"Let's get something straight here... e is real, 10 is just the number of fingers we have."

- Prof. Nima Arkani-Hamed, UC Berkeley

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

If you liked problem 1 and you're interested in astrophysics, general relativity, and cosmology, you should check out a paper by Saul Perlmutter, Michael S. Turner, and Martin White (Physical Review Letters, July 26, 1999, Volume **83**, Issue 4, pp. 670-673). This article and references therein describe an ongoing study of type Ia supernovae which have "standard candle" light output and have enabled these scientists to measure large-scale cosmological parameters. One of the most interesting results is that their data is consistent with a universe that is expanding at an *accelerating* rate! Saul Perlmutter's group is here at Berkeley and works at LBL.

Problem 1

First, let's consider the implications of the difference in light intensity of the two supernovae (SN1 and SN2). These particular supernovae are known to have identical "standard candle" light output, i.e. the total light power P emitted is the same for SN1 and SN2. A small solid angle $d\Omega$ of the total light is detected on earth, so the intensity of light I detected is given by:

$$I = \frac{d\Omega \cdot P}{4\pi R^2}, \tag{1}$$

where R is the distance from a supernova to the earth at the time the light is emitted. Therefore the ratio of light intensities tells us the ratio of distances:

$$\frac{I_1}{I_2} = \frac{R_2^2}{R_1^2} = 4. \tag{2}$$

(a)

An astronomer theorizes that SN1 causes SN2, and that they are both at rest with respect to the earth. Since the two events SN1 $= (ct_1, x_1)$ and SN2 $= (ct_2, x_2)$ are causally related, there must be a timelike or lightlike separation between the events:

$$c^2\Delta t^2 - \Delta x^2 \geq 0, \tag{3}$$

where $\Delta t = t_2 - t_1$ and $\Delta x = x_2 - x_1$.

From Eq. (2), we see that if the distance between SN2 and SN1 is Δx, the distance between SN1 and earth is also Δx. Then the elapsed time Δt_{earth} between detection of the two supernovae on earth, taking into account the propagation time of the light to the earth, is given by:

$$c\Delta t_{earth} = c\Delta t + c\left(\frac{2\Delta x}{c}\right) - c\left(\frac{\Delta x}{c}\right) = c\Delta t + \Delta x. \tag{4}$$

From Eq. (3), we know that $c\Delta t \geq \Delta x$, so we find that:

$$\Delta x \leq \frac{c\Delta t_{earth}}{2}, \tag{5}$$

or that $\Delta x_{max} = 5$ light years.

(b)

A physicist theorizes that the two supernovae were traveling away from the earth at some velocity and occurred at the same proper time. In this case the two events have a spacelike or lightlike separation:

$$c^2\Delta t^2 - \Delta x^2 \leq 0. \tag{6}$$

Thus, in the earth frame there is an observed time difference Δt_{obs} between SN1 and SN2, which from Eq. (6) must satisfy:

$$c\Delta t_{obs} \leq \Delta x_{obs}, \tag{7}$$

where Δx_{obs} is the distance between SN1 and SN2 as observed in the earth frame. As in part (a) we include the light propagation time, and find that:

$$c\Delta t_{earth} = c\Delta t_{obs} + \Delta x_{obs} \leq 2\Delta x_{obs}. \tag{8}$$

So in this case we find that $\Delta x_{obs,min} = 5$ light years.

Problem 2

(a)

This is just the traditional Lorentz matrix, only in 3D, so it is similar to the expression (1.12) in Prof. Strovink's notes on relativity,

$$\begin{pmatrix} ct' \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} ct \\ x \\ y \end{pmatrix}. \tag{9}$$

(b)

The general idea is to rotate to a system where we know the correct transform (from part (a)), and then rotate back. So we begin with:

$$\boldsymbol{r}' = \mathfrak{L} \cdot \boldsymbol{r}. \tag{10}$$

Then we rotate the coordinate system with a rotation matrix R so that $\vec{\beta}$ is along $\hat{x}$:

$$R\boldsymbol{r}' = R\mathfrak{L} \cdot \boldsymbol{r} = R\mathfrak{L}R^{-1}(R\boldsymbol{r}). \tag{11}$$

In this frame we know the Lorentz transform Λ from part (a), so we find that:

$$R\mathfrak{L}R^{-1} = \Lambda. \tag{12}$$

In other words,

$$\mathfrak{L} = R^{-1}\Lambda R. \tag{13}$$

The math can be made a little easier in these cases because rotations are described by orthogonal matrices which satisfy $R^{-1} = R^T$ where R^T is the transpose of R.

