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## DUALITY IN FINITE ELEMENT METHODS

## by

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#### Abstract

The duality that exists between the basic equations and the dependent variables of the problems of stretching and of bending of plates is applied to the finite element method. A displacement method in the stretching problem is a stress function method in the bending problem and vice-versa.

The displacement method for the stretching problem and the dual stress function method for the bending problem are applied to orthotropic triangular elements. The derivation of the dual purpose stiffness-flexibiltity matrix is carried out explicitly.

Various boundary conditions are considered including elastic supports and edge beams in both stretching and bending problems. It is found that elastic supports in one problem are dual of an edge beam in the other problem and vice-versa. A treatment of dislocations in the stretching of multiplyconnected plates is also included.

The stress function method uses two equations per node whereas three equations per node are used in displacement methods. It has the same properties of accuracy and convergence as the well established dual displacement method for plate stretching. The availability for the same problem of both a stress function method and of a displacement method allows obtaining upper and lower bounds on flexibility coefficients.

A program originally written for the analysis of plane stress and plane strain problems by the displacement method is used to solve plate bending problems. Results show that a high degree of accuracy may be achieved for the stress couples. The determination of the deflection and slopes is made from the curvatures and involves no loss of accuracy. Comparisons with results of the fully compatible displacement method are presented.


## I. Introduction

The analysis of plate and shell structures by the finite element method is generally associated with the use of a stiffness matrix for triangular or rectangular elements and with the determination of the displacements at the vertices or nodes of the elements into which the structure is subdivided. In the problem of stretching of plates satisfactory results are reported (1) with use of triangular elements and of element displacements that are linear functions of the cartesian coordinates, whereas some difficulties seem to be encountered in obtaining as satisfactory results in the problem of plate bending (2)(3) (4)(5).

In contrast to the displacement method, force methods where stresses or stress functions are the unknowns have recieved comparatively little or no attention. A stress method based on the discretization of the stress field and the use of the theorem of stationary complementary potential energy is presented in reference (6) but stress functions have apparently not been used yet in formulating a finite element method. In view of the duality that exists, however, between the plate stretching and plate bending problems(7) a stiffness method for the stretching problem may be interpreted as a flexibility method for the bending problem if the displacements are replaced by stress functions. Similarly, a stiffness method for plate bending may be used to solve plate stretching problems if the deflection of the plate is interpreted as Airy!s stress function.

The use of stress functions in the finite element method is presented here within the context of the stretching-bending duality and of the application of one mathematical method to the solution of both stretching and bending problems. Because of this duality the use of stress functions for the analysis of plates in bending leads to a finite element method that has the same properties and characteristics as
the dual displacement method for the analysis of plates in stretching. In particular, the use of triangular elements and of piece-wise linear stress functions in the bending problem should lead to as satisfactory a finite element method as the dual displacement method for the stretching problem. This latter method has been tested for accuracy(l) and is monotonically convergent(8). It involves two equations per node whereas three equations per node are used in displacement methods for plate bending.

The derivation of the stiffness matrix and of the dual flexibility matrix for the stretching and bending, respectively, of an orthotropic triangular plate is here based on dual variational formulations and is carried out explicitly.

The duality between the basic equations, dependent variables, loadings and boundary conditions of the stretching and bending problem is presented in detail in reference 7. It is here extended and applied to plates supported elastically or bounded by edge beams.
2. Variational Formulation of the Stretching Problem in Terms of the Displacements

Consider a plate, Fig. 1, in equilibrium under a surface load of vector intensity

$$
\begin{equation*}
\overline{\mathrm{p}}=\mathrm{p}_{\mathrm{x}} \overline{\mathrm{i}}+\mathrm{p}_{\mathrm{y}} \overline{\mathrm{j}} \tag{I}
\end{equation*}
$$

and an edge load of vector intensity

$$
\begin{equation*}
\bar{N}_{n}=N_{n x} \bar{i}+N_{n y} \bar{j} \tag{2}
\end{equation*}
$$

The displacement vector

$$
\begin{equation*}
\overline{\mathrm{u}}=\mathrm{u} \overline{\mathrm{i}}+\mathrm{v} \overline{\mathrm{j}} \tag{3}
\end{equation*}
$$

makes stationary the potential energy of the plate considered as a functional of the displacements.

We consider linearly elastic orthotropic plates having the coordinate planes as planes of elastic and thermal symmetry and for which the strain energy density function has the form

$$
\begin{align*}
W= & \frac{E_{x} E_{y}^{h}}{2\left(1-\nu_{x} \nu_{y}\right)}\left[\frac{\varepsilon_{x x}^{2}}{E_{y}}+\frac{\varepsilon_{y y}^{2}}{E_{x}}+\left(\frac{\nu_{x}}{E_{y}}+\frac{\nu_{y}}{E_{x}}\right) \varepsilon_{x x} \varepsilon_{y y}\right]+2 G h \varepsilon_{x y}^{2} \\
& +N_{x}^{\circ} \varepsilon_{x x}+N_{y}^{\circ} \varepsilon_{y y} \tag{4}
\end{align*}
$$

where $E_{x}$ and $E_{y}$ are Young's moduli, $G$ is the shear modulus and $\nu_{x}$ and $\nu_{y}$ are Poisson's ratios. These are related through the relation

$$
\begin{equation*}
\frac{\nu_{y}}{E_{x}}=\frac{\nu_{x}}{E_{y}} \tag{5}
\end{equation*}
$$

$\varepsilon_{x x}, \varepsilon_{y y}$ and $\varepsilon_{x y}$ are the linear components of strain and are related to the displacements through the relations

$$
\begin{align*}
& \varepsilon_{x x}=u, x  \tag{6}\\
& \varepsilon_{y y}=v, y  \tag{7}\\
& 2 \varepsilon_{x y}=u, y+v, x \tag{8}
\end{align*}
$$

$N_{x}^{\circ}$ and $N_{y}^{\circ}$ are initial stress resultants related to thermal strains $\varepsilon_{x}^{\circ}$ and $\varepsilon_{\mathrm{y}}^{\circ}$ through the relations

$$
\begin{align*}
& N_{x}^{\circ}=-\frac{E_{x} h}{1-\nu_{x} \nu_{y}}\left(\varepsilon_{x}^{\circ}+\nu_{x} \varepsilon_{y}^{\circ}\right)  \tag{9}\\
& N_{y}^{\circ}=-\frac{E_{y}^{h}}{1-\nu_{x} \nu_{y}}\left(\varepsilon_{y}^{\circ}+\nu_{y} \varepsilon_{x}^{\circ}\right) \tag{10}
\end{align*}
$$

We consider boundary conditions of the form

$$
\begin{align*}
& N_{n x}=-\frac{\partial B}{\partial u}  \tag{11}\\
& N_{n y}=-\frac{\partial B}{\partial v} \tag{12}
\end{align*}
$$

where $B$ is a function of the displacements. If $N_{n x}$ and $N_{n y}$ are specified at the boundary then

$$
\begin{equation*}
B=-N_{n x} u-N_{n y}{ }^{v} \tag{13}
\end{equation*}
$$

In the case of elastic boundary conditions of the form

$$
\begin{align*}
& N_{n x}=k_{x x}\left(u^{s}-u\right)+k_{x y}\left(v^{s}-v\right)  \tag{14}\\
& N_{n y}=k_{y x}\left(u^{s}-u\right)+k_{y y}\left(v^{s}-v\right) \tag{15}
\end{align*}
$$

where the stiffness coefficients $k_{x x}, k_{x y}=k_{y x}$ and $k_{y y}$ and the support displacements $u^{s}$ and $v^{s}$ are given functions of position on the boundary, $B$ takes the form

$$
\begin{equation*}
B=\frac{1}{2} k_{x x}\left(u-u^{s}\right)^{2}+k_{x y}\left(u-u^{s}\right)\left(v-v^{s}\right)+\frac{1}{2} k_{y y}\left(v-v^{s}\right)^{2} \tag{16}
\end{equation*}
$$

Letting

$$
\begin{align*}
& \mathrm{P}=-\mathrm{p}_{\mathrm{x}}^{\mathrm{u}}-\mathrm{p}_{\mathrm{y}} \mathrm{v}  \tag{17}\\
& \mathrm{~W}_{\mathrm{t}}=\iint \mathrm{WdA}  \tag{18}\\
& \mathrm{P}_{\mathrm{t}}=\iint \mathrm{PdA} \tag{19}
\end{align*}
$$

and

$$
\begin{equation*}
B_{t}=\oint \mathrm{Bd} s \tag{20}
\end{equation*}
$$

where the area integrals extend over the domain of the plate and the curvilinear integral extends over the boundary, the total potential energy of the plate is

$$
\begin{equation*}
\Pi=W_{t}+P_{t}+B_{t} \tag{21}
\end{equation*}
$$

The displacements of the plate satisfy the variational equation

$$
\begin{equation*}
\delta \pi=0 \tag{22}
\end{equation*}
$$

## 3. Stretching of a Triangular Plate

An approximate solution of the problem of stretching of a triangular plate, Fig. 2, is now obtained by applying a direct method to the variational equation 22.

