APPROXIMATING 3D RADIATIVE TRANSFER IN CLOUDS WITH DIFFUSION THEORY

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Abstract: We overview diffusion theory as applied to shortwave radiation transport in clouds for the benefit of the Intercomparison of 3D Radiation Codes (I3RC) community. Partial results for the square-wave cloud model are offered in illustration and we argue that, for applications such as cloud modeling, the price in accuracy is offset by the inherent efficiency of diffusion methods.

1. Introduction and Outline

There is no easy and general solution to 3D radiative transfer problems in dense clouds. Straightforward Monte Carlo (MC) methods are quick to implement but have well-documented limitations. More advanced MC approaches like "local estimation" have more flexibility but require more care. All MC methods are slow to converge and can never give a complete solution in photon phase-space. Explicit solutions can provide the comprehensive information sometimes required about the radiance field and they converge faster but their implementation is even more time-consuming. If this situation was not frustrating enough, computational radiative transfer hardly ever offers any real insight into the "dynamics" of cloud-radiation interaction. Under these circumstances, it is no surprise that practitioners have sought relief in photon diffusion theory since the most early forays into 3D radiative transfer [e.g., *Giovanelli*, 1959]. It is indeed relatively easy to encode, has extremely fast convergence, analytical solutions are possible in some cases, even some general results, a vast "culture" is inherited from statistical physics, hydrology and engineering applications, it incorporates the intuitive appeal of random-walk theory, and so on.

In the next section, we set up the general albedo problem of interest to the I3RC study group, we define the quantities of interest in I3RC that are primarily based on radiance fields sampled at the boundaries, and we describe the independent-pixel approximation or "IPA" which is as far as plane-parallel theory will go to help 3D radiative transfer and is therefore the main point of reference for I3RC. In section 3, we survey photon diffusion theory starting from the basic equations, their derivation from first principles and their interpretation in terms of random walks, standard techniques to overcome their limitations, their numerical implementation, specific results for the I3RC square-wave cloud model (Case #1), analytical solutions and insights into 3D radiation transport in clouds. Finally, we summarize and look at applications in section 4.

2. Elements of Three-Dimensional Radiative Transfer Theory

2.1 Setting up the General 3D Albedo Problem

We are given an open three-dimensional domain $\mathrm{M} = (0,L_x)\otimes(0,L_y)\otimes(0,L_z)$ representing an optical medium with spatially variable extinction $\sigma(\boldsymbol{r})$ where we denote $\boldsymbol{r} = (x,y,z)^\mathrm{T}$, superscript "T" meaning transpose. We assume here that the single-scattering albedo ϖ_0 is spatially uniform and similarly for the phase function $p(\Omega\bullet\Omega')$ where $\Omega^{(')}$ is the unit-vector indicating propagation direction after (before) a scattering event. We wish to solve the monochromatic integro-differential radiative transfer equation (RTE) [*Chandrasekhar*, 1950]

$$\Omega\bullet\nabla I = -\sigma(\boldsymbol{r})I(\boldsymbol{r},\Omega) + \varpi_0\sigma(\boldsymbol{r})\int p(\Omega'\bullet\Omega)I(\boldsymbol{r},\Omega)\mathrm{d}\Omega' + S(\boldsymbol{r},\Omega) \tag{1}$$

for $I(\boldsymbol{r},\Omega)$ where $\|\Omega\| = \sqrt{\Omega^2} = \sqrt{\Omega\bullet\Omega} = 1$ and $\boldsymbol{r} \in \underline{\mathrm{M}}$ which designates the "closure" of the open set M (for the present purposes, this means "M and its boundary set ∂M"). The last term in Eq. (1), $S(\boldsymbol{r},\Omega)$, represents internal sources. In the neutron transport community, Eq. (1) is known as the one-group linear transport equation.

We will represent the unit vector $\Omega = (\Omega_x,\Omega_y,\Omega_z)^T$ in polar coordinates $\theta = \cos^{-1}\mu$ and φ; so $\Omega_x = \cos\varphi\sqrt{1-\mu^2}$, $\Omega_y = \sin\varphi\sqrt{1-\mu^2}$, and $\Omega_z = \mu$. The phase function is assumed azimuthally symmetric and normalized so that $\int p(\Omega'\bullet\Omega)d\Omega = 1$ where $\Omega'\bullet\Omega = \mu_s$ is the cosine of the scattering angle, thus $d\Omega = 2\pi d\mu_s$. Physically, this means the scatterers are either spherical (like cloud droplets) or randomly oriented. Extinction $\sigma(\boldsymbol{r})$, also assumed independent of direction of propagation, is related to the photon mean free path (MFP) $\lambda(\boldsymbol{r}) = 1/\sigma(\boldsymbol{r})$ which, importantly, is a local quantity in 3D radiative transfer. These optical properties are dependent on the density of scatterers and their cross-sections for scattering and absorption, appropriately averaged over the droplet population. Similarly, ϖ_0 and $p(\mu_s)$ are related to the population-averaged differential cross-section for scattering; an important parameter of the phase function is the mean value of μ_s or asymmetry factor $g = 2\pi\int\mu_s p(\mu_s)d\mu_s$, of considerable importance further on.