Now for the actual math. From (a) and Eq. (13) we find:

$$\mathfrak{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{pmatrix} \cdot \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{pmatrix}, \tag{14}$$

where we use straightforward 3D extensions of the usual rotation matrices. Multiplying these matrices gives us:

$$\mathfrak{L} = \begin{pmatrix} \gamma & -\gamma\beta cos\theta & -\gamma\beta sin\theta \\ -\gamma\beta cos\theta & \gamma cos^2\theta + sin^2\theta & -cos\theta sin\theta + \gamma cos\theta sin\theta \\ -\gamma\beta sin\theta & -cos\theta sin\theta + \gamma cos\theta sin\theta & cos^2\theta + \gamma sin^2\theta \end{pmatrix}. \tag{15}$$

We are given that:

$$\boldsymbol{V} = \beta c \frac{\hat{x} + \hat{y}}{\sqrt{2}}, \tag{16}$$

so $\theta = \pi/4$. Then $\mathfrak{L}$ is given by:

$$\mathfrak{L} = \begin{pmatrix} \gamma & \frac{-\gamma\beta}{\sqrt{2}} & \frac{-\gamma\beta}{\sqrt{2}} \\ \frac{-\gamma\beta}{\sqrt{2}} & \frac{1+\gamma}{2} & \frac{1}{2}(\gamma - 1) \\ \frac{-\gamma\beta}{\sqrt{2}} & \frac{1}{2}(\gamma - 1) & \frac{1+\gamma}{2} \end{pmatrix} \tag{17}$$

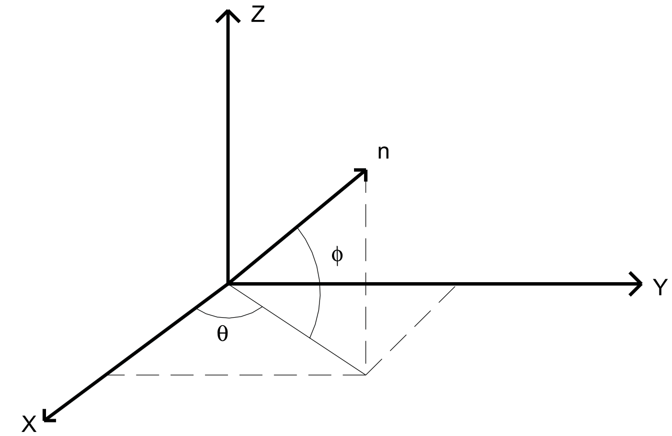


Figure 1: Relationship between $\hat{n}$ and the angles θ and ϕ employed in problem (3) in the first approach.

As you can see by inspection, this matrix is symmetric under interchange of x and y and reduces to the identity matrix as $\beta \to 0$.

Problem 3

Here are two common approaches to this problem. The first method involves matrix multiplication in a manner similar to that employed in problem 2. The second involves determining a general vector formula for the Lorentz transform.

Approach 1

First, we'll determine the rotation matrix R which will take us into the frame where $\vec{\beta}$ is along $\hat{x}$. For me, it's easier to think of this in terms of the angles θ and ϕ as defined in Fig. 1.

From Fig. 1, we notice that there is a natural correspondence between (n_x, n_y, n_z) and θ, ϕ given by

$$\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} cos\phi cos\theta \\ cos\phi sin\theta \\ sin\phi \end{pmatrix}. \tag{18}$$

Later, these relations will be used to express $\mathfrak{L}$ in terms of n_x, n_y and n_z.

R is given by the multiplication of two rotation matrices A and B,

$$R = B \cdot A, \tag{19}$$

where A rotates the axes about $\hat{z}$ by θ

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & sin\theta & 0 \\ 0 & -sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{20}$$

and B rotates the axes about a new $\hat{y}'$ (the y-axis after rotation by A) by ϕ so that $\hat{x}$ is along $\hat{n}$:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\phi & 0 & sin\phi \\ 0 & 0 & 1 & 0 \\ 0 & -sin\phi & 0 & cos\phi \end{pmatrix}. \tag{21}$$

Applying Eq. (13), we can now solve for $\mathfrak{L}$:

$$\mathfrak{L} = A^{-1} \cdot B^{-1} \cdot \Lambda \cdot B \cdot A = A^T \cdot B^T \cdot \Lambda \cdot B \cdot A, \tag{22}$$

where Λ is given by Eq. (1.12) in Prof. Strovink's notes on relativity.