The stress resultants are specified on the three sides of the triangle. On side $m$ we let

$$
\begin{align*}
& N_{n x}=N_{x}^{m}  \tag{23}\\
& N_{n y}=N_{y}^{m} \tag{24}
\end{align*}
$$

The displacements are sought as linear functions of the coordinates. It will be convenient to use triangular coordinates $\boldsymbol{\xi}_{1}, \boldsymbol{\xi}_{2}$, and $\xi_{3}$. They are related to the cartesian coordinates through the linear relations

$$
\begin{equation*}
\xi_{i}=\frac{a_{i} y-b_{i} x+c_{i}}{2 A} \quad i=1,2,3 \tag{25}
\end{equation*}
$$

where $a_{i}$ and $b_{i}$ are the components of side (i) of the triangle considered as a vector and oriented counterclockwise when seen from the positive side of the $Z$ axis, Fig. 3. A is the area of the triangle. $c_{i} / 2 A$ are the triangular coordinates of the origin and are not needed in what follows. The triangular coordinates are non-dimensional distances as shown in Fig. 3. They satisfy the relations

$$
\begin{align*}
& \xi_{1}+\xi_{2}+\xi_{3}=1  \tag{26}\\
& \iint \xi_{i} \mathrm{dA}=\frac{\mathrm{A}}{3}  \tag{27}\\
& \iint \xi_{i}^{2} \mathrm{dA}=\frac{\mathrm{A}}{6}  \tag{28}\\
& \iint \xi_{i} \xi_{j} \mathrm{dA}=\frac{A}{12} \quad i \neq j \tag{29}
\end{align*}
$$

It is also noted that

$$
\begin{align*}
& a_{1}+a_{2}+a_{3}=0  \tag{30}\\
& b_{1}+b_{2}+b_{3}=0  \tag{31}\\
& c_{1}+c_{2}+c_{3}=2 A \tag{32}
\end{align*}
$$

The displacements $u$ and $v$ are expressed in terms of their values at the nodes through the relations

$$
\begin{align*}
& \mathrm{u}=\mathrm{u}_{\mathrm{i}} \boldsymbol{\xi}_{\mathrm{i}}  \tag{33}\\
& \mathrm{v}=\mathrm{v}_{\mathrm{i}} \boldsymbol{\xi}_{\mathrm{i}} \tag{34}
\end{align*}
$$

in which $u_{i}$ and $v_{i}$ are the displacements at node $i$ and the summation convention pertaining to a repeated index is used.

The total potential energy $T_{\text {will }}$ now be expressed in terms of the nodal values of the displacements and will be made stationary with regard to them. In computing the strains the chain rule of partial differentiation is used in the form

$$
\begin{align*}
& ()_{, x}=\frac{\partial()}{\partial \xi_{i}} \xi_{i, x}=-\frac{b_{i}}{2 A} \frac{\partial()}{\partial \xi_{i}}  \tag{35}\\
& ()_{, y}=\frac{\partial()}{\partial \xi_{i}} \xi_{i, y}=\frac{a_{i}}{2 A} \frac{\partial()}{\partial \xi_{i}} \tag{36}
\end{align*}
$$

yielding

$$
\begin{equation*}
\varepsilon_{x x}=-\frac{b_{i} u_{i}}{2 A} \tag{37}
\end{equation*}
$$

$$
\begin{equation*}
\varepsilon_{y y}=\frac{a_{i} v_{i}}{2 \mathrm{~A}} \tag{38}
\end{equation*}
$$

$$
\begin{equation*}
2 \varepsilon_{x y}=\frac{a_{i} u_{i}-b_{i} v_{i}}{2 A} \tag{39}
\end{equation*}
$$

The potential energies of the surface load and of the boundary load, Eqs. 19 and 20, and the term involving the initial stress resultants in

Eq. 18, take the form

$$
\begin{align*}
& P_{t}=-P_{x i} u_{i}-P_{y i} v_{i}  \tag{40}\\
& B_{t}=-R_{x i} u_{i}-R_{y i} v_{i}  \tag{41}\\
& \iint\left(N_{x}^{\circ} \varepsilon_{x x}+N_{y}^{\circ} \varepsilon_{y y}\right) d A=-\Theta_{x i} u_{i}-\Theta_{y i} v_{i} \tag{42}
\end{align*}
$$

where

$$
\begin{align*}
& P_{x i}=\iint p_{x} \xi_{i} d A  \tag{43}\\
& P_{y i}=\iint p_{y} \xi_{i} d A  \tag{44}\\
& \ell_{m} \quad \ell_{m} \\
& R_{x i}=\int_{0} N_{x}^{m} \xi_{i} d s_{m}=\frac{1}{\ell m} \int_{0} N_{x}^{m} s_{m} d s_{m}, m \neq i  \tag{45}\\
& R_{y i}=\int_{0}^{l m} N_{y}^{m} \xi_{i} d s_{m}=\frac{1}{l m} \int_{0}^{l m} N_{y}^{m} s_{m} d s_{m}, m \neq i  \tag{46}\\
& \Theta_{x i}=\frac{b_{i}}{2 A} \iint_{x}^{\circ} d A  \tag{47}\\
& \Theta_{y i}=-\frac{a_{i}}{2 A} \iint N_{y}^{\circ} d A \tag{48}
\end{align*}
$$

In Eqs. 45 and 46 m refers to the two sides of the triangular plate intersecting at node $i$, and on each side $s_{m}$ is oriented positively towards node i.

The potential energy of the plate, Eq. 21, may now be wiriten in the form

$$
\begin{align*}
& \left.\Pi=\frac{E_{x} E_{y} h}{8 A\left(l-\nu_{x} \nu_{y}\right.}\right)\left[\frac{\left(b_{i} u_{i}\right)^{2}}{E_{y}}+\frac{\left(a_{i} v_{i}\right)^{2}}{E_{x}}-\left(\frac{\nu_{x}}{E_{y}}+\frac{\nu_{y}}{E_{x}}\right) b_{i} a_{j} u_{i} v_{j}\right] \\
& +\frac{G h}{A}\left(a_{i} u_{i}-b_{i} v_{i}\right)^{2}-\left(P_{x i}+R_{x i}+\Theta_{x i}\right) u_{i}-\left(P_{y i}+R_{y i}+\Theta_{y i}\right) v_{i} \tag{49}
\end{align*}
$$

The variational equation $\delta \pi=0$ yields at each node $k$ the two equations

$$
\begin{align*}
& \frac{\partial \pi}{\partial u_{k}}=0  \tag{50}\\
& \frac{\partial \Pi}{\partial v_{k}}=0 \tag{51}
\end{align*}
$$

or, explicitly

$$
\begin{align*}
& \frac{h}{4 A\left(1-\nu_{x} \nu_{y}\right)}\left\{\left[E_{x} b_{k} b_{i}+G\left(1-\nu_{x} \nu_{y}\right) a_{k} a_{i}\right] u_{i}-\left[E_{x} \nu_{x} b_{k} a_{i}+G\left(1-\nu_{x} \nu_{y}\right) a_{k} b_{i}\right] v_{i}\right\}= \\
& P_{x k}+R_{x k}+\Theta_{x k} \tag{52}
\end{align*}
$$

$\frac{h}{4 A\left(1-\nu_{x} \nu_{y}\right)}\left\{-\left[E_{y} \nu_{y} a_{k} b_{i}+G\left(1-\nu_{x} \nu_{y}\right) b_{k} a_{i}\right] u_{i}+\left[E_{y} a_{k} a_{i}+G\left(1-\nu_{x} \nu_{y}\right) b_{k} b_{i}\right] v_{i}\right\}=$

$$
\begin{equation*}
P_{y k}+R_{y k}+\Theta_{y k} \tag{53}
\end{equation*}
$$

In the case where $p_{x}, P_{y}, N_{x}^{\circ}$ and $N_{y}^{\circ}$ are linear in $x$ and $y$ the integrals in Eqs. 43, 44, 47 and 48 may be expressed in terms of the nodal values $p_{x i}, p_{y i}, N_{x i}^{\circ}$ and $N_{y i}^{\circ}$, where irefers to any of the three nodes, according to the relations

$$
\begin{align*}
& P_{x i}=\iint p_{x j} \xi_{j} \xi_{i} d A=\frac{A}{12}\left(p_{x i}+p_{x l}+p_{x 2}+p_{x 3}\right)  \tag{54}\\
& P_{y i}=\iint p_{y j} \xi_{j} \xi_{i} d A=\frac{A}{12}\left(p_{y i}+p_{y l}+p_{y 2}+p_{y 3}\right)  \tag{55}\\
& \Theta_{x i}=\frac{b_{i}}{2 A} \iint N_{x j}^{\circ} \xi_{j} d A=\frac{b_{i}}{6}\left(N_{x l}^{\circ}+N_{x 2}^{\circ}+N_{x 3}^{\circ}\right)  \tag{56}\\
& \Theta_{y i}=-\frac{a_{i}}{2 A} \iint N_{y j}^{\circ} \xi_{j} d A=-\frac{a_{i}}{6}\left(N_{y l}^{\circ}+N_{y 2}^{\circ}+N_{y 3}^{\circ}\right) \tag{57}
\end{align*}
$$

It is convenient to call Eqs. 52 and 53 equilibrium equations and the terms on their right hand sides generalized nodal forces at node $k$. The six equilibrium equations at the three nodes of the triangular plate form a system in six unknown nodal displacements. This system of equations is singular, however, because the homogeneous system admits non-trivial solutions that are the nodal displacements in an arbitrary rigid body displacement of the plate. It may be verified, as it is to be expected, that the thermal generalized nodal forces are self equilibrating, and that for the non-homogeneous system to admit a solution the generalized nodal forces associated with the surface and edge loads must be statically equivalent to zero. This condition is equivalent to the condition that the surface and edge load be self-equilibrating. In that case the general solution of the nonhomogeneous system consists of a particular solution superimposed on an arbitrary rigid body displacement. By fixing the latter a definite solution is obtained.

In the case where only the thermal generalized forces are not zero the solution for the displacements represents a free deformation of the triangular plate.

## 4. Application to a Plate of Arbitrary Shape

The derivation of Eqs. 52 and 53 may be viewed as a step in applying the variational equation $\delta \Pi=0$ for a plate of arbitrary shape. The domain of the plate is , Fig. 5, subdivided arbitrarily into triangular elements and the displacements are sought as piecewise linear functions. The boundary of the plate, if curved, is replaced by an approximating polygon and the boundary conditions are formulated for the polygonal boundary. Known displacements at the boundary are replaced over each side joining two consecutive nodes by approximating linear functions.

The potential energy of the plate is equal to the sum of the potential energies of the triangular elements. In forming this sum the potential energies of the boundary loads of the triangles cancel each other over all sides in the interior of the plate because the stress resultants along a side common to two elements act in opposite directions on the two elements.