The solution we seek is uniquely determined as soon as we specify $S(\boldsymbol{r},\Omega)$ and the boundary conditions (BCs) for $I(\boldsymbol{r},\Omega)$ on $\partial M = \underline{M}\backslash M$. In the I3RC study, we are interested in so-called "albedo" problems where $S(\boldsymbol{r},\Omega) \equiv 0$ in Eq. (1) and $\theta_0 = \cos^{-1}\mu_0$ is the solar zenith angle ($\varphi_0 = \pi$ in I3RC conventions): for $(x,y)^T \in (0,L_x)\otimes(0,L_y)$, we set

$$I(x,y,L_z,\Omega) = F_0\delta(\Omega+\Omega_0),\ \Omega_z < 0, \tag{2a}$$

where F_0 is the solar constant, and

$$I(x,y,0,\Omega) = 0,\ \Omega_z > 0, \tag{2b}$$

in absence of surface reflection. Horizontally, we use cyclical BCs: for $(y,z)^T \in (0,L_y)\otimes(0,L_z)$,

$$I(L_x,y,z,\Omega) = I(0,y,z,\Omega),\ \forall\Omega, \tag{2c}$$

and similarly in $(0,L_x)\otimes(0,L_z)$. Figure 1 illustrates the geometry and notations.

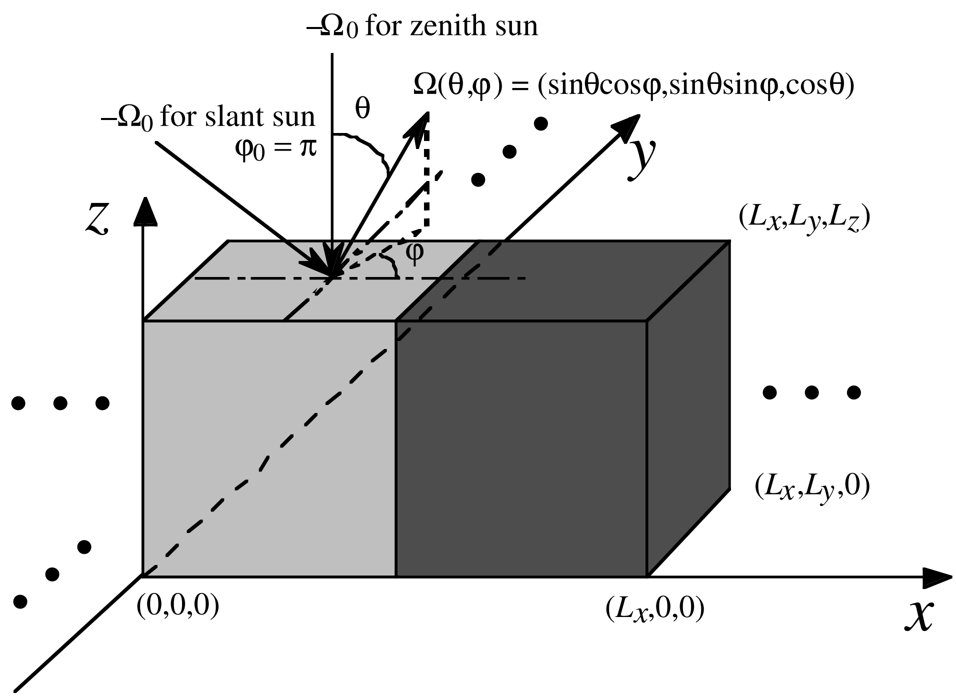

Figure 1: *Schematic of Albedo Problem Illustrated with Square-Wave Cloud Model (Case #1).* In absence of y-variability, grid size in this direction, L_y, is arbitrarily set equal to its counterpart for z, $L_z = 0.25$ km $= \ell_x = L_x/2$ ($N_x = 2$). Infinite periodic replication of the basic element in the picture is represented by "$\cdots$" (x-direction) and "$\therefore$" (y-direction). Both zenith and slant solar illuminations are shown along with an arbitrary reflected beam. In this particular model, we have $\sigma(\boldsymbol{r}) = [10 + 8\times\text{sign}(x-L_x/2)]/L_z$, that is a wave of amplitude 8 over a mean value of 10 in optical thickness $\tau(x)$, as defined in Eqs. (5) and (17).

