbf{x}) \in \{1, 1e-2\}$

Fracture cell
$\mathbf{K}_h(\mathbf{x}) \in \{1, 1e-4\}$



... compute *effective* $\mathbf{K}_H$, $H \approx 1x1$

method	heterogeneous	fracture 1e-2	fracture 1e-4
(A)rithmetic	89.5002	0.6436	0.6400
(H)armonic	79.3953	0.0273	0.000273
(R)enormalization (6x)	78.6652	0.4799	0.4703
(P)ressure based	86.5225, 81.5955	0.455825	0.446895

MsFEM

Idea: solve on scale H

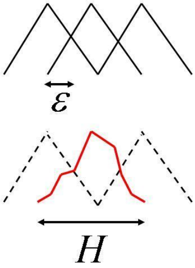
incorporate ϵ-scale of $\mathbf{K}^\epsilon(\mathbf{x})$ into basis functions *[Babuska, Osborn'83]*, *[Hou, Wu'97-]*,*[Efendiev, Hou, Wu, '00]*

Given original test functions $\{\phi_1(\mathbf{x}), \ldots \phi_{N_H}(\mathbf{x})\}$...

solve local problem on T_H

$$-\nabla K^\epsilon(\mathbf{x})\nabla\psi_i = 0, \quad \mathbf{y} \in T_H$$
$$\psi_i|_{\partial T_H} = \phi_i$$

Use ψ_i to construct basis for V_H.



MsFEM and oversampling

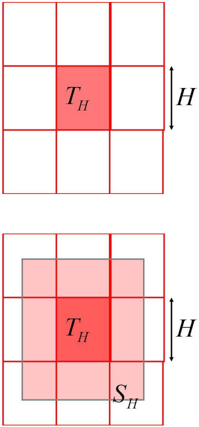
MsFEM: solve on T_H

$$-\nabla K^{\epsilon}(\mathbf{x})\nabla\psi_i^T = 0, \quad \mathbf{y} \in T_H$$
$$\psi_i|_{\partial T_H} = \phi_i$$

Use ψ_i to construct basis for V_H.

Oversampling MsFEM: solve on S_H

$$-\nabla K^{\epsilon}(\mathbf{x})\nabla\psi_i^S = 0,$$
$$\psi_i|_{\partial S} = \phi_i$$



Analysis of MsFEM (periodic **K**)

Error estimates *[Hou, Wu'97-],[Efendiev, Hou, Wu, '00]*:

$$\begin{aligned} \| p - p_H \|_{L^2(\Omega)} &\leq C(H^2 + \frac{\epsilon}{H}) \\ \| p - p_H \|_{H^1(\Omega)} &\leq C(H + \epsilon + \sqrt{\frac{\epsilon}{H}}) \end{aligned}$$

Difficulties:

- Resonance effect partially removed by oversampling.
- Efficiency ... still solving on every $\mathbf{T}_H$ ...
- Applied in practice to non-periodic problems
- Extensions to nonlinear and transient cases *[Efendiev,Pankov]*

Also *[Chen'05, Chen, Hou'02]*, for mixed FE methods.

Heterogeneous multiscale FEM

Idea: *[E, Engquist'02]*, *[Ming, Yue'03, E, Ming, Zhnag'04]*

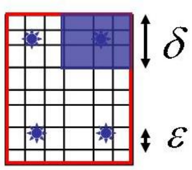
Solve with H, incorporate ϵ-scale of $\mathbf{K}^\epsilon(\mathbf{x})$ at quadrature points.

$$\int_T (\mathbf{K}(\mathbf{x})\nabla\phi_i(\mathbf{x})\nabla\phi_j(\mathbf{x})dA)_h := \sum_m w_m \mathbf{K}(\mathbf{x}_m)\nabla\phi_i(\mathbf{x}_m)\nabla\phi_j(\mathbf{x}_m)$$

$$\approx \sum_m w_m \mathbf{K}^\epsilon_{ij}(\mathbf{x}_m)\nabla\hat{\phi}_i(\mathbf{x}_m)\nabla\hat{\phi}_j(\mathbf{x}_m)$$

Need to capture variation in $\mathbf{K}^\epsilon(\mathbf{x})$: solve for $\hat{\phi}$

$$-\nabla\cdot\mathbf{K}^\epsilon(\mathbf{x})\nabla\hat{\phi} = 0, \quad \mathbf{x}\in T_\delta$$
$$\hat{\phi}-\phi \text{ periodic on } T_\delta$$



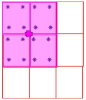
Heterogeneous multiscale FEM: analysis

- Analysis: (periodic $\mathbf{K}^{\epsilon}(\mathbf{x})$)

$$\| p - p_H \|_{L^2\Omega} \leq C \left(H^2 + \begin{cases} O(\epsilon), & \delta = \epsilon \\ O(\delta + \frac{\epsilon}{\delta}), & \delta > \epsilon \end{cases} \right)$$

- Special error estimates for random **K**.