The result of this rather tedious matrix multiplication, after some simplification using basic trigonometric identities, is given by $\mathfrak{L} =$

$$\begin{pmatrix} \gamma & -\beta\gamma cos\theta cos\phi & -\beta\gamma sin\theta cos\phi & -\beta\gamma sin\phi \\ -\beta\gamma cos\theta cos\phi & 1+(\gamma-1)cos^2\theta cos^2\phi & (\gamma-1)sin\theta cos\theta cos^2\phi & (\gamma-1)cos\theta sin\phi cos\phi \\ -\beta\gamma sin\theta cos\phi & (\gamma-1)sin\theta cos\theta cos^2\phi & 1+(\gamma-1)sin^2\theta cos^2\phi & (\gamma-1)sin\theta sin\phi cos\phi \\ -\beta\gamma sin\phi & (\gamma-1)cos\theta sin\phi cos\phi & (\gamma-1)sin\theta sin\phi cos\phi & 1+(\gamma-1)sin^2\phi \end{pmatrix}. \tag{23}$$

If we then use the relations given in Eq. (18) to re-express Eq. (23) in terms of n_x, n_y and n_z we find that:

$$\mathfrak{L} = \begin{pmatrix} \gamma & -\beta\gamma n_x & -\beta\gamma n_y & -\beta\gamma n_z \\ -\beta\gamma n_x & 1+(\gamma-1)n_x^2 & (\gamma-1)n_x n_y & (\gamma-1)n_x n_z \\ -\beta\gamma n_y & (\gamma-1)n_y n_x & 1+(\gamma-1)n_y^2 & (\gamma-1)n_y n_z \\ -\beta\gamma n_z & (\gamma-1)n_z n_x & (\gamma-1)n_z n_y & 1+(\gamma-1)n_z^2 \end{pmatrix}. \tag{24}$$

Approach 2

In this approach, we work out a vector formula for the Lorentz transformation using the fact that length contraction occurs only in the direction of $\vec{\beta}$. If, for example, $\hat{n}$ is along $\hat{x}$, Λ from Eq. (1.12) in Strovink's notes is applicable and we find:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma(ct-\beta x) \\ x+(\gamma-1)x-\gamma\beta ct \\ y \\ z \end{pmatrix}. \tag{25}$$

Let $\vec{r} \equiv (x, y, z)$, then by analogy with Eq. (25) we find that:

$$ct' = \gamma(ct - \beta\vec{r}\cdot\hat{n}) \tag{26}$$

and

$$\vec{r}' = \vec{r} + \hat{n}((\gamma-1)\vec{r}\cdot\hat{n} - \gamma\beta ct). \tag{27}$$

We can then express these equations in terms of n_x, n_y and n_z:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma ct - \gamma\beta(n_x x + n_y y + n_z z) \\ x + n_x(\gamma-1)(n_x x + n_y y + n_z z) - \gamma\beta ct \\ y + n_y(\gamma-1)(n_x x + n_y y + n_z z) - \gamma\beta ct \\ z + n_z(\gamma-1)(n_x x + n_y y + n_z z) - \gamma\beta ct \end{pmatrix}. \tag{28}$$

These expressions can be re-written in matrix form, yielding $\mathfrak{L}$ from Eq. (10) to be:

$$\mathfrak{L} = \begin{pmatrix} \gamma & -\beta\gamma n_x & -\beta\gamma n_y & -\beta\gamma n_z \\ -\beta\gamma n_x & 1+(\gamma-1)n_x^2 & (\gamma-1)n_x n_y & (\gamma-1)n_x n_z \\ -\beta\gamma n_y & (\gamma-1)n_y n_x & 1+(\gamma-1)n_y^2 & (\gamma-1)n_y n_z \\ -\beta\gamma n_z & (\gamma-1)n_z n_x & (\gamma-1)n_z n_y & 1+(\gamma-1)n_z^2 \end{pmatrix}, \tag{29}$$

which you will notice is the same result as the one obtained in approach 1.