One can also use $I(\boldsymbol{r},\Omega)$ to model only the "diffuse" component of the radiance field:

$$I(\boldsymbol{r},\Omega) \text{ becomes } I(\boldsymbol{r},\Omega) + F_0\delta(\Omega+\Omega_0)\exp[-\tau(\boldsymbol{r};\boldsymbol{r}_0(\boldsymbol{r},\Omega_0))] \tag{3}$$

where the latter term is the “direct” (or “non-scattered” or “un-collided”) component with

$$\boldsymbol{r}_0(\boldsymbol{r},\Omega_0) = \boldsymbol{r} + \Omega_0 s_0,\ s_0 = (L_z - z)/\Omega_{0z} \tag{4}$$

being the piercing point at the upper (illuminated) boundary, $z = L_z$, for the solar beam reaching $\boldsymbol{r}$. As it appears in Eq. (3), we also need to define the non-dimensional quantity

$$\tau(\boldsymbol{r};\boldsymbol{r}') = \int_0^{\|\boldsymbol{r}'-\boldsymbol{r}\|} \sigma(\boldsymbol{r}+s[(\boldsymbol{r}'-\boldsymbol{r})/\|\boldsymbol{r}'-\boldsymbol{r}\|])\mathrm{d}s \tag{5}$$

as the optical distance between two points in $\underline{\mathrm{M}}$. In this re-formulation, we set $I(x,y,L_z,\Omega) = 0$ in the upper BC in (2a) and, in the RTE (1), we use the following internal source term:

$$S(\boldsymbol{r},\Omega) = \sigma(\boldsymbol{r})F_0\exp[-\tau(\boldsymbol{r};\boldsymbol{r}_0(\boldsymbol{r},\Omega_0))]\varpi_0 p(-\Omega\bullet\Omega_0) \tag{6}$$

which describes solar-beam penetration to point $\boldsymbol{r}$ followed by a single scattering from $-\Omega_0$ to Ω.

TWO REMARKS:

a. Formal Solution of RTE. In absence on scattering ($\varpi_0 = 0$), the 3D RTE in (1) is integrable in closed form:

$$I(\boldsymbol{r},\Omega) = \int_0^{s_{\partial\mathrm{M}}(\boldsymbol{r},\Omega)} S(\boldsymbol{r}-s\Omega,\Omega)\exp[-\tau(\boldsymbol{r}-s\Omega;\boldsymbol{r})]\mathrm{d}s \tag{7}$$

where $s_{\partial\mathrm{M}}(\boldsymbol{r},\Omega)$ defines the boundary piercing point for the beam arriving at $\boldsymbol{r}$ in direction Ω which can be computed as in Eq. (4) for the solar beam. Note that Ω in (7) is clearly more like a parameter, indice, or label for the radiance field than an independent variable. This no-scattering assumption is often justifiable in thermal IR and microwave spectral regions where, furthermore, $S(\boldsymbol{r},\Omega)$ is simply $(1-\varpi_0)\sigma(\boldsymbol{r})$ times Planck’s function. In this example, $S(\boldsymbol{r},\Omega)$ is isotropic, i.e., does not depend explicitly on Ω.

As an example of a non-isotropic source term, using (6) in (7) leads to the 1st-order scattering approximation of the solution of the RTE which is often used for optically thin media, i.e., optical distances in (5) are generally small. Equation (7) thus acquires meaning for the I3RC cases where $0 < \varpi_0 \leq 1$ too, but the optical distances are generally large so the 1st-order scattering approximation fails and the diffusion approximation described in §3 becomes more accurate. Finally, one can interpret $S(\boldsymbol{r},\Omega)$ in (7) as the (Ω-integral) multiple scattering source term in the RTE (1). Equation (7) is then called the “formal solution” of the RTE.

b. Expansion in Spherical Harmonics. As soon as $\varpi_0 > 0$, the integro-differential RTE in (1) is, mathematically speaking, equivalent to an infinite system of coupled 1st-order partial differential equations (PDEs). Form this standpoint, the spatial variable $\boldsymbol{r}$ is independent while the angular variable Ω determines the label of dependent variable I, the coefficients of the PDEs, and their respective BCs. The complexity of this situation can be greatly reduced by noting that the angular coupling in (1) is actually a convolution product of $p(\Omega'\bullet\Omega)$ and $I(\boldsymbol{r},\Omega)$ in the spherical coordinates of Ω, so the angular coupling is simply tri-diagonal for the spherical-harmonic decompositions of $p(\Omega'\bullet\Omega)$ and $I(\boldsymbol{r},\Omega)$, respectively, p_l and $I_{lm}(\boldsymbol{r})$ with $l \geq 0$ ($|m| \leq l$).