The variational equation $\delta \Pi=0$ for the plate yields a set of simultaneous equations for determining the nodal displacements. At each node $k$ belonging to $n$ elements two equations are obtained by superposition of $n$ pairs of equations such as Eqs. 52 and 53. These two equations are conveniently referred to as equilibrium equations and their right hand sides as generalized nodal forces at node $k$. Let $F_{x k}$ anf $F_{y k}$ be the generalized nodal forces due to the edge loads, ie.,

$$
\begin{align*}
& F_{x k}=\sum R_{x k}  \tag{58}\\
& F_{y k}=\sum R_{y k} \tag{59}
\end{align*}
$$

where the summation extends over the elements having node $k$ in common. At an interior node we obtain

$$
\begin{align*}
& F_{x k}=0  \tag{60}\\
& F_{y k}=0 \tag{61}
\end{align*}
$$

because the stress resultants on a side common to two elements contribute opposite quantities to the sums in Eqs. 58 and 59. At a boundary node k let m and n denote the two sides issuing from that node, Fig. 6. From Eqs. 45 and 46 we can write

$$
\begin{equation*}
F_{x k}=\frac{1}{\ell_{m}} \int_{0}^{l_{m}} N_{x}^{m} s_{m} d s_{m}+\frac{1}{\ell_{n}} \int_{0}^{\ell_{n}} N_{x}^{n} s_{n} d s_{n} \tag{62}
\end{equation*}
$$

$$
\begin{equation*}
F_{y k}=\frac{1}{\ell_{m}} \int_{0}^{\ell_{m}} N_{y}^{m} s_{m} d s_{m}+\frac{1}{\ell_{n}} \int_{0}^{\ell_{n}} N_{y}^{n} s_{n} d s_{n} \tag{63}
\end{equation*}
$$

In Eqs. 62 and 63 sides $m$ and $n$ are oriented towards node $k$, Fig. 6.
Writing two equations per node yields as many equations as nodal displacements. The right hand sides of these equations are known at all interior nodes if the externally applied load is known. At boundary nodes $F_{x k}$ and $F_{y k}$ may or may not be known depending on the boundary conditions.
a) Stress Boundary Conditions

If the stress resultants are specified on the two sides issuing from node $k$, the two equilibrium equations at that node have known right hand sides.

In the case where the stress resultants are known on all the boundary, it is necessary to specify the elements of a rigid body displacement in order to avoid treating with a singular matrix. This will be examined in more detail subsequently.
b) Displacement Boundary Conditions

If the displacements of a boundary node are specified, the two equilibrium equations associated with that node are not part of the system of simultaneous equations. They are used after solving this system to compute the unknown edge reactions $F_{x k}$ and $F_{y k}$. It may be noted that the preceding applies as well to an interior node whose displacements are specified and at which there are therefore unknown reactions.
c) Mixed Boundary Conditions

If the stress resultant component in a direction (a) is specified on the two sides issuing from node $k$ and if the displacement of node $k$ in a different direction ( $\beta$ ) is specified, one equation at node k is obtained by projecting the two equilibrium equations on the direction (a) and a second equation is obtained by expressing the known displace-
ment component in terms of the cartesian components $u_{k}$ and $v_{k}$.
In the case where stress boundary conditions are specified on all the boundary, a rigid body displacement must also be specified if the displacements are to be determined in a definite manner. This may be done by specifying as zero the displacements of a node $i$ and the rotation about that node. In order to specify zero rotation the displacement component of a node $j$ in the direction perpendicular to the line ij may be specified as zero. The two equilibrium equations at node $i$ and the equilibrium equation at node $j$ in the direction perpendicular to line $i j$ are deleted from the system of simultaneous equations. They may be used to perform a statical check by computing the reactions.
d) Elastic Boundary Conditions

Elastic boundary conditions are specified by means of the function B, Eq. 16, to which corresponds the potential energy $B_{t}$, Eq. 20. Using piecewise linear functions to represent $u, v, u s$ and $v^{s}$ and referring to the notation defined in Fig. 7 we obtain at node 2 generalized nodal forces of the form

$$
\begin{align*}
& -\frac{\partial B_{t}}{\partial u_{2}}=\sum_{i=1}^{3} k_{2 i}^{x x}\left(u_{i}^{s}-u_{i}\right)+\sum_{i=1}^{3} k_{2 i}^{x y}\left(v_{i}^{s}-v_{i}\right)  \tag{64}\\
& -\frac{\partial B_{t}}{\partial v_{2}}=\sum_{i=1}^{3} k_{2 i}^{y x}\left(u_{i}^{s}-u_{i}\right)+\sum_{i=1}^{3} k_{2 i}^{y y}\left(v_{i}^{s}-v_{i}\right) \tag{65}
\end{align*}
$$

where

$$
\begin{align*}
& k_{22}^{x x}=\frac{1}{\ell_{1}^{2}} \int_{0}^{\ell_{1}} k_{x x} s_{1}^{2} d s_{1}+\frac{1}{\ell_{3}^{2}} \int_{0}^{\ell_{3 x}} k_{x x} s_{3}^{2} d s_{3}  \tag{66}\\
& k_{2 i}^{x x}=\frac{1}{\ell_{i}^{2}} \int_{0}^{\ell_{i}} k_{x x} s_{i}\left(\ell_{i}-s_{i}\right) d_{i} \quad \tag{67}
\end{align*}
$$

By using in turn $\mathrm{k}_{\mathrm{yy}}, \mathrm{k}_{\mathrm{xy}}$, and $\mathrm{k}_{\mathrm{yx}}$ instead of $\mathrm{k}_{\mathrm{xx}}$ in Eqs. 66 and 67 we obtain the remaining stiffness coefficients. It is noted that

$$
\begin{equation*}
k_{2 i}^{x y}=k_{2 i}^{y x}, \quad i=1,3 \tag{68}
\end{equation*}
$$

The right hand sides of Eqs. 64 and 65 take the place in the equilibrium equations at node 2 of the generalized nodal forces $F_{x_{2}}$ and $F_{y_{2}}$ due to the edge load. ${ }_{i}^{s}$ and $v_{i}^{s}$ are assumed known. If, instead, the spring forces are known the boundary conditions are then of the stress type.

## 5. Plate Bounded by an Edge Beam

Formulated analytically the boundary conditions for a plate with an edge beam are the differential equations of equilibrium of the edge beam expressed in terms of plate edge displacements. It will be assumed that the edge of the plate coincides with the centerline of the beam.

The problem may be discretized by adding to the potential energy of the plate the potential energy of the edge beam and applying the direct variational method. The piece-wise linear displacements need however a special interpretation when used to compute the potential energy of the beam because they imply no change of curvature of a side joining two adjacent boundary nodes. In order to compute the strain energy due to bending a numerical integration may be made where the change of curvature is computed in terms of nodal displacements by means of differences of side rotations.

The strain energy of the beam is taken in the form

$$
\begin{equation*}
W^{b}=\frac{1}{2} \oint\left[E A\left(\varepsilon-\varepsilon^{\circ}\right)^{2}+E I\left(x-x^{\circ}\right)^{2}\right] d s \tag{69}
\end{equation*}
$$

where $\varepsilon$ and $X$ are the extensional and curvature strains, respectively.
$\mathcal{E}^{\circ}$ and $\mathcal{X}^{\circ}$ are thermal strains, $A$ is the cross sectional area, I is the moment of inertia with regard to the axis perpendicular to the plane of the beam and passing through the centroid and $E$ is Young's modulus. The potential energy of the distributed load applied on the beam is of the form

$$
\begin{equation*}
P^{b}=-\oint\left(\Delta N_{x}^{u}+\Delta N_{y} v\right) d s \tag{70}
\end{equation*}
$$

In Eq. $70 \Delta N_{x}$ and $\Delta N_{y}$ are the components of the distributed load which consists of the load applied externally on the beam and of the stress resultants at the plate edge. Using piece-wise linear displacements and a piece-wise constant thermal strain to compute the strain energy due to axial stretching we obtain

$$
\begin{equation*}
\frac{1}{2} \oint E A\left(\varepsilon-\varepsilon^{\circ}\right) d s=\frac{1}{2} \sum_{i} E_{i} A_{i}\left(\varepsilon_{i}-\varepsilon_{i}^{\circ}\right)^{2} \ell_{i} \tag{71}
\end{equation*}
$$

where i refers to a boundary segment and the summation extends over all such segments. Referring to Fig. 8 we can write

$$
\begin{equation*}
\varepsilon_{i} \ell_{i}=-\left(u_{i+1}-u_{i}\right) \sin \phi_{i}+\left(v_{i+1}-v_{i}\right) \cos \phi_{i} \tag{72}
\end{equation*}
$$

Letting $\omega_{i}$ denote the rotation of side $i$ we have

$$
\begin{equation*}
\omega_{i} \ell_{i}=-\left(u_{i+1}-u_{i}\right) \cos \phi_{i}-\left(v_{i+1}-v_{i}\right) \sin \phi_{i} \tag{73}
\end{equation*}
$$

and letting at node $k$

$$
\begin{equation*}
x_{k}=\frac{2}{\ell_{k}+\ell_{k-1}}\left(\omega_{k}-\omega_{k-1}\right) \tag{74}
\end{equation*}
$$

The strain energy due to bending is expressed in the form

$$
\begin{equation*}
\frac{1}{2} \oint_{E I\left(x-x^{\circ}\right)^{2} d s=\frac{1}{2} \sum_{k} E_{k} I_{k}\left(x_{k}-x_{k}^{\circ}\right)^{2} \frac{\ell_{k}+\ell_{k-1}}{2},{ }^{2}}^{2} \tag{75}
\end{equation*}
$$

where the summation extends over the boundary nodes.
The potential energy of the applied load is obtained as a sum of products of generalized nodal forces and nodal displacements in the form

$$
\begin{equation*}
P^{b}=-\sum_{k} \Delta F_{x k} u_{k}+\Delta F_{y k} v_{k} \tag{76}
\end{equation*}
$$

where

$$
\begin{align*}
& \Delta F_{x k}=\bar{F}_{x k}-F_{x k}  \tag{77}\\
& \Delta F_{y k}=\bar{F}_{y k}-F_{y k} \tag{78}
\end{align*}
$$

$F_{x k}$ and $F_{y k}$ are the generalized nodal forces arising from the stress resultants acting on the plate edge, Eqs. 62 and $63 . \bar{F}_{x k}$ and $\bar{F}_{y k}$ are the generalized nodal forces arising from the load applied externally on the edge beam. Denoting by $\overline{\mathrm{N}} \mathrm{m}_{\mathrm{x}}^{\mathrm{m}}$ and $\overline{\mathrm{N}}_{\mathrm{y}}^{\mathrm{m}}$ the components of that load on side $m, \bar{F}_{x k}$ and $\bar{F}_{y k}$ are obtained through formulas similar to Eqs. 62 and 63.