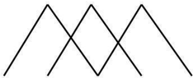
- Computational cost: smaller than using h, competitive with MsFEM.
- Extensions to nonlinear cases



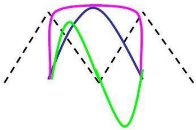
Variational and subgrid multiscale FEM

Idea of Bubbles:

provide enrichment of FE spaces *[Hughes]* used to stabilize convection-diffusion problems, CFD, or elasticity FE formulations, to resolve nonpolynomial behavior such as: boundary layers, numerical Green's functions



usual FE basis functions



basis functions plus bubbles

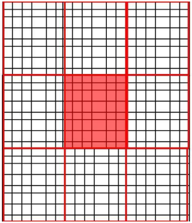
Write $p_h = \bar{p} + p'$

Subgrid methods for mixed FE methods *[Arbogast, Keenan, Minkoff', Arbogast'00]*

Idea: write $p_h = p_H + \delta p$, same for velocity variables.

Subgrid upscaling

- Write $p_h = p_H + \delta p$, $\mathbf{v}_h = \mathbf{v}_H + \delta \mathbf{v}$
- Decouple coarse problems by the closure assumptions: no flow for $\delta \mathbf{v}$ on the coarse boundaries ∂T_H.
- Use higher order FE for velocity v_H (BDDF) and lower order (RT) for δv.
- Use numerical Green's functions to precompute for influence of coarse on fine scale.

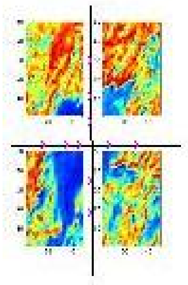


Error estimates and experiments for $\frac{H}{h}$ fixed

$$\| p - p_H \|_{L^2(\Omega)} = O(H), \quad \| \mathbf{v} - \mathbf{v}_H \|_{L^2(\Omega)} = O(H^2)$$

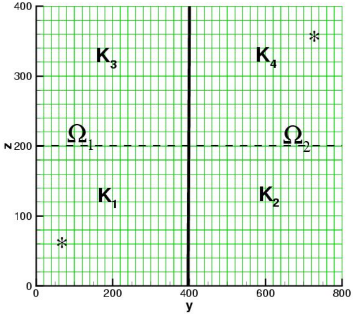
Recent extensions by *[Aarnes et al]* of Mixed MsFEM and Subgrid methods

Mortar based methods



- Complete decoupling of problems on T_H: they are connected by mortars on interfaces
- optimal convergence: a-priori analysis *[ACWY'00]* and a-posteriori analysis *[P05]*, *[APWY'07]* available
- mortar upscaling *[PWY02]*
- Computational savings achieved when efficient interface solvers are available
- No reconstruction necessary: solution at grid h available
- Automatic implementation of transient nonlinear problems

Example: adaptive mortar modeling *[P05]*



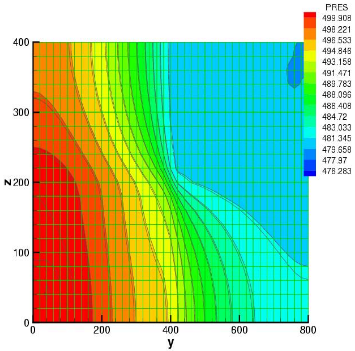
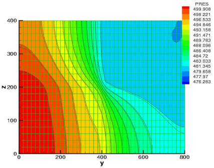


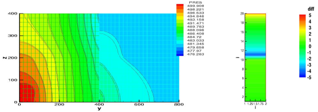
Figure: Left: $\boxed{\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma}$. Right: solution $\Theta_h \equiv p_h$, no mortars

Results: mortars for single phase flow in porous media

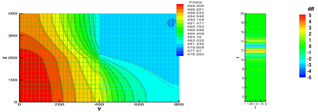


Solution $\Theta_h \equiv p_h$
(no mortars)

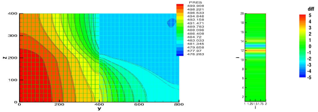
$n_m = 1$



$n_m = 5$



$n_m = 15$



Solution $\tilde{\Theta}_h = \tilde{p}_h$ (with mortars). Right: $\eta_{\Gamma,*}$

Flow coupled to transport $\mathcal{F}(\Theta) = 0$ with $\Theta = (\mathbf{u}, p, c)$

Flow

$$\mathbf{u} = -\mathbf{K}\nabla p, \quad \nabla \cdot \mathbf{u} = 0$$

Diffusive-dispersive transport

$$\phi \frac{\partial c}{\partial t} \quad + \quad \nabla \cdot (\mathbf{u}c - \mathbf{D}(\mathbf{u})\nabla c) = 0$$

Definitions

$$\begin{aligned}
\mathbf{D}(\mathbf{u}) \quad &:= \quad \text{diffusion} + \text{dispersion} \\
&:= \quad d_{mol}\mathbf{I} + |\mathbf{u}|(d_{long}\mathbf{E}(\mathbf{u}) + d_{transv}(\mathbf{I} - \mathbf{E}(\mathbf{u}))). \\
\mathbf{E}(\mathbf{u}) \quad &= \quad \frac{1}{|\mathbf{u}|^2} u_i u_j \\
\mathbf{D}(\mathbf{u}) \quad &\approx \quad d_{mol}\mathbf{I} + d_{long}|\mathbf{u}|\mathbf{E}(\mathbf{u})
\end{aligned}$$