The first two spherical harmonics have a special significance: at $l = 0$, we find a scalar measure of radiant energy (or photon) density

$$J(\boldsymbol{r}) = \int I(\boldsymbol{r},\Omega)\mathrm{d}\Omega; \tag{8}$$

at $l = 1$ ($m = -1,0,1$), we find the radiant energy flux vector

$$\boldsymbol{F}(\boldsymbol{r}) = \int \Omega I(\boldsymbol{r},\Omega)\mathrm{d}\Omega. \tag{9}$$

Radiant energy conservation is expressed by

$$\nabla\bullet\boldsymbol{F} + (1-\varpi_0)\sigma(\boldsymbol{r})J(\boldsymbol{r}) = 0, \tag{10}$$

obtained simply by integrating over all angles the RTE in (1) when $S(\boldsymbol{r},\Omega) \equiv 0$; if this is not the case, a source term appears on the r.h. side of (10).

2.2 Definitions of 2D Radiance-Related Quantities Relevant to I3RC

In I3RC – Phase 1, we are interested primarily in the "escaping" radiance fields, namely,

$$\pi I(x,y,L_z,\Omega) \,/\, \mu_0 F_0 \text{ with } \Omega_z \geq 0, \tag{11a}$$

$$\pi I(x,y,0,\Omega) \,/\, \mu_0 F_0 \text{ with } \Omega_z \leq 0, \tag{11b}$$

for $(x,y)^{\mathrm{T}} \in (0,L_x)\otimes(0,L_y)$, respectively for reflected and transmitted radiation. The quantities are expressed here in non-dimensional "bi-directional reflection distribution function" (BRDF) units as a ratio of radiance I to $\mu_0 F_0/\pi$, i.e., the radiance from a perfect Lambertian reflector.

In radiant energy budget applications, one is interested in normalized hemi-spherical fluxes at boundaries:

$$R(x,y) = \int_{\Omega_z \geq 0} \Omega_z I(x,y,L_z,\Omega)\mathrm{d}\Omega \,/\, \mu_0 F_0, \tag{12a}$$

$$T(x,y) = \int_{\Omega_z \leq 0} |\Omega_z| I(x,y,0,\Omega)\mathrm{d}\Omega \,/\, \mu_0 F_0, \tag{12b}$$

which are respectively the reflectance (or albedo) and transmittance fields. Note that, if the direct-diffuse separation in (3) is invoked, then directly transmitted contributions are required in Eqs. (11b), (12b), and (10).

Another local quantity of interest is (true) column absorption,

$$A(x,y) = (1-\varpi_0)\int_0^{L_z} \sigma(\boldsymbol{r})J(\boldsymbol{r})\mathrm{d}z \,/\, \mu_0 F_0 = -\int_0^{L_z} \nabla\bullet\boldsymbol{F}(\boldsymbol{r})\mathrm{d}z \,/\, \mu_0 F_0, \tag{13}$$

where $-\nabla\bullet\boldsymbol{F}(\boldsymbol{r})$ is proportional to the local solar heating rate. "Apparent" column absorption,

$$A_{\mathrm{app}}(x,y) = 1 - R(x,y) - T(x,y) = [-F_z(x,y,L_z)+F_z(x,y,0)] \,/\, \mu_0 F_0, \tag{14}$$

is also worth looking at. The quantity in Eq. (14) can characteristically be inferred from the radiation sampled as it escapes from the upper and lower cloud boundaries whereas its counterpart in Eq. (13) calls for internal radiation fields that are not as easy to measure or to compute. It is therefore important to investigate the difference between the apparent and true column absorptions which is equal to the vertically-integrated horizontal flux divergence (normalized by $\mu_0 F_0$):

$$H(x,y) = 1 - [R(x,y) + T(x,y)] - A(x,y) = \int_0^{L_z} [\partial_x F_x + \partial_y F_y]\mathrm{d}z \,/\, \mu_0 F_0, \tag{15}$$

using the divergence theorem to cancel the z-integral of $\partial_z F_z$. Note that in typical 3D geometries H can be quite small, especially when averaged over some horizontal area, because of cancellation effects in the vertical and horizontal integrations. In general, none of its non-negative components (A, R, T) or its signed ones (A_{app}, F_x, F_y) are small by any standard of comparison such as $\mu_0 F_0$.

Because of the cyclic horizontal BCs in Eq. (2c), the integral of (15) over $(0,L_x)\otimes(0,L_y)$ vanishes identically. This is expected, simply to conserve total incoming radiant energy over the entire computational domain:

$$A + R + T \text{ (absorbed or escaped)} = 1 \text{ (incoming from above)}, \tag{16a}$$

where

$$F = \iint F(x,y)\mathrm{d}x\mathrm{d}y/L_x L_y \tag{16b}$$

for $F = A,R,T$. It is reasonable to assume there exists such an averaging scale in real cloud/surface systems as well and this is the basic premise in ARESE-type experiments; see an account of the observations by *Valero et al.* [1997] and simulations by *Marshak et al.* [1997].