By making stationary the potential energy of the beam with regard to the displacements of node 3, Fig. 9, force-displacement relations are obtained in the form

$$
\begin{align*}
& \Delta F_{x 3}=F_{x 3}^{\circ}+\sum_{i=1}^{5} k_{3 i}^{x x} u_{i}+\sum_{i=1}^{5} k_{3 i}^{x y} v_{i}  \tag{79}\\
& \Delta F_{y 3}=F_{y_{3}}^{0}+\sum_{i=1}^{5} k_{3 i}^{y x} u_{i}+\sum_{i=1}^{5} k_{3 i}^{y y} v_{i} \tag{80}
\end{align*}
$$

Introducing the notation

$$
\begin{array}{ll}
d_{k}=\frac{2 E_{k} I_{k}}{l_{k}+l_{k-1}} & k=2,3,4 \\
a_{i}=E_{i} A_{i} l_{i} & i=1, \ldots, 4 \tag{82}
\end{array}
$$

$$
\begin{array}{ll}
s_{i}=\frac{\sin \phi_{i}}{l_{i}}=-\frac{a_{i}}{l_{i}^{2}} & i=1, \ldots, 4 \\
c_{i}=\frac{\cos \phi_{i}}{l_{i}}=\frac{b_{i}}{l_{i}^{2}} & \tag{84}
\end{array}
$$

The stiffness coefficients in Eqs. 79 and 80 are obtained in the form

$$
\begin{align*}
& k_{33}^{x x}=a_{2} s_{2}^{2}+a_{3} s_{3}^{2}+d_{3}\left(c_{2}+c_{3}\right)^{2}+d_{2} c_{2}^{2}+d_{4} d_{3}^{2}  \tag{85}\\
& k_{32}^{x x}=-a_{2} s_{2}^{2}-d_{3} c_{2}\left(c_{2}+c_{3}\right)-d_{2} c_{2}\left(c_{2}+c_{1}\right)  \tag{86}\\
& k_{34}^{x x}=-a_{3} s_{3}^{2}-d_{3} c_{3}\left(c_{2}+c_{3}\right)-d_{4} d_{3}\left(c_{3}+c_{4}\right)  \tag{87}\\
& k_{31}^{x x}=d_{2} c_{1} c_{2}  \tag{88}\\
& k_{35}^{x x}=d_{4} c_{3} c_{4}  \tag{89}\\
& k_{33}^{x y}=-a_{2} s_{2} c_{2}-a_{3} s_{3} c_{3}+d_{3}\left(c_{2}+c_{3}\right)\left(s_{2}+s_{3}\right)+d_{2} s_{2} c_{2}+d_{4} s_{3} c_{3} \\
& k_{32}^{x y}=a_{2} s_{2} c_{2}-d_{3} s_{2}\left(c_{2}+c_{3}\right)-d_{2} c_{2}\left(s_{2}+s_{1}\right)  \tag{91}\\
& k_{34}^{x y}=a_{3} s_{3} c_{3}-d_{3} s_{3}\left(c_{2}+c_{3}\right)-d_{4} c_{3}\left(s_{3}+s_{4}\right)  \tag{92}\\
& k_{31}^{x y}=d_{2} s_{1} c_{2}  \tag{93}\\
& k_{35}=d_{4} s_{4} c_{3} \tag{94}
\end{align*}
$$

$k_{3 i}^{y y}$ and $k_{3 i}^{y x} i^{y}=1, \ldots, 5$ are obtained from Eqs. 85 to 94 by interchanging $x$ and $y$ and $s_{j}$ and $c_{j}, j=1, \ldots, 4$.

The quantities $F_{x 3}^{\circ}$ and $F_{y 3}^{\circ}$ in Eqs. 79 and 80 arise from the thermal strains and are obtained in the form

$$
\begin{align*}
\mathrm{F}_{\mathrm{x} 3}^{\circ}= & \mathrm{a}_{2} \mathrm{~s}_{2} \varepsilon_{2}^{\circ}-\mathrm{a}_{3} \mathrm{~s}_{3} \varepsilon_{3}^{\circ}-\frac{1}{2} \mathrm{~d}_{3}\left(\mathrm{c}_{2}+\mathrm{c}_{3}\right)\left(\ell_{2}+\ell_{3}\right) x_{3}^{\circ} \\
& +\frac{1}{2} \mathrm{~d}_{2} c_{2}\left(\ell_{1}+\ell_{2}\right) x_{2}^{\circ}+\frac{1}{2} d_{4} c_{3}\left(l_{3}+\ell_{4}\right) x_{4}^{\circ}  \tag{95}\\
\mathrm{F}_{\mathrm{y} 3}^{\circ}= & -a_{2} c_{2} \varepsilon_{2}^{\circ}+a_{3} c_{3} \varepsilon_{3}^{\circ}-\frac{1}{2} d_{3}\left(s_{2}+s_{3}\right)\left(l_{2}+\ell_{3}\right) x_{3}^{\circ} \\
& +\frac{1}{2} d_{2} s_{2}\left(\ell_{1}+\ell_{2}\right) x_{2}^{\circ}+\frac{1}{2} d_{4} s_{3}\left(l_{3}+\ell_{4}\right) x_{4}^{\circ} \tag{96}
\end{align*}
$$

When writing the two equilibrium equations for the plate at a boundary node $k$ the generalized nodal forces $F_{x k}$ and $F_{y k}$ are replaced by

$$
\begin{align*}
& F_{x k}=\bar{F}_{x k}-\Delta F_{x k}  \tag{97}\\
& F_{y k}=\bar{F}_{y k}-\Delta F_{y k} \tag{98}
\end{align*}
$$

and $\Delta F_{x k}$ and $\Delta F_{y k}$ are expressed in terms of the displacements through Eqs. 79 and 80.

## 6. Multiply-connected Plate with Dislocations

Consider a multiply-connected plate bounded on the outside by a curve ( $C$ ) and on the inside by curves $\left(C_{1}\right),\left(C_{2}\right), \ldots,\left(C_{n}\right)$. Such a plate may have, in the absence of external loads, initial stresses corresponding to $n$ independent dislocations. It will be sufficient to consider one dislocation such as shown in Fig. 10. The positive face of the dislocation may be brought to coincide with the negative face through a rigid body displacement that may be defined by means of the translation components $\delta u^{\circ}$ and $\delta v^{\circ}$ at the origin of coordinates and of
the rotation angle $\delta \omega^{\circ}$. It is known that the closing of the dislocation results in multivalued displacements and in singlevalued stresses (10).

Geometrically the 2 faces of the dislocation are considered as one curve as shown in Fig. lla. Consider a cut made in the stressed plate along the dislocation curve and a node $q$ on that curve. Denote by $\mathrm{q}^{-}$the node belonging to the negative face of the cut and by $\mathrm{q}^{+}$the node belonging to the positive face, Fig. 11 b . The displacements of nodes $q^{-}$and $q^{+}$ that arise from closing the dislocation are related through the relations

$$
\begin{align*}
& u_{q}^{+}=u_{q}^{-}+\delta u^{\circ}-v_{q} \delta \omega^{\circ}  \tag{99}\\
& v_{q}^{+}=v_{q}^{-}+\delta v^{\circ}+x_{q} \delta \omega^{\circ} \tag{100}
\end{align*}
$$

where $x_{q}, y_{q}$ are the coordinates of node $q$. Letting

$$
\begin{align*}
& \delta u_{q}=u_{q}^{+}-u_{q}^{-}  \tag{101}\\
& \delta v_{q}=v_{q}^{+}-v_{q}^{-} \tag{102}
\end{align*}
$$

Eqs. 99 and 100 may be written in the form

$$
\begin{align*}
& \delta_{u_{q}}=\delta_{u^{\circ}}-y_{q} \delta \omega^{\circ}  \tag{103}\\
& \delta_{v_{q}}=\delta_{v^{\circ}}+x_{q} \delta \omega^{\circ} \tag{104}
\end{align*}
$$

Consider now the two equilibrium equations associated with node $\mathrm{k}^{-}$,
Fig. 12. They may be written using matrix notation in the form

$$
\begin{equation*}
\mathrm{K}_{\mathrm{kk}}^{-} \mathrm{U}_{\mathrm{k}}^{-}+\mathrm{K}_{\mathrm{ki}}^{-} \mathrm{U}_{\mathrm{i}}^{-}+\mathrm{K}_{\mathrm{kj}}^{-} \mathrm{U}_{\mathrm{j}}^{-}+\sum_{\mathrm{m}} \mathrm{~K}_{\mathrm{km}} \mathrm{U}_{\mathrm{m}}=\mathrm{F}_{\mathrm{k}}^{-} \tag{105}
\end{equation*}
$$

Similarly at node $\mathrm{k}^{+}$we may write

$$
\begin{equation*}
\mathrm{K}_{\mathrm{kk}}^{+} \mathrm{U}_{\mathrm{k}}^{+}+\mathrm{K}_{\mathrm{ki}}^{+} \mathrm{U}_{\mathrm{i}}^{+}+\mathrm{K}_{\mathrm{kj}}^{+} \mathrm{U}_{\mathrm{j}}^{+}+\sum_{\mathrm{P}} \mathrm{~K}_{\mathrm{kp}} \mathrm{U}_{\mathrm{p}}=\mathrm{F}_{\mathrm{k}}^{+} \tag{106}
\end{equation*}
$$

where $\mathrm{F}_{\mathrm{k}}^{-}$and $\mathrm{F}_{\mathrm{k}}^{+}$are the matrices of the generalized nodal forces due to the stress resultants acting on the negative and positive faces of the dislocation, respectively. If surface forces or thermal effects are present the corresponding generalized nodal forces are added to the right hand sides of Eqs. 105 and 106. Because the stress resultants are continuous across the dislocation we have

$$
\begin{equation*}
\mathrm{F}_{\mathrm{k}}^{-}+\mathrm{F}_{\mathrm{k}}^{+}=0 \tag{107}
\end{equation*}
$$