Problem 4

(a) The current I is the charge per second traveling through the channel, given by:

$$I = nA(+e)(+\beta c) + nA(-e)(-\beta c) = 2nAe\beta c. \tag{30}$$

(b)

Ampere's law (in SI units, feel free to use whatever units you like of course) is:

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enclosed}. \tag{31}$$

So in our case, assuming an infinitely long channel and using I from Eq. (30):

$$B_\phi \cdot 2\pi r = \mu_0 2nAe\beta c, \tag{32}$$

therefore

$$B_\phi = \frac{\mu_0 nAe\beta c}{\pi r}. \tag{33}$$

(c)

Let's solve this both suggested ways... first using length contraction. The density of positrons n_+ is given by:

$$n_+ = \frac{N_+}{A \cdot d}, \tag{34}$$

where d is a unit length of the channel in the lab frame S and N_+ is the total number of positrons contained in this volume. This new frame S' is the rest frame of the positrons, so $d' = \gamma d$ (sort of length un-contraction). Therefore the observed positron density in S' is given by:

$$n'_+ = \frac{N_+}{A\gamma d} = \frac{n_+}{\gamma}. \tag{35}$$

We can arrive at the same conclusion using the fact that $\left(c\rho, \vec{j}\right)$ is a four-vector. Considering only the z-direction, we have the relation:

$$\begin{pmatrix} c\rho' \\ j'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \cdot \begin{pmatrix} c\rho \\ j_z \end{pmatrix}, \tag{36}$$

where the charge density in the lab frame S satisfies $c\rho = n_+ ec$ and the current density in S is given by $j_z = n_+ e\beta c$. Thus from Eq. (36) we find that:

$$c\rho' = n'_+ ec = \gamma\left(n_+ ec - \beta^2 n_+ ec\right). \tag{37}$$

Consequently,

$$n'_+ = \frac{n_+}{\gamma} \tag{38}$$

as above.

(d)

Here we'll just stick to the four-vector method. The relationship between the charge density of electrons seen in S' and S is given by:

$$\begin{pmatrix} c\rho' \\ j'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \cdot \begin{pmatrix} c\rho \\ j_z \end{pmatrix}, \tag{39}$$

from which we find the charge density:

$$c\rho' = \gamma c\rho - \beta\gamma j_z. \tag{40}$$

The electron charge density in S satisfies $c\rho = n_-(-e)c$ and the current density in S is given by $j_z = n_-(-e)(-\beta)c$. Plugging these into Eq. (40) allows us to solve for n'_-:

$$n'_- = \gamma\left(1 + \beta^2\right)n_-. \tag{41}$$

(e)

First we solve the problem using Gauss's law:

$$\int E'_r \cdot dA = \int \frac{\rho}{\epsilon_0} dV. \tag{42}$$

Choosing a cylindrical Gaussian surface centered on the z-axis with radius r and length d, we find:

$$E'_r \cdot 2\pi rd = Ad\left(\rho'_+ + \rho'_-\right)/\epsilon_0. \tag{43}$$

Using the relations for the charge density of positrons and electrons in S' from Eqs. (38) and (41), we find that:

$$\rho'_+ + \rho'_- = \frac{ne}{\gamma}\left(1 - \gamma^2(1 + \beta^2)\right) = -2\beta^2\gamma ne. \tag{44}$$

Combining these results, we find for the radial electric field E'_r seen in S':

$$E'_r = -\hat{r}\frac{\beta^2\gamma neA}{\pi\epsilon_0 r}. \tag{45}$$

We can also solve this problem using the relativistic field transformations for E and B given in Prof. Strovink's notes (Eq. (1.33)), in particular:

$$E'_\perp = \gamma\left(E_\perp + c\vec{\beta} \times \vec{B}\right). \tag{46}$$

Employing B_ϕ from Eq. (33) and noting that $E_r = 0$ in S, we find that:

$$E'_r = -c\beta\gamma B_\phi \hat{r} = -\hat{r}\frac{\beta^2\gamma neA}{\pi\epsilon_0 r}. \tag{47}$$

Where we use the fact that $\epsilon_0\mu_0 = 1/c^2$. This, of course, agrees with our result from Eq. (45) using Gauss's law.