Reconstruction and downscaling

Flow coupled to transport

$$\mathbf{u} = -\mathbf{K}\nabla p, \quad \nabla \cdot \mathbf{u} = 0$$
$$\phi \frac{\partial c}{\partial t} + \nabla \cdot (\mathbf{u}c - \mathbf{D}(\mathbf{u})\nabla c) = 0$$

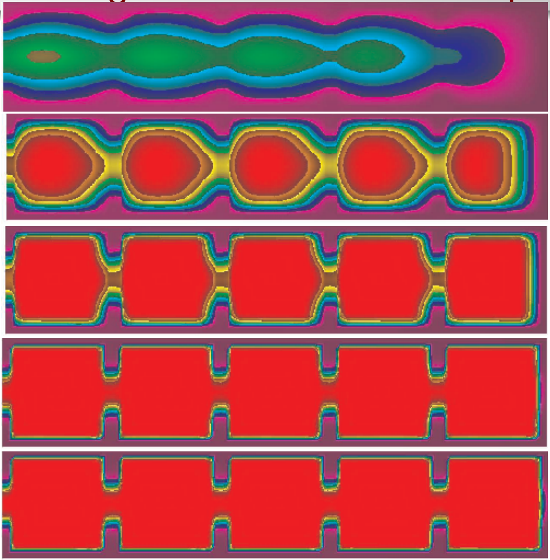
Need accurate fine scale $\mathbf{u}_h$!

... have to reconstruct $\mathbf{u}_h$ from $\mathbf{u}_H$

Ideas

- *[Oden, Vemaganti]*: use the coarse solution p_H as boundary conditions for the local problem solved for p_h
- global-local upscaling *[Durlofsky, Chen, Gerritsen]*
- use global information *[Efendiev'06,'07]*

Small/large contrast diffusive-dispersive transport



Multi-phase flow problems, phases $\alpha = w, n$

$$\frac{\partial}{\partial t}\phi S_\alpha \rho_\alpha - \nabla \cdot \mathbf{K}\lambda_\alpha \nabla P_\alpha = q_\alpha, \quad (1)$$

$$P_n - P_w = P_c(S_w) \quad (2)$$

$$S_n + S_w \equiv 1 \quad (3)$$

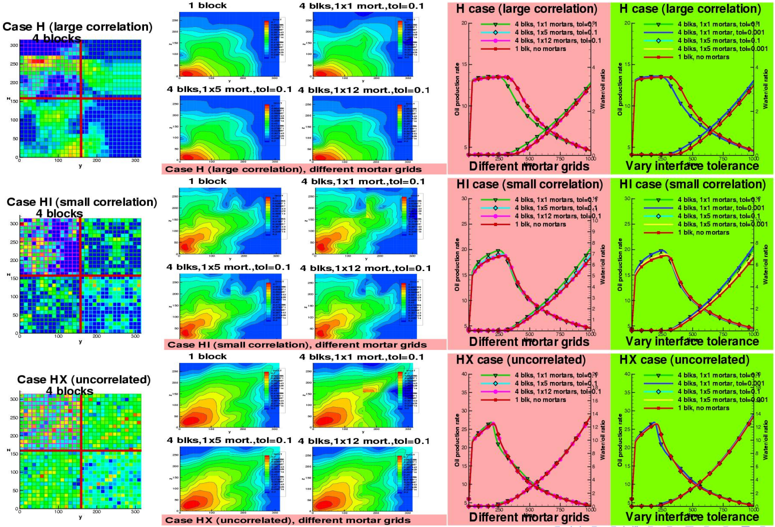
Well–posedness results and character of solutions

degenerate parabolic/elliptic + parabolic/hyperbolic *[altdibene,Arbogast,ChenI]*

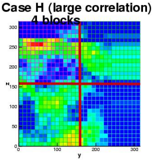
Handling transient nonlinearities

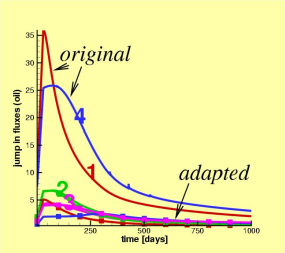
- use upscaled **K**; how to handle λ_α, P_c ? [*Durlofsky et al,Trykozko and Zijl*]
- if heterogeneity is $\approx$ periodic, use homogenization *[Arbogast97,Bourgeat,BourKozMik95]*

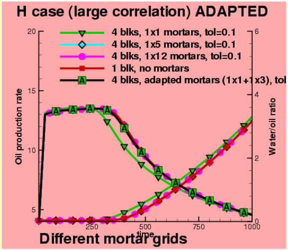
Example: mortar upscaling for multi-phase flow



Example: mortar adaptivity *[P05]*







Recap

- Elliptic problems with multiscale coefficients lead to large linear systems
- Multiscale FE aim to reduce computational complexity
- Not all are applicable and/or perform equally well when applied to transient coupled nonlinear problems