Letting

$$
\begin{equation*}
\delta_{U}=U_{q}^{+}-U_{q}^{-} \quad q=i, j, k \tag{108}
\end{equation*}
$$

we can write

$$
\begin{align*}
& \mathrm{U}_{\mathrm{q}}^{+}=\mathrm{U}_{\mathrm{q}}+\frac{1}{2} \delta_{\mathrm{U}_{\mathrm{q}}}  \tag{109}\\
& \mathrm{U}_{\mathrm{q}}^{-}=\mathrm{U}_{\mathrm{q}}-\frac{1}{2} \delta_{U_{\mathrm{q}}} \tag{110}
\end{align*}
$$

where from Eqs. 103 and 104

$$
\delta_{U_{q}}=\left\{\begin{array}{c}
\delta u_{q} \\
\delta_{v_{q}}
\end{array}\right\}=\left\{\begin{array}{l}
\delta u^{\circ} \\
\delta_{v^{\circ}}
\end{array}\right\}+\delta \omega^{\circ}\left\{\begin{array}{c}
-y_{q} \\
x_{q}
\end{array}\right\}
$$

and $U_{q}$ is the matrix of the average displacements.
Superimposing Eqs. 105 and 106 and substituting for $\mathrm{U}_{\mathrm{q}}^{+}$and $\mathrm{U}_{\mathrm{q}}^{-}$, using Eqs. 109 and 110 obtain a matrix equation of the form

$$
\begin{equation*}
\mathrm{K}_{k k} \mathrm{U}_{\mathrm{k}}+\mathrm{K}_{\mathrm{ki}} \mathrm{U}_{\mathrm{i}}+\mathrm{K}_{\mathrm{kj}} \mathrm{U}_{j}+\sum_{\mathrm{m}} \mathrm{~K}_{\mathrm{km}} \mathrm{U}_{\mathrm{m}}+\sum_{\mathrm{p}} \mathrm{~K}_{\mathrm{kp}} \mathrm{U}_{\mathrm{p}}=\delta_{\mathrm{F}} \tag{112}
\end{equation*}
$$

where

$$
\begin{array}{ll}
\mathrm{K}_{\mathrm{kq}}=\mathrm{K}_{\mathrm{kq}}^{+}+\mathrm{K}_{\mathrm{kq}}^{-} & \mathrm{q}=\mathrm{i}, \mathrm{j}, \mathrm{k} \\
\delta_{\mathrm{F}_{\mathrm{k}}}=\frac{1}{2} \sum_{\mathrm{q}}\left(\mathrm{~K}_{\mathrm{kq}}^{-}-\mathrm{K}_{\mathrm{kq}}^{+}\right) \delta U_{\mathrm{q}} \tag{114}
\end{array}
$$

In writing the two equilibrium equations at a node $k$ not situated on the dislocation but belonging to elements having one or two nodes such as node $q$ on the dislocation, the displacements of node $q$ are replaced by the right hand side of one of Eqs. 109 and 110. Eq. 109 is used if node k is on the positive side of the dislocation and Eq. 110 is used if node k is on the negative side.

In summary, the presence of dislocations does not alter the procedure of forming the stiffness matrix of the structure. The dislocations contribute only fictitious generalized nodal forces.

## 7. Variational Formulation of the Bending Problem in Terms of Stress Functions

A variational formulation of the bending problem and its application to establish what may be called a flexibility matrix for a triangular element may be directly obtained from the preceding by replacing the dependent variables and the elastic constants of the stretching problem by their dual dependent variables and elastic constants in the bending problem according to the correspondence established in reference (7). Because the resulting variational formulation seems to differ, however, from the form it takes when based on the principle of virtual forces it is established here directly starting from this latter form.

Consider a plate subjected to a normal distributed load of intensity q , an edge deflection w and an edge rotation $\mathrm{w}, \mathrm{n}$ in the plane normal to the boundary curve, Fig. 13. The stress couples in the plate and the stress couples and transverse shear on its boundary make stationary
the functional

$$
\begin{equation*}
\Pi^{\prime}=\iint W^{\prime} d A+\oint B^{\prime} d s \tag{115}
\end{equation*}
$$

where $W^{\prime}$ is the complementary strain energy density and $B^{\prime}$ may be called the complementary potential energy density of the boundary forces. $\Pi^{\prime}$ is a functional of the stress couples and transverse shears whose variations are constrained to satisfy the homogeneous differential equations of equilibrium but are otherwise arbitrary.

For a plate made of the same material as in the stretching problem and with no transverse shear deformability $W^{\prime}$ takes the form

$$
\begin{align*}
W^{\prime}= & \frac{6}{h^{3}}\left[\frac{M_{x x}^{2}}{E_{x}}+\frac{M_{y y}^{2}}{E_{y}}-\left(\frac{\nu_{y}}{E_{x}}+\frac{\nu_{x}}{E_{y}}\right) M_{x x} M_{y y}+\frac{M_{x x}^{2}}{G}\right] \\
& +x_{x}^{0}{ }_{x}{ }_{x x}+x_{y}^{\circ} M_{y y} \tag{116}
\end{align*}
$$

where $M_{x x}, M_{y y}$ and $M_{x y}$ are the stress couples, Fig. 14, and $X_{x}^{\circ}$ and $X_{y}^{\circ}$ are thermal curvatures. To $X_{x}^{\circ}$ and $X_{y}^{\circ}$ correspond the initial stress couples

$$
\begin{align*}
& M_{x}^{\circ}=-D_{x}\left(x_{x}^{\circ}+\nu_{x} x_{y}^{\circ}\right)  \tag{117}\\
& M_{y}^{\circ}=-D_{y}\left(x_{y}^{\circ}+\nu_{y} x_{x}^{\circ}\right) \tag{118}
\end{align*}
$$

where

$$
\begin{equation*}
D_{x}=\frac{E_{x} h^{3}}{12\left(1-\nu_{x} \nu_{y}\right)} \tag{119}
\end{equation*}
$$

and

$$
\begin{equation*}
D_{y}=\frac{E y^{3}}{12\left(1-\nu_{x} \nu_{y}\right)} \tag{120}
\end{equation*}
$$

For prescribed $w$ and $w, n$ at the boundary $B^{\prime}$ is expressed in terms of the bending stress couple $M_{n n}$, the twisting stress couple $M_{n s}$ and the transverse shear $Q_{n}$, Fig. 15a, in the form

$$
\begin{equation*}
B^{\prime}=M_{n n} w, n+M_{n s} w, s-Q_{n} w \tag{121}
\end{equation*}
$$

Alternatively, using the cartesian components $M_{n x}$ and $M_{n y}$ of the stress couple vector, Fig. 15b, and the cartesian components ${ }^{w}, x$ and $w, y^{\prime} B^{\prime}$ takes the form

$$
\begin{equation*}
B^{\prime}=-M_{n x} w, y+M_{n y} w_{x}-Q_{n} w \tag{122}
\end{equation*}
$$

In order to vary arbitrarily the internal forces in $\Pi^{\prime}$ without violating the statical constraints the stress couples and transverse shears are expressed in terms of two stress functions $U$ and $V$ and a particular solution of the differential equations of equilibrium. These are

$$
\begin{align*}
& M_{x x, x}+M_{y x, y}-Q_{x}=0  \tag{123}\\
& M_{x y, x}+M_{y y, y}-Q_{y}=0  \tag{124}\\
& Q_{x, x}+Q_{y, y}+q=0 \tag{125}
\end{align*}
$$

The particular solution of Eqs. 123 and 124 is taken in the form

$$
\begin{align*}
& M_{x y}^{p}=M_{y x}^{p}=0  \tag{126}\\
& Q_{x}^{p}=M_{x x, x}^{p}=-\left[D_{x}\left(K_{y}+\nu_{x} K_{x}\right)\right], x  \tag{127}\\
& Q_{y}^{P}=M_{y y, y}^{p}=-\left[D_{y}\left(K_{x}+\nu_{y} K_{y}\right)\right] \tag{128}
\end{align*}
$$

$$
\begin{align*}
& M_{x x}^{p}=-D_{x}\left(K_{y}+\nu_{x} K_{x}\right)  \tag{129}\\
& M_{y y}^{p}=-D_{y}\left(K_{x}+\nu_{y} K_{y}\right) \tag{130}
\end{align*}
$$

and in order to satisfy Eq. $125, \mathrm{~K}_{\mathbf{x}}$ and $\mathrm{K}_{\mathrm{y}}$ must satisfy the differential equation

$$
\begin{equation*}
\left[D_{x}\left(K_{y}+\nu_{x} K_{x}\right)\right], x x+\left[D_{y}\left(K_{x}+\nu_{y} K_{y}\right)\right], y y-q=0 \tag{131}
\end{equation*}
$$

The stress couples and transverse shears of this particular solution are those that would occur in two families of strips parallel to the coordinate axes. The load may be subdivided arbitrarily between the two families of strips and the end conditions of the strips are also arbtirary. It will be assumed that the particular solution for a given problem is determined in a definite manner. The reason for introducing the functions $K_{x}$ and $K_{y}$ is that $K_{x, x}$ and $K_{y, y}$ are dual of the load components $p_{x}$ and $P_{y}$, respectively, in the stretching problem. $K_{x}$ and $K_{y}$ have the dimensions of curvatures and are parts of the actual curvatures of the plate in the $y$ and $x$ directions respectively. The remaining parts are contributed by the stress functions and are such that the curvatures and twist of the plate are geometrically compatible and the boundary conditions are satisfied.