2.3 The Independent Pixel Approximation (IPA) to 3D Radiative Transfer

Although it is insufficient to describe the 3D structure of a real cloud, often all that is available from remote sensing data (cf. I3RC "Landsat" case) or in simple stochastic models (such as bounded cascades [*Cahalan et al.*, 1994a]) is the optical depth as a function of position in the horizontal plane, i.e.,

$$\tau^*(x,y) = \tau(x,y,0;x,y,L_z) \tag{17}$$

from Eq. (5) for $(x,y)^T \in (0,L_x)\otimes(0,L_y)$. In the IPA, all horizontal derivatives are explicitly ignored in Eq. (1), so the standard change of variables $d\tau(z) = \sigma(z)dz$ can be made, holding (x,y) constant. Therefore $\tau^*(x,y)$ is all that is needed to obtain $I(\boldsymbol{r},\Omega) = I(\tau^*(x,y);z,\Omega)$ and all the required quantities in (11–13) using any plane-parallel computational model. Notice that the whole (x,y) dependence in I is now collapsed into the single parameter τ^*. In contrast, the I3RC study is focused entirely on deviations from the IPA, locally and over the entire (x,y)-domain which *Cahalan et al.* [1994b] call the "IPA bias."

Since horizontal derivatives are set to zero in the IPA, Eq. (15) for $H(x,y)$ vanishes identically. There is no difference between A and A_{app}: true column absorption can be reliably inferred from boundary quantities alone. Looking back at Eqs. (16a–b), this does not imply however that a *single* IPA (plane-parallel) computation works at scales large enough for (16a–b) to apply, only that lateral boundary effects can be neglected in the domain-average energy balance equation. However, the IPA has a good chance of working when *many* plane-parallel computations are averaged over the (x,y)-variability [*Cahalan et al.*, 1994a]. This in turn does not imply that IPA predictions for radiative processes on smaller scales (e.g., local heating rates required in numerical simulations of cloud dynamics) are adequate because of the above-mentioned cancellation effects.

3. The Diffusion or "P_1" Approximation

3.1 Diffusion Basics

The most direct route to diffusion theory is to complement the exact law of radiant energy (or photon number) conservation in Eq. (10) with a so-called constitutive equation that links the flux $\boldsymbol{F}$ back to the density J. This is precisely the role of Fick's law of diffusion:

$$\boldsymbol{F}(\boldsymbol{r}) = -D(\boldsymbol{r})\nabla J \tag{18}$$

where $D(\boldsymbol{r})$ is local radiative diffusivity, adopting units where the velocity of light $c = 1$. More generally, $\boldsymbol{F} = -D\nabla U$ where $U = J/c$ is radiant energy density and the units of D are length^2/time. Fick's law was actually formulated for particles with mass and associated with the expression of mass conservation (in the so-called continuity equation). Equation (18) is also an analog of Fourier's famous law of heat conduction (complementing the energy conservation equation) or Ohm's law (and charge conservation) or Darcy's law of incompressible flow in porous media (and again conservation of fluid mass), and so on. There are many situations were diffusion laws arise.

By substitution of (18) into Eq. (10), we now have a steady-state 3D diffusion problem (2nd-order elliptical PDE) to solve for $J(\boldsymbol{r})$:

$$[-\nabla\bullet D(\boldsymbol{r})\nabla + (1-\varpi_0)\sigma(\boldsymbol{r})]J = 0, \tag{19}$$

a considerable simplification over the 3D RTE. This scheme was first proposed by *Eddington* [1916] as a closure of the hierarchy of equations corresponding to the successive Ω-moments of the RTE. The 0th-order moment of (1) yields Eq. (10); along with Eddington's closure hypothesis, the 1st-order equation leads to an important expression for diffusivity in terms of known quantities:

$$D(\boldsymbol{r}) = 1/[3(1-\varpi_0 g)\sigma(\boldsymbol{r})]. \tag{20}$$

At least three other routes have been charted from radiative transfer theory to diffusion:

- 1st-order truncation in a spherical harmonic expansion, hence the name "P_1" approximation used by some authors [e.g., *Case and Zweifel*, 1967];
- through asymptotic radiative transfer theory [*van de Hulst*, 1980] leading to (18) and (20), the latter with higher-order correction terms if necessary;
- through discrete-angle radiative transfer, where a formal choice of a phase-function parameter is used [*Lovejoy at al.*, 1990].