Problem 5

(a)

Note the interesting fact that the number of seconds in a year is approximately $\pi \times 10^7$, a useful fact at cocktail parties and for back-of-the-envelope calculations.

If you naively multiply the acceleration by the time, you find:

$$v = gt \approx 10c. \tag{48}$$

So, if you're a believer in relativity, this can't be right...

(b)

This part is basically worked out in the text of the problem, so there's nothing to say...

(c)

We start with

$$dx = c \cdot sinh(\eta)d\tau = c \cdot sinh\left(\frac{g\tau}{c}\right)d\tau, \tag{49}$$

where we use the expression

$$c\eta = g\tau. \tag{50}$$

Next we integrate the small displacements from $0 \to \tau_f$ where τ_f is the final "astronaut time."

$$\int_0^{x_f} dx = \int_0^{\tau_f} c \cdot sinh\left(\frac{g\tau}{c}\right)d\tau \tag{51}$$

We can make a straightforward change of variable $\xi = g\tau/c$:

$$x_f = \frac{c^2}{g}\int_0^{g\tau_f/c} sinh(\xi)d\xi. \tag{52}$$

Finally arriving at the solution:

$$x_f = \frac{c^2}{g}(cosh(g\tau_f/c) - 1). \tag{53}$$

(d)

If we plug in the numbers we find that $x_f \approx 10^{20}$ meters or 10^4 light years. This is just the first leg of the journey, so the furthest distance the astronaut can reach is twice this, or 20,000 light years away! So the engineer was right...

Problem 6

(a)

The relativistic expression for energy E of particles with non-zero mass is given by Eq. (1.23) in Strovink's notes:

$$E = \gamma mc^2, \tag{54}$$

where m is the rest mass of the particles. Since $\gamma = cosh(\eta)$, the boost η is given by:

$$\eta = cosh^{-1}\left(\frac{E}{mc^2}\right) \tag{55}$$

Knowing from the problem that $mc^2 = 0.5 \times 10^6$ eV and $E_{final} = 5 \times 10^{10}$ eV, we can solve for η:

$$\eta = 12.2 \tag{56}$$

(b)

We can use the result obtained in problem 5, namely Eq. (53), replacing g with some constant acceleration a. We also replace $cosh(g\tau_f/c)$ with γ_f, which from part (a) we find is $\gamma_f \approx 10^5$. This gives us:

$$x_f \approx \frac{c^2}{a}\gamma_f \tag{57}$$

Solving for a and making the appropriate substitutions yields:

$$a \approx 3 \times 10^{17} g \tag{58}$$

(c)

We can use the relation between proper time $d\tau$ and time in the laboratory frame dt from problem 5:

$$dt = cosh(\eta)d\tau. \tag{59}$$

If we apply the relation $c\eta = a\tau$, then integrating this expression yields:

$$t_{lab} = \frac{c}{a}sinh(\eta_f) = \frac{c}{a}\beta_f\gamma_f \approx 10^{-5}\ s \tag{60}$$

where t_{lab} is the time interval in the lab frame.

From $c\eta = a\tau$ we can quickly calculate the proper time interval:

$$\tau = \frac{c}{a}\eta_f \approx 10^{-9}\ s \tag{61}$$

So, taking the ratio gives an "average" γ factor of 10^4.

Problem 7

(a)

We know that photons satisfy $E^2 - p^2c^2 = 0$. Then, if we substitute the appropriate values from the problem into the equation describing the Lorentz transformation for the four-momentum (ignoring the z-direction), we find:

$$\begin{pmatrix} E'/c \\ (E'/c)cos\phi' \\ (E'/c)sin\phi' \end{pmatrix} = \begin{pmatrix} cosh(\eta) & -sinh(\eta) & 0 \\ -sinh(\eta) & cosh(\eta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} E/c \\ (E/c)cos\phi \\ (E/c)sin\phi \end{pmatrix}. \tag{62}$$

Solving for E' gives us:

$$E' = E(cosh(\eta) - sinh(\eta)cos\phi) = Ecosh(\eta)(1 - \beta cos\phi). \tag{63}$$