The general solution of the equilibrium equations may be written in the form

$$
\begin{align*}
& M_{x x}=M_{x x}^{*}+M_{x x}^{p}  \tag{132}\\
& M_{y y}=M_{y y}^{*}+M_{y y}^{p}  \tag{133}\\
& M_{x y}=M_{x y}^{*} \tag{134}
\end{align*}
$$

$$
\begin{align*}
& Q_{x}=Q_{x}^{*}+Q_{x}^{p}  \tag{135}\\
& Q_{y}=Q_{y}^{*}+Q_{y}^{p} \tag{136}
\end{align*}
$$

where

$$
\begin{align*}
& M_{x x}^{*}=V, y  \tag{137}\\
& M_{y y}^{*}=U, x  \tag{138}\\
& M_{x y}^{*}=-\frac{1}{2}\left(U, y+V_{, x}\right)  \tag{139}\\
& Q_{x}^{*}=\Omega_{z, y}  \tag{140}\\
& Q_{y}^{*}=-\Omega_{z, x} \tag{141}
\end{align*}
$$

and

$$
\begin{equation*}
\Omega_{z}=\frac{1}{2}(\mathrm{~V}, \mathrm{x}-\mathrm{U}, \mathrm{y}) \tag{142}
\end{equation*}
$$

At the boundary we have the relations

$$
\begin{align*}
& M_{n x}=-y_{, s} M_{x y}+x, s M_{y y}  \tag{143}\\
& M_{n y}=y_{, s} M_{x x}-x_{, s} M_{y x}  \tag{144}\\
& Q_{n}=y_{, s} Q_{x}-x, s_{y}^{Q} \tag{145}
\end{align*}
$$

wheres denotes the arclength of the boundary curve.
The functional $\pi^{\prime}$ may now be expressed in terms of the stress functions. The complementary strain energy density, Eq. 116, takes the form

$$
\begin{align*}
W^{\prime}= & W^{\prime \prime}-K_{x}^{U}, x-K_{y} V_{y}+\frac{1}{2} D_{x} K_{y}^{2}+\frac{1}{2}\left(\nu_{x} D_{x}+\nu_{y} D_{y}\right) K_{x} K_{y} \\
& +\frac{1}{2} D_{y} K_{x}^{2}-D_{x} x_{x}^{\circ}\left(K_{y}+\nu_{x} K_{x}\right)-D_{y} x_{y}^{\circ}\left(K_{x}+\nu_{y} K_{y}\right) \quad(1 \tag{146}
\end{align*}
$$

where

$$
\begin{align*}
W^{\prime \prime}= & \frac{6}{E^{3}}\left[\frac{V^{2}, y}{E_{x}}+\frac{U^{2}, x}{E_{y}}-\left(\frac{\nu_{y}}{E_{x}}+\frac{\nu_{x}}{E_{y}}\right) U, V_{, y}+\frac{\left(U, y+V, x^{\prime}\right.}{4 G}\right] \\
& +x_{x}^{\circ} V_{, y}+x_{y}^{\circ} U, x \tag{147}
\end{align*}
$$

The terms that do not involve the stress functions in Eq. 146 may be deleted when forming the functional $\pi^{\prime}$ because they do not vary. The area integral of the linear terms in $U$ and $V$ in Eq. 146 may be expressed through use of Green's theorem, assuming the stress functions to be singlevalued, in the form

$$
\begin{align*}
& \iint\left(-K_{x} U, x-K_{y} V, y\right) d A=\iint\left(K_{x, x} U+K_{y, y} V\right) d A \\
& \quad+\oint_{\left(K_{y} V d x-K_{x} U d y\right)} \tag{148}
\end{align*}
$$

Multivalued stress functions are considered subsequently.
Similarly, the terms not involving the stress functions may be deleted from the boundary integral in the functional $\Pi^{\prime}$. The remaining integral, after successive integrations by parts, is expressed in the form

$$
\begin{equation*}
\oint\left(M_{n x}^{*} w, y+M_{x y}^{*} w, x_{n}^{-} Q_{n}^{*} w\right) d s=\oint\left(w, y s s^{U}-w, V\right) d s \tag{149}
\end{equation*}
$$

Finally, the functional takes the form

$$
\begin{equation*}
\Pi^{\prime \prime}=\iint\left(W^{\prime \prime}+P^{\prime \prime}\right) d A+\oint B^{\prime \prime} d s \tag{150}
\end{equation*}
$$

where W!' is given through Eq. 147 and

$$
\begin{align*}
& P^{\prime \prime}=K_{x, x} U+K_{y, y} V  \tag{151}\\
& B^{\prime \prime}=-X_{s y}^{*} U+X_{s x}^{*} V \tag{152}
\end{align*}
$$

where

$$
\begin{align*}
& x_{s y}^{*}=-w_{, y s}+K_{x} y_{, s}  \tag{153}\\
& x_{s x}^{*}=-w_{, x s}+K_{y} x_{, s} \tag{154}
\end{align*}
$$

It may be readily observed that $-\pi^{\prime \prime}$ is obtained from the potential energy $T$ of the stretching problem if in the latter the interchange indicated below is made.

Table 1.a

$$
\begin{array}{ll}
u, v & U, V \\
N_{x}^{\circ}, N_{y}^{\circ} & -x_{y}^{\circ},-x_{x}^{\circ} \\
p_{x}, p_{y} & K_{x, x}, K_{y, y} \\
N_{n x}, N_{n y} & -x_{s y}^{*}, x_{s x}^{*} \\
E_{x} h, E_{y} h, G h & -D_{y}^{-1},-D_{x}^{-1}-\left(G h^{3} / 3\right)^{-1} \\
\nu_{x}, \nu_{y} & -\nu_{x},-\nu_{y}
\end{array}
$$

The duality indicated above may be completed to include all the dependent variables of the stretching and of the bending problem (7). For this purpose let $\boldsymbol{X}_{\mathrm{yy}}^{*}, \boldsymbol{X}_{\mathrm{xx}}^{*}$, and $\boldsymbol{X}_{\mathrm{xy}}$ be related to $\mathrm{M}_{\mathrm{yy}}^{*} \underset{\mathrm{xx}}{\mathrm{M}_{*}^{*}}$ and $\mathrm{M}_{\mathrm{xy}}$ as shown below. Obtain

$$
\begin{equation*}
x_{y y}^{*}=\frac{12}{E_{y} h^{3}}\left(M_{y y}^{*}-\nu_{x} M_{x x}^{*}\right)=-w, y y K_{x} \tag{155}
\end{equation*}
$$

$$
\begin{align*}
& x_{x x}^{*}=\frac{12}{E_{x} h^{3}}\left(M_{x x}^{*}-\nu_{y}^{M} M_{y y}^{*}\right)=-w, x x+K_{y}  \tag{156}\\
& x_{x y}=\frac{12}{G h^{3}} M_{x y}=-w_{, x y} \tag{157}
\end{align*}
$$

The superscript $*$ is associated with the solution of the homogeneous equations of equilibrium in terms of stress functions. The quantities $x_{y y}, x_{x x}$ and $x_{x y}$ are the curvatures and twist of the plate, i.e. $-w_{, ~ y y},-w, x x^{\text {and }-w, x y}$, respectively.

The table below indicates the dual correspondence between dependent variables of the stretching and of the bending problem.

Table 1.b

$$
\begin{aligned}
& N_{x x}, N_{x y}, N_{y y} \\
& \varepsilon_{\mathrm{xx}}, \varepsilon_{\mathrm{xy}}, \varepsilon_{\mathrm{yy}} \\
& \mathrm{M}_{\mathrm{yy}}^{*},-\mathrm{M}_{\mathrm{xy}}, \mathrm{M}_{\mathrm{xx}}^{*} \\
& \omega_{z}=\frac{1}{2}\left(v_{, x}-u, y\right) \\
& -x_{y y}^{*}, x_{\mathrm{xy}},-x_{\mathrm{xx}}^{*} \\
& \Omega_{z}=\frac{1}{2}\left(V_{, x}-U, y\right) \\
& x_{y z}=\varepsilon_{y y, x}-\varepsilon_{x y, y}=\omega_{z, y} \quad Q_{x}^{*}=M_{x x, x}^{*}+M_{y x, y}^{*}=\Omega_{z, y} \\
& x_{\mathrm{xz}}=\varepsilon_{\mathrm{yx}, \mathrm{x}}-\varepsilon_{\mathrm{xx}, \mathrm{y}}=\omega_{\mathrm{z}, \mathrm{x}} \quad-\mathrm{Q}_{\mathrm{y}}^{*}=-\underset{\mathrm{xy}, \mathrm{x}}{\mathrm{M}^{*}}-\mathrm{M}_{\mathrm{yy}, \mathrm{y}}^{*}=\Omega_{\mathrm{z}, \mathrm{x}} \\
& \varepsilon_{\mathrm{ss}}, \chi=\left(\omega_{\mathrm{z}}-\varepsilon_{\mathrm{sn}}\right), \mathrm{s} \\
& M_{n n}^{*}{ }_{n} Q_{n e}=\left(\Omega_{z}+M_{n s}^{*}\right), s
\end{aligned}
$$

If subscripts $n$ and $s$ indicate the directions of the outward normal and of the tangent, respectively, at a point of the boundary curve, the correspondence between quantities at the boundary defined with respect to the directions $n$ and $s$ is obtained by changing in the table above subscripts $x$ and $y$ into $n$ and $s$, respectively. Derivatives with regard to $x$ and $y$ may not in general be changed into derivatives with regard to $n$ and $s$ if the boundary is curved. The appropriate formulas obtained with use of curvilinear coordinates may be found in reference (7). Of
interest is the duality between the extensional strain $\varepsilon_{s s}$ of the boundary curve and the stress couple $\mathrm{M}_{\mathrm{nn}}^{*}$ and the duality between the in-plane change of curvature $\mathcal{X}$ of the boundary curve and the effective transverse shear $Q^{*}$ ne . This is indicated in the last line of Table l.b.