However, none of these alternatives conveys the physical meaning of the photon diffusion assumption in Eqs. (18) and (20), namely, that the photon's "transport" MFP

$$\lambda_t(\boldsymbol{r}) = 3D(\boldsymbol{r}) = [(1-\varpi_0 g)\sigma(\boldsymbol{r})]^{-1} = \lambda(\boldsymbol{r})/(1-\varpi_0 g) \tag{21a}$$

is small with respect to the outer dimensions of the system (i.e., cloud thickness L_z), and possibly even all the scales of σ/D-variability down to the grid constants ℓ_i ($i = x,y,z$) [*Davis et al.*, 1998]. Indeed, the conceptual model here is that the optical medium is so dense that photons are transported via convoluted random (drunkards') walks with each "step," on the order of λ_t, small with respect to L_z. This contrasts strongly with the idea of photons in quasi-ballistic streaming motion through a semi-transparent medium. Note that the stated limitation on the σ-variability can be overcome by allowing for "anomalous" photon diffusion [*Davis and Marshak*, 1997], which does not at present have a complete mathematical description.

If we adopt Eq. (19) to describe photon transport, then the simple physical random-walk picture applies exactly only to situations where $\varpi_0 = 1$, i.e., the second term in (19) vanishes identically. In presence of absorption ($\varpi_0 < 1$), there is another important length scale in diffusion theory, the (local) "diffusion" length defined as the square-root of the ratio of the two coefficients in (19), namely,

$$\Lambda_d(\boldsymbol{r}) = \sqrt{D(\boldsymbol{r})/(1-\varpi_0)\sigma(\boldsymbol{r})} = \lambda(\boldsymbol{r})/[3(1-\varpi_0)(1-\varpi_0 g)]^{1/2}. \tag{21b}$$

In radiative transfer terminology, this random-walk picture we have conjured up means that, collectively, the angular distribution of photons $I(\boldsymbol{r},\Omega)$ is almost everywhere amenable to representation by a scalar term $J(\boldsymbol{r})$ and a vector term $\boldsymbol{F}(\boldsymbol{r})$; more specifically, we have:

$$I(\boldsymbol{r},\Omega) \approx [J(\boldsymbol{r})+3\Omega\bullet\boldsymbol{F}(\boldsymbol{r})]/4\pi. \tag{22}$$

The "almost" qualifier is important here because this approximation will certainly to break down at (even just near) the open boundaries of the medium, $z = 0$ and L_z in the slab geometry considered in I3RC, and possibly at internal "boundaries" too where the local extinction value becomes too small (λ_t increases beyond, say, L_z). As absorption increases with $(1-\varpi_0)$, we can anticipate more difficulties for diffusion/random-walk theory when $\Lambda_d(\boldsymbol{r})$ in (21b) becomes comparable to and eventually smaller than $\lambda_t(\boldsymbol{r})$ in (21a), even in absence of boundaries. In other words, the non-dimensional quantity $3(1-\varpi_0)/(1-\varpi_0 g)$ reaches and exceeds $O(1)$ values which occurs when $\varpi_0 < 2/(3-g) \approx 0.93$ for $g \approx 0.85$ (the canonical value for dense boundary-layer clouds).

3.2 BCs in Diffusion Theory and Escaping Fluxes

Boundaries and BCs are a mathematical as well as a physical complication in radiative diffusion theory, unlike in its aforementioned thermal, electrical and hydrological analogs. Indeed, neither "potential" (J) nor "current" ($\boldsymbol{n}\bullet\boldsymbol{F}$ where $\boldsymbol{n}$ is the normal unit-vector at the boundary point) are imposed. These would lead respectively to Dirichlet or von Neumann BCs. Instead, the z-axis BCs in Eqs. (2a) and (2b) can only be approximated by mixed or Robin BCs [e.g., *Case and Zweifel*, 1967]:

$$\frac{1}{2}\,[1-\chi\lambda_t(\boldsymbol{r})\partial_z]J(\boldsymbol{r})\big|_{z=L_z} = 1, \tag{23a}$$

adopting radiance units where $\mu_0 F_0 = 1$, and

$$\frac{1}{2}\,[1+\chi\lambda_t(\boldsymbol{r})\partial_z]J(\boldsymbol{r})\big|_{z=0} = 0. \tag{23b}$$

In the above, χ is an $O(1)$ numerical factor that determines the local extrapolation length $\chi\lambda_t(x,y)$ that is generally determined by matching in some respect exact calculations. For instance, the Milne problem (which pertains to very opaque media), suggests $\chi = 0.7104$ whereas Eq. (22) predicts 2/3 and asymptotic arguments for tenuous media lead to 4/3.