If we then find the equation for p'_x we can solve for $cos\phi'$:

$$cos\phi' = \frac{E}{E'}(cosh(\eta)cos\phi - sinh(\eta)). \tag{64}$$

Substituting in the expression for E' from Eq. (63) yields:

$$cos\phi' = \frac{cos\phi - \beta}{1 - \beta cos\phi}. \tag{65}$$

(b)

Now we use the inverse Lorentz transform:

$$\begin{pmatrix} E/c \\ (E/c)cos\phi \\ (E/c)sin\phi \end{pmatrix} = \begin{pmatrix} cosh(\eta) & sinh(\eta) & 0 \\ sinh(\eta) & cosh(\eta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} E'/c \\ (E'/c)cos\phi' \\ (E'/c)sin\phi' \end{pmatrix}. \tag{66}$$

If we perform calculations similar to those in part (a), we find:

$$E = E'(cosh(\eta) + sinh(\eta)cos\phi') = E'cosh(\eta)(1 + \beta cos\phi'). \tag{67}$$

and

$$cos\phi = \frac{E'}{E}(cosh(\eta)cos\phi' + sinh(\eta)) = \frac{cos\phi' + \beta}{1 + \beta cos\phi'}. \tag{68}$$

(c)

Here, we can employ the relationship between energy and frequency of a photon, namely:

$$E = h\nu, \tag{69}$$

where h is Planck's constant. Thus from Eq. (63) we solve for ν', finding the relativistic Doppler shift formula:

$$\nu' = \nu cosh(\eta)(1 - \beta cos\phi). \tag{70}$$

If an observer knows only the frequency as observed in a given frame, one cannot figure out what the frequency of light was in the rest frame of the source. Thus a measurement of light frequency in a particular frame does not directly tell us about the velocity of the source. However, if we have prior knowledge of what the frequency of light at rest should be (for example, well-known atomic transitions in hydrogen or helium), we can tell something about the motion of the source.

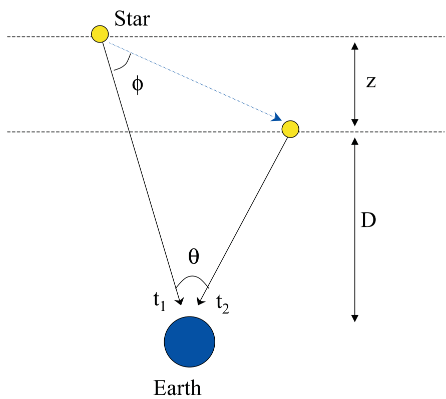
Problem 8

This situation is physically reasonable, here is one example of how it could happen...
Could the star be moving toward or away from us, even though the spectral features are not redshifted or blueshifted? The answer is yes, as can be seen from the relativistic Doppler shift given in Eq. (70). We demand that $\nu' = \nu$, and then

find a condition on the velocity of the source βc and the angle between $\vec{\beta}$ and the direction to earth ϕ:

$$cos\phi = \frac{\gamma - 1}{\beta\gamma}. \tag{71}$$

So long as this condition is satisfied, there is no restriction on the motion of the source (save that the source, if massive, cannot move at the speed of light!).



Next we consider if some particular type of motion could increase the apparent velocity of the star across the sky. Once again the answer is yes. Consider the situation depicted in the figure to the right. Of course, the drawing is greatly exaggerated in dimensions since $z \ll D$ and $t_2 - t_1$ is differentially small, but hopefully it will give you the basic idea. Suppose the astronomer makes two measurements with which she determines the motion of the star across the sky. The star is moving toward the earth in this case, so it takes the light detected in the first measurement longer to get to the earth. Suppose that the star gets closer to the earth by z between the times it emits the detected light. Then the time between the two light measurements on earth is:

$$t_2 - t_1 = \Delta t - z/c. \tag{72}$$

where Δt is the time it takes the star to move to the new location in the earth frame. Then the apparent angular motion is given by:

$$D\frac{d\theta}{dt} = \frac{D\Delta\theta}{\Delta t - z/c}. \tag{73}$$

So in fact (which is clear if you try some reasonable numbers), this apparent velocity can exceed the real velocity of the source by quite a bit, enough to make the star look like it's going c or faster. There are real cases of this in astronomy... for example at the center of the galaxy there are stars whose apparent velocity greatly exceeds c (of course they're redshifted and blueshifted all over the place, but you get the idea...)!