## 8. Bending of a Triangular Element

The duality mentioned above is now used to apply to the bending problem of a triangular element the equations established for the stretching problem. The equations dual of Eqs. 52 and 53 are

$$
\begin{equation*}
\frac{3}{A h^{3}}\left\{\left[\frac{b_{k}^{b_{i}}}{E_{y}}+\frac{a_{k} a_{i}}{4 G}\right] U_{i}+\left[\frac{\nu_{x}^{b_{k} a_{i}}}{E_{y}}-\frac{a_{k} b_{i}}{4 G}\right] V_{i}\right\}=-P_{x k}^{\prime}-R_{x k}^{\prime \cdots}-\Theta_{x k}^{\prime} \tag{158}
\end{equation*}
$$

$$
\begin{equation*}
\frac{3}{A h^{3}}\left\{\left[\frac{\nu_{y} a_{k} b_{i}}{E_{x}}-\frac{b_{k} a_{i}}{4 G}\right] U_{i}+\left[\frac{a_{k} a_{i}}{E_{x}}+\frac{b_{k} b_{i}}{4 G}\right] V_{i}\right\}=-P_{y k}^{\prime}-R_{y k}^{\prime *}-\Theta_{y k}^{\prime} \tag{159}
\end{equation*}
$$

where $P_{x k}^{\prime}, P_{y k}^{\prime}, R_{x k}^{\prime *}, R_{y k}^{\prime *}, \Theta_{x k}^{\prime}$ and $\Theta_{y k}^{\prime}$ are dual of $P_{x k}, P_{y k}$, $R_{x k}, R_{y k}, \Theta_{x k}$ and $\Theta_{y k}$, respectively, and may be expressed through equations dual of Eqs. 43 to 48 . It appears appropriate to call the terms on the right-hand sides of Eqs. 158 and 159 generalized nodal rotations.

The three pairs of equations associated with the three nodes of the triangular plate form a system of six equations in six unknowns. This system is singular, however, because the homogeneous system has a non-trivial solution that is dual of the arbitrary rigid body displacement in the stretching problem. Such a solution for the stress functions yield zero stress couple's. For the system of six equations to have a solution the generalized nodal rotations must satisfy three
compatibility equations that are dual of the three equilibrium equations to be satisfied by the generalized nodal forces in the stretching problem. Two of these equations are conditions of singlevaluedness of $w, x$ and $w, y$ and the third equation is a condition of singlevaluedness for $w$.

## 9. Application to a Plate of Arbitrary Shape

As in the stretching problem two equations are associated with each node $k$. They are obtained by superposition of $n$ pairs of equations such as Eqs. 158 and 159 , $n$ being the number of elements having node $k$ in common. These two equations may conveniently be referred to as compatibility equations. At an interior node the superposition of $R_{x k}^{1 \cdot /}$ and $R_{y k}^{1 *}$ yields zero values because each of the quantities $\chi_{s y}^{*}=-w, y s+K_{x} y, s$ and $\mathcal{X}_{s x}^{*}=-w_{, x s}+K_{y} x, s$ take two opposite values on an edge common to two elements. At a boundary node, Fig. 16, the superpositions of $R_{x k}^{1 *}$ and of $R_{y k}^{\prime *}$, respectively, yields the generalized nodal rotations.
$\mathrm{F}_{\mathrm{xk}}^{\mathbf{*}^{*}}$ and $\mathrm{F}_{\mathrm{yk}}^{\prime *}$ are dual of the generalized nodal forces $\mathrm{F}_{\mathrm{xk}}$ and $\mathrm{F}_{\mathrm{yk}}$, respectively.

We let

$$
\begin{align*}
& F_{x k}^{\prime *}=F_{x k}^{\prime}-F_{x k}^{\prime p}  \tag{162}\\
& F_{y k}^{\prime *}=F_{y k}^{\prime}-F_{y k}^{\prime p} \tag{163}
\end{align*}
$$

where

$$
\begin{align*}
& F_{x k}^{\prime}=\frac{1}{l_{m}} \int_{0}^{\operatorname{lm}_{m}}(w, y s)_{m}^{s}{ }_{m}^{-30-}{ }^{d s}{ }_{m}+\frac{1}{\ell_{n}} \int_{o}^{l_{n}}(w, y s)_{n} s_{n} d s_{n}  \tag{164}\\
& F_{y k}^{\prime}=\frac{1}{l_{m}} \int_{0}^{l_{m}}\left(-w, x s_{m}\right)_{m} s_{m}+\frac{1}{l_{n}} \int_{0}^{l_{n}}\left(-w, x s_{n} s_{n} d s_{n}\right.  \tag{165}\\
& \text { and } \\
& F_{x k}^{\prime P}=\frac{1}{\ell_{m}} \int_{0}^{\ell_{m}}\left(K_{x} y, s\right)_{m} s_{m}^{d s}{ }_{m}+\frac{1}{\ell_{n}} \int_{0}^{\ell_{n}}\left(K_{x} y, s_{n}\right)_{n}{ }_{n} d_{n}  \tag{166}\\
& \ell_{\mathrm{m}} \quad \ell_{\mathrm{n}} \\
& F_{y k}^{\prime{ }^{p}}=\frac{1}{\ell_{m}} \int_{0}\left(-K_{y} x, s_{m}\right)_{m} s_{m}{ }^{d s}{ }_{m}+\frac{1}{\ell_{n}} \int_{0}\left(-K_{y} x, s_{n}\right)_{n} s_{n} s_{n} \tag{167}
\end{align*}
$$

It is possible to express $F_{x k}^{\prime}$ and $F_{y k}^{\prime}$ in terms of the slopes $w, y$ and ${ }^{w}$, x , respectively, through integration by parts in Eqs. 164 and 165. In doing this it is noted that the positive sense of $s$ coincides with that of $s_{m}$ but is opposite to that of $s_{n}$, Fig. 16. Obtain

$$
\begin{align*}
& \mathrm{F}_{\mathrm{xk}}^{\prime}=\frac{1}{\ell_{\mathrm{n}}} \int_{\mathrm{o}}^{\ell_{\mathrm{n}}} \mathrm{w}, \mathrm{y} \mathrm{~d}-\frac{1}{\ell_{\mathrm{m}}} \int_{\mathrm{o}}^{\ell_{\mathrm{m}}^{\mathrm{n}}} \mathrm{w}, \mathrm{y} \mathrm{ds} \\
& \mathrm{~F}_{\mathrm{yk}}^{\prime}=-\frac{1}{\ell_{\mathrm{n}}} \int_{\mathrm{o}}^{\ell_{\mathrm{n}}} \mathrm{w}, \mathrm{x} \mathrm{~d}+\frac{1}{\ell_{\mathrm{m}}} \int_{\mathrm{o}}^{\ell_{\mathrm{m}}} \mathrm{w}, \mathrm{x} \mathrm{ds} \tag{168}
\end{align*}
$$

It is seen that $F_{x k}^{\prime}$ is the difference between the averages over sides n and m , respectively, of the slope with regard to the y axis. A similar statement holds for $F_{y k}^{\prime}$.

Various types of boundary conditions are now examined.
a. Displacement Boundary Conditions

From specified values of $w$ and $w, n$ at the boundary it is possible to compute $w, x$ and $w, y$ through the relations

$$
\begin{align*}
& w_{, x}=w_{,} n^{y}, s^{+w}, s^{x}, s  \tag{170}\\
& w_{, y}=-w^{x}, n^{x}+w, s^{y}, s \tag{171}
\end{align*}
$$

then $w, x s$ and $w, y s$ through differentiation
with regard to s. $X_{s x}^{*}$ and $X_{s y}^{*}$ are computed through Eqs. 153 and 154 and finally the generalized nodal rotations $F_{x k}^{\prime *}$ and $F_{\text {yk }}^{\prime *}$ are computed through Eqs. 160 and 161.

The computation of the generalized nodal rotations from $X_{s X}^{*}$ and $x_{s y}^{*}$ is dual of the computation of the generalized nodal forces from specified stress resultants. An alternate way, however, is to compute $F_{x k}^{\prime P}$ and $F_{y k}^{\prime p}$ through Eqs. 166 and 167 , then $F_{x k}^{\prime}$ and $F_{y k}^{\prime}$ through Eqs. 168 and 169 and finally $F_{x k}^{\prime *}$ and $F^{\prime} \%$ yk through Eqs. 162 and 163. In this alternate method it is not necessary to determine the curvature components $w, x s$ and $w, y s$.

It is recalled that in the stretching problem with stress conditions on all the boundary, three equilibrium equations are deleted and are replaced by three corresponding conditions specifying the elements of a rigid body displacement. A dual procedure is followed in the bending problem if displacement conditions are given on all the boundary.
b. Stress Boundary Conditions

For a plate without transverse shear deformability stress boundary conditions specify the values of the bending stress couple $M_{n n}$ and of the effective transverse shear $Q_{n e}$. Having determined the particular solution of the equilibrium equations, $M_{n n}^{*}$ and $Q_{n e}^{*}$ are determined through the relations

$$
\begin{align*}
& M_{n n}^{*}=M_{n n}-M_{n n}^{p}  \tag{172}\\
& Q_{n e}^{*}=Q_{n e}-Q_{n e}^{p} \tag{173}
\end{align*}
$$

From the known values of $M_{n n}^{*}$ and $Q_{n e}^{*}$ the stress functions $U$ and $V$ are determined and the boundary conditions are then treated as the displacement boundary conditions are in the stretching problem. For determining $U$ and $V$ consider the boundaryanc $A_{o} A$, oriented positively from $A_{0}$ to A, Fig. 17, and let

$$
\begin{align*}
& P_{z}^{*}=\int_{0}^{s} Q_{n e}^{*} d s  \tag{174}\\
& C_{x}^{*}=-y_{z}^{*}+\int_{0}^{s}\left(y Q_{n e}^{*}+M_{n n}^{*} x_{s}^{*}\right) d s  \tag{175}\\
& C_{y}^{*}=\underset{z}{x}+\int_{0}^{*}\left(-x Q_{n e}^{*}+M_{n n}^{*} y, s\right) d s \tag{176}
\end{align*}
$$