The reflected and transmitted flux fields of interest in I3RC, Eqs. (16a–b), are given by:

$$R(x,y) = \frac{1}{2}\,[1+\chi\lambda_t(\boldsymbol{r})\partial_z]J(\boldsymbol{r})\big|_{z=0} = J(x,y,L_z)-1, \tag{24a}$$

$$T(x,y) = \frac{1}{2}\,[1-\chi\lambda_t(\boldsymbol{r})\partial_z]J(\boldsymbol{r})\big|_{z=0} = J(x,y,0), \tag{24b}$$

using (23a–b) while (13) can be applied directly to obtain $A(x,y)$ from the local rate of absorption:

$$-\nabla \bullet \boldsymbol{F} = (1-\varpi_0)\sigma(\boldsymbol{r})J(\boldsymbol{r}) \tag{24c}$$

from (10). Note that μ_0 has vanished entirely from the problem, meaning that this formulation is equivalent to modeling a diffuse source at the upper boundary, a limitation to be lifted shortly.

3.3 Enhancements to Diffusion Theory

Because we are trying to represent with Eq. (22) a radiance field that is not everywhere that smooth in Ω-space, the separation of direct and diffuse components in Eq. (3) leads not to an equivalent formulation as for the RTE but to an actual improvement in diffusion theory. So $J(\boldsymbol{r})$ now models only the diffuse component of the radiation field. Using Eq. (6) for $S(\boldsymbol{r},\Omega)$ to re-derive (10) from (1) by Ω-integration of each term, we find the PDE to be solved is now

$$[-\nabla \bullet D(\boldsymbol{r})\nabla + (1-\varpi_0)\sigma(\boldsymbol{r})]J = \varpi_0\sigma(\boldsymbol{r})F_0\exp[-\tau(\boldsymbol{r};\boldsymbol{r}_0(\boldsymbol{r},\Omega_0))]. \tag{25}$$

As for BCs, the r.h. side of (23a) is now set to 0, and the "–1" is dropped from (24a) for local reflectance. Finally, a direct component $\exp[-\tau(x,y,0;\boldsymbol{r}_0(x,y,0,\Omega_0))]$ is added to the transmittance field in (24b). In particular, this improved formulation restores the important dependence on μ_0.

By the same token, it is advantageous to add a δ-component to the phase function to emulate the quasi-forward streaming of light due to the diffraction peak in Mie-type scattering. Following *Joseph et al.* [1976], we take

$$p(\mu_s) = f\delta(1-\mu_s) + (1-f)[1+3g'\mu_s], \tag{26}$$

hence

$$g' = (g-f)/(1-f). \tag{27}$$

Single-scattering albedo is rescaled as

$$\varpi_0' = \varpi_0(1-f)/(1-\varpi_0 f), \tag{28}$$

and the extinction field as

$$\sigma'(\boldsymbol{r}) = (1-\varpi_0 f)\sigma(\boldsymbol{r}). \tag{29}$$

This is the basis of the δ-Eddington approximation where it is customary to set $f = g^2$. We thus solve the PDE in (25) with homogeneous BCs using the new coefficients. We note however that neither absorptivity $(1-\varpi_0)\sigma(\boldsymbol{r})$ nor diffusivity $D(\boldsymbol{r})$ on the l.h. side of (25) change under the mappings in Eqs. (27–29), nor does $\lambda_t(x,y)$ in the BCs expressed in (23a–b). The only changes are in $\varpi_0\sigma(\boldsymbol{r})$ and $\tau(\boldsymbol{r};\boldsymbol{r}_0(\boldsymbol{r},\Omega_0))$ that appear in the internal source term on the r.h. side of (25): they are respectively rescaled by $(1-f)$ and $(1-\varpi_0 f)$, and we recall that $\tau(\boldsymbol{r};\boldsymbol{r}_0(\boldsymbol{r},\Omega_0))$ also appears in the direct transmission term that is now required in addition to (24b) for local transmittance. Because of the exponential function in (25), this rescaling has non-trivial consequences on $J(\boldsymbol{r})$.

Being entirely based on the scalar- and vector-flux quantities in Eqs. (8–9), diffusion theoretical methods do not naturally yield radiance fields $I(\boldsymbol{r},\Omega)$. However, one can use the diffusion solution to approximate the Ω-integral (multiple-scattering source) term in the RTE from Eq. (22) with the phase function in (26) or a more realistic one, then use the formal solution of the RTE in Eq. (7) to obtain radiances anywhere. This procedure is known in the neutron transport literature as "long-characteristics." From there, hemi-spherical boundary fluxes in Eqs. (12a–b), already estimated using Eqs. (24a–b), can then be re-evaluated by standard angular quadrature rules to obtain improved agreement with detailed RTE solutions.

