EXACT SEQUENCES, DE RHAM DIAGRAM, MAXWELL EQUATIONS, hp ADAPTIVITY

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The gradient and curl operators, along with the divergence operator, form an *exact sequence*,

$$H^1 \xrightarrow{\boldsymbol{\nabla}} \boldsymbol{H}(\text{curl}) \xrightarrow{\boldsymbol{\nabla}\times} \boldsymbol{H}(\text{div}) \xrightarrow{\boldsymbol{\nabla}\circ} L^2 .$$

In an exact sequence of operators, the range of each operator coincides with the null space of the operator next in the sequence. The spaces above may be modified to incorporate homogeneous essential boundary conditions. The exact sequence property can be reproduced at the discrete level for polynomial spaces,

$$W_p \xrightarrow{\boldsymbol{\nabla}} \boldsymbol{Q}_p \xrightarrow{\boldsymbol{\nabla}\times} \boldsymbol{V}_p \xrightarrow{\boldsymbol{\nabla}\circ} Y_p .$$

The polynomial spaces correspond to the three fundamental master elements.

Tetrahedral element. We use standard spaces $\mathcal{P}^p$ of polynomials of (group) order less or equal p, e.g. $x^2y^3z^2 \in \mathcal{P}^7$,

$$\begin{aligned} W_p &= \mathcal{P}^p, \\ \boldsymbol{Q}_p &= \mathcal{P}^{p-1} \times \mathcal{P}^{p-1} \times \mathcal{P}^{p-1}, \\ \boldsymbol{V}_p &= \mathcal{P}^{p-2} \times \mathcal{P}^{p-2} \times \mathcal{P}^{p-2}, \\ Y_p &= \mathcal{P}^{p-3} . \end{aligned}$$

With differentiation, the order of polynomials simply drops by one.

Hexahedral element. We use tensor product spaces $\mathcal{Q}^{(p,q,r)} = \mathcal{P}^p \otimes \mathcal{P}^q \otimes \mathcal{P}^r$ of polynomials of order p, q, r with respect to individual variables x, y, z, e.g. $x^2y^3z^2 \in \mathcal{Q}^{(2,3,2)}$.

$$\begin{aligned} W_p &= Q^{(p,q,r)} \\ \boldsymbol{Q}_p &= Q^{(p-1,q,r)} \times Q^{(p,q-1,r)} \times Q^{(p,q,r-1)} \\ \boldsymbol{V}_p &= Q^{(p,q-1,r-1)} \times Q^{(p-1,q,r-1)} \times Q^{(p-1,q-1,r)} \\ Y_p &= Q^{(p-1,q-1,r-1)} . \end{aligned}$$

Prismatic element. We use tensor products of polynomials of (group) order less or equal p in (x, y) with polynomials of order less or equal q in z, $\mathcal{Q}^{(p,q)} = \mathcal{P}^p \otimes \mathcal{P}^q$, e.g. $x^2y^3z^2 \in \mathcal{Q}^{(5,2)}$.

$$\begin{aligned} W_p &= Q^{(p,q)} \\ \boldsymbol{Q}_p &= Q^{(p-1,q)} \times Q^{(p-1,q)} \times Q^{(p,q-1)} \\ \boldsymbol{V}_p &= Q^{(p-1,q-1)} \times Q^{(p-1,q-1)} \times Q^{(p-2,q)} \\ Y_p &= Q^{(p-2,q-1)} . \end{aligned}$$

For tetrahedra and prisms, the construction can be generalized to *incomplete polynomial spaces*. For spaces $\boldsymbol{Q}_p$ this corresponds to Nedelec's elements of the first type. By employing the incomplete polynomial spaces, we accomplish an affect that the order of involved polynomials drops at the end of the sequence only by one, similarly to the hexahedral element. More importantly, the construction can be generalized to arbitrary parametric elements and meshes with elements of variable order, enabling p and hp-adaptivity.
With a slightly increased regularity that in the original sequence (expressed in terms of fractional Sobolev spaces), it is possible to introduce the so called *projection-based* interpolation operators,

$$\Pi : \mathcal{W} := H^{\frac{1}{2}+r} \to W_p, \qquad \Pi^{curl} : \mathcal{Q} := \boldsymbol{H}^{\frac{1}{2}+r}(\mathrm{curl}) \to \boldsymbol{Q}_p,$$
$$\Pi^{div} : \mathcal{V} := \boldsymbol{H}^{r}(\mathrm{div}) \to \boldsymbol{V}_p, \qquad P : L^2 \to Y_p\ ,$$

where $r > 0$. The interpolation operators are optimal in element order p and element size h. The corresponding interpolation errors display (almost) identical p- and h-convergence rates as the best approximation errors. The interpolation operators make the following de Rham diagram commute.

$$\begin{array}{ccccccc}
\mathcal{W} & \xrightarrow{\boldsymbol{\nabla}} & \mathcal{Q} & \xrightarrow{\boldsymbol{\nabla}\times} & \mathcal{V} & \xrightarrow{\boldsymbol{\nabla}\circ} & L^2 \\
\downarrow \Pi & & \downarrow \Pi^{\mathrm{curl}} & & \downarrow \Pi^{\mathrm{div}} & & \downarrow P \\
W_{hp} & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{Q}_{hp} & \xrightarrow{\boldsymbol{\nabla}\times} & \boldsymbol{V}_{hp} & \xrightarrow{\boldsymbol{\nabla}\circ} & Y_{hp}
\end{array} \ .$$

I will discuss the importance of the exact sequences and the commuting diagram property for the discretization of Maxwell equations, focusing on *stability, approximability, and optimal hp mesh refinements.* The automatic hp-adaptivity enabling *exponential convergence* has been a subject of an intensive research for over a decade but only recently we managed to produce first convincing multidimensional results for elliptic problems, and the very first results for Maxwell equations. Figure 1 presents the convergence history for the hp-adaptive algorithm applied to the solution of a 2D diffraction problem. The error is reported on an algebraic (problem size) vs. the logarithmic (error) scale. The straight line indicates the exponential convergence. The hp-methodology allows for achieving unprecedented accuracy up to 0.01 percent error measured in (relative) energy norm.

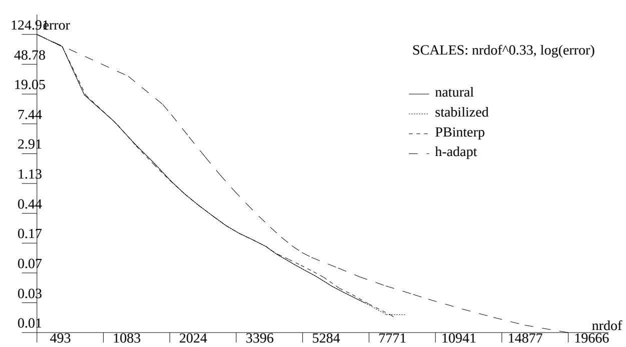

Figure 1: Convergence history for a 2D diffraction problem. Three (slightly) different versions of the hp-algorithm are compared with h-adaptivity using quadratic elements

References

[1] L. Demkowicz, "Finite element methods for Maxwell equations", *Encyclopedia of Computational Mechanics*, Edited by Erwin Stein, René de Borst and Thomas J.R. Hughes, 2004 John Wiley & sons, Ltd.