3.4 Numerical Implementation

Numerical solution of a scalar PDE such as (19) or (25) poses no particular problem or, rather, it has been extensively studied due to the many applications in statistical physics partially listed above and the numerous engineering problems associated with these. Many efficient and robust methods have this been developed for regular grids (including MC-like methods using "particles" with quite different propagation rules than photons) as well as unstructured meshes (for finite element approaches). *Qu* [1999] has developed a full-multigrid solver for Eq. (25) with a variety of lateral BCs, including cyclical, and he has implemented the long-characteristics procedure in his Eddington Delta-approximation in 3 Dimensions model (ED3D). Figure 2 shows ED3D output for the local reflectance and transmittance fields for the square-wave cloud model.

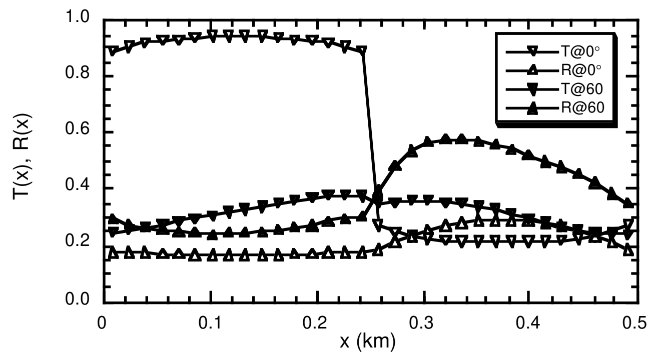

Figure 2: *Diffusion Solutions for the Square-Wave Cloud Model (I3RC case #1).* $T(x)$ and $R(x)$ results for both zenith and 60° sun problems are plotted for the absorbing case ($\varpi_0 = 0.99$).

3.5 Analytical Results

We are not aware of an analytical solution of the square-wave cloud problem in radiative diffusion theory. However, the plane-parallel "sine-wave" cloud problem has been worked out [*Weinman and Swartztrauber*, 1968] as well as several isolated cloud problems: finite cylinders [*Barkstrom and Arduini*, 1977]; cuboids [*Davies*, 1978] which are, in a sense, one half-oscillation in the train of square waves; and spheres [*Davis*, 1992], both full and hollow.

The problem set-up in Eqs. (19) and (23a,b) is simple enough that at least one general 3D result can be derived, so far, only for slab geometry with $\varpi_0 = 1$. Indeed, *Davis et al.* [1998] relate the relative change in domain-average albedo, $\delta R/R$ where R is obtained as in (16b), is to the given fluctuation in extinction $\delta\sigma(\boldsymbol{r})$ relative to the 3D domain-average extinction,

$$\langle\sigma\rangle = \iint \sigma(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z/L_xL_yL_z, \tag{30}$$

and the resulting perturbation in vertical flux normalized by its incoming counterpart, $\delta F_z(\boldsymbol{r})/\mu_0 F_0$. More precisely, the authors show that

$$\frac{-\delta R}{R} = \frac{1}{\langle\sigma\rangle\mu_0 F_0}\iint \delta\sigma(\boldsymbol{r})\delta F_z(\boldsymbol{r})\mathrm{d}\boldsymbol{r}/L_xL_yL_z. \tag{31}$$

In other words, δR is proportional to the spatial cross-correlation function of $\sigma(\boldsymbol{r})$ and $F_z(\boldsymbol{r})$. Furthermore, they prove that δR is always negative. The only requirement in the derivation is that the spatial integral of $\delta\sigma(\boldsymbol{r})$ vanishes which, recalling that σ is proportional to scattering particle density, is equivalent to a mass conservation constraint. This establishes that, at least under illumination conditions symmetric enough to be represented by (23a), albedo is decreased no matter how the density of scatterers is redistributed in the 3D volume. For instance, if this 3D density redistribution is such that it leaves column integrals (local optical depth) invariant, the IPA predicts no change in radiative quantities but 3D diffusion theory does and the IPA bias always lowers the domain-average albedo. The general result in Eq. (31) can be used as a numerical benchmark for any radiative diffusion code that yields internal fields.

4. Summary and Outlook

We have surveyed the mathematical and physical technicalities of photon diffusion theory and its background in radiative transfer theory, emphasizing the nature of the approximation as well as the benefits it offers over computational radiative transfer. In the context of I3RC which is interested in dense clouds, the most notable advantage of diffusion is the extreme numerical efficiency of some common implementations which more than offsets the inherent loss of accuracy with respect to Monte Carlo or explicit methods of solving the radiative transfer equation.

Three-dimensional diffusion-based codes are likely to find applications in the short term in the dynamical cloud modeling community. Indeed, computer time is more important than high accuracy in the required solar heating rates when running Large-Eddy Simulation (LES) codes, given that their current shortwave radiation parameterizations totally neglect horizontal fluxes in spite of the effort to resolve intricate 3D cloud structure. This is a much rougher approximation than 3D diffusion versus 3D radiative transfer that can lead to significant biases in the local energy budget estimations, on the order of the horizontal flux values. These fluxes are generally not small with respect to the incoming broadband flux.

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