$P_{z}^{*}$ is the resultant of the effective transverse shear $Q_{n e}^{*}$ on arc $A_{o} A$. $C_{x}^{*}$ and $C_{y}^{*}$ are the moments, with regard to axes $x$ and $y$, respectively, passing through $A$, of the transverse shear $Q *$ ne and of the stress couple $\mathrm{M}_{\mathrm{nn}}^{*}$ on $\operatorname{arc} \mathrm{A}_{\mathrm{o}} \mathrm{A}$. Letting

$$
\begin{equation*}
\Omega=\Omega_{z}+M_{n s}^{*} \tag{177}
\end{equation*}
$$

It is found that (11)

$$
\begin{align*}
& \Omega=\Omega_{0}+P_{z}^{*}  \tag{178}\\
& U=U_{o}-\left(y-y_{o}\right) \Omega_{o}+C_{x}^{*}  \tag{179}\\
& V=V_{o}+\left(x-x_{o}\right) \Omega_{o}+C_{y}^{*} \tag{180}
\end{align*}
$$

where $U_{0}, V_{o}$, and $\Omega_{0}$ are the values of $U, V$, and $\Omega$ at point $A_{0}$.
It may be noted that the determination of $U$ and $V$ from $M_{n n}^{*}$ and $Q_{n}^{*}$ is dual of the determination of $u$ and $v$ from the extensional strain $\varepsilon_{s s}$ and the change of in-plane curvature $X$ of the boundary curve. The constants $U_{0}$ and $V_{o}$ are dual of the displacements of point $A_{o}$ and $\Omega_{0}$ is dual of the rotation of the boundary curve at point $A_{0}$. For a simply connected domain $U_{o}, V_{o}$, and $\Omega_{o}$ may be chosen arbitrarily at one point $A_{0} . U$ and $V$ are then fully determined if there are not more than one point such as $A_{o}$ i.e. if the stress boundary conditions are given
either on all the boundary curve or on only one continuous arc of it.
The determination of $U$ and $V$ through Eqs. 179 and 180 yields singlevalued functions. if $Q_{n e}^{*}$ and $M_{n n}^{*}$ form a system statically equivalent to zero. This is always the case in a simply connected domain without singularities since the applied distributed load is equilibrated by the particular solution of the equilibrium equations. The cases of a multiply-connected domain and of stress boundary conditions given on more than one arc of the boundary are treated subsequently.

An alternate way of treating stress boundary conditions is to express average values over the boundary sides of $M_{n n}^{*}$ and $Q_{n e}^{*}$ in terms of the nodal values of the stress functions. This is done through formulas dual of those expressing the extensional strain and the in-plane change of curvature of a boundary curve in terms of nodal displacements, Eqs. 72 to 74 . The equation for $M_{n n}^{*}$ is associated with a side and uses the nodal values of $U$ and $V$ at the two nodes defining the side. The equation for $Q_{n e}^{*}$ is associated with a node and uses the values of $\Omega$ at the two sides issuing from the node. Deleting the subscripts in $M_{n n}^{*}$ and $Q_{n e}^{*}$ and letting $M_{i}^{*}$ be associated with side $i$ and $Q_{k}^{*}$ be associated with node k, Fig. 18, the equations dual of Eqs. 72 to 74 take the form

$$
\begin{align*}
& M_{i}^{*} \ell_{i}=-\left(U_{i+1}-U_{i}\right) \sin \phi_{i}+\left(V_{i+1}-V_{i}\right) \cos \phi_{i}  \tag{181}\\
& \Omega_{i} \ell_{i}=-\left(U_{i+1}-U_{i}\right) \cos \phi_{i}-\left(V_{i+1}-V_{i}\right) \sin \phi_{i}  \tag{182}\\
& Q_{k}^{*}=\frac{2}{\ell_{k}+\ell_{k-1}}\left(\Omega_{k}-\Omega_{k-1}\right) \tag{183}
\end{align*}
$$

c. Displacement Boundary Conditions Alternating with Stress Boundary Conditions

Consider the boundary curve shown in Fig. 19 and let displacement conditions be given on arcs $\left(D_{1}\right)\left(D_{2}\right), \ldots\left(D_{n}\right)$ including
points $A_{1}, B_{1}, A_{2}, B_{2}, \ldots, A_{n}, B_{n}$ and let stress conditions be given on $\operatorname{arcs}\left(S_{1}\right),\left(S_{2}\right), \ldots,\left(S_{n}\right)$. The total number of boundary conditions must be equal to twice the number of boundary nodes. At each node of arc ( $D_{i}$ ) except the end nodes $B_{i}$ and $A_{i+1}$ two compatibility equations are written as explained in (a). The same cannot be done at $B_{i}$ and $A_{i+1}$ because $w$ and $w, n$ must be known on both sides of a node in order to be able to compute the generalized nodal rotations. At each node of $\operatorname{arc}\left(S_{i}\right)$ and at node $B_{i}$ the stress functions $U$ and $V$ may be expressed, as explained in (b), in terms of their values $U_{i}$ and $V_{i}$ at node $A_{i}$ and in terms of the value $\Omega_{i}$ of $\Omega$ at the side joining $A_{i}$ to the next node. There remains therefore to write three conditions per $\operatorname{arc}\left(S_{i}\right)$ corresponding to the three unknowns $U_{i}, V_{i}$, and $\Omega_{i}$. In order to write these three conditions we will make use of a property of $w$ that is more conveniently presented in its dual statical form as a property of Airy's stress function.

Consider an arc $A B$ of a boundary curve oriented positively from A to $B$, Fig. 20. Let $P_{x}^{A B}$ and $P_{y}^{A B}$ be the resultants and $C_{z}^{A B}$ be the moment with regard to point $B$ of the stress resultants $N_{n x}^{*}$ and $N_{n y}^{*}$ acting on $\operatorname{arc} A B$ and due only to the stress function $\Psi$. The following relations may be established (12).

$$
\begin{align*}
& \Psi_{, x}^{B}-\Psi_{, x}^{A}=-P_{y}^{A B}  \tag{184}\\
& \Psi_{, y}^{B}-\Psi_{, y}^{A}=+P_{x}^{A B}  \tag{185}\\
& \psi^{B}-\Psi^{A}-\left(x_{B}-x_{A}\right) \psi_{, x}^{A}-\left(y_{B}-y_{A}\right) \Psi_{, y}^{A}=C_{z}^{A B} \tag{186}
\end{align*}
$$

If $A$ and $B$ are appropriately placed on sides $i_{0} i_{1}$ and $i_{p} i_{p+1}$, respectively, Fig. 20, then $P_{x}^{A B}, P_{y}^{A B}$ and $C_{z}^{A B}$ are statically equivalent
to the generalized nodal forces, due to $\mathrm{N}_{\mathrm{nx}}^{*}$ and $\underset{n y}{\mathrm{~N} *}$, at the nodes situated between $A$ and $B$. We will assume for simplicity that $A$ and $B$ are at the mid-sides, which implies that $N_{n x}^{*}$ and $N_{n y}^{*}$ are constanton sides $i_{0} i_{l}$ and $i_{p} i_{p+1}$. In the bending problem the quantities dual of $\psi, N_{n x}^{*}$ and $N_{n y}^{*}$ are $w, w, y s$ and $-w,{ }_{x s}$, respectively, and the quantities dual of thegeneralized nodal forces due to $\mathrm{N}_{\mathrm{nx}}^{*}$ and $N_{\text {ny }}^{*}$ are the generalized nodal rotations $F_{x k}^{\prime}$ and $F_{y k}^{\prime}$, Eqs. 164 and 165 or Eqs. 168 and 169.

Letting $\operatorname{arc} i_{1} i_{p}$ in Fig. 20 represent an $\operatorname{arc}\left(S_{i}\right)$, Fig. 19, with $i_{1}$ and $i_{p}$ coinciding with $A_{i}$ and $B_{i}$, respectively, we can write the three relations

$$
\begin{equation*}
w_{, x}^{B}-w_{, x}^{A}=-\sum_{k=i_{l}}^{i} F_{y k}^{\prime} \tag{187}
\end{equation*}
$$

$$
\begin{equation*}
w_{, y}^{B}-w_{, y}^{A}=\sum_{k=i_{1}}^{i_{p}} F_{x k}^{\prime} \tag{188}
\end{equation*}
$$

$$
\begin{align*}
& w^{B}-w^{A}-\left(x_{B}-x_{A}\right) w^{A}, x^{A}-\left(y_{B}-y_{A}\right) w_{y}^{A}=\sum_{k=i}^{i}\left[\left(x_{k}-x_{B}\right) F_{y k}^{\prime}-\right. \\
& \left.\quad\left(y_{k}-y_{B}\right) F_{x k}^{\prime}\right] \tag{189}
\end{align*}
$$

In using Eqs. 187 to 189 as part of the simultaneous equations of the finite element method the right hand sides are replaced in terms of the nodal values of the stress functions and in terms of the known generalized nodal rotations due to the $\mathrm{K}_{\mathrm{x}}$ and $\mathrm{K}_{\mathrm{y}}$ terms and to thermal effects, if any. The left hand sides are replaced by their known values.

The three equations above are written for all arcs ( $S_{i}$ ) except an arbitrary one for which $U, V$, and $\Omega$ are specified arbitrarily at one point.
d. Multiply-connected Plate

The boundary forces due to the stress functions are selfequilibrating on the boundary as a whole but not necessarily on each boundary curve if the plate is multiply-connected.

Consider one closed boundary curve and an arbitrary point $A_{o}$ on that curve. Let $R_{z}^{*}, M_{x}^{*}$, and $M_{y}^{*}$ be the resultant and the moments, respectively, that are statically equivalent at point $A_{o}$ to the boundary forces $Q_{n e}^{*}$ and $M_{n n}^{*}$. Eqs. 178 to 180 show that $U, V$, and $\Omega$ are multivalued and experience at $A_{o}$ after a turn in the positive sense around the boundary curve the discontinuities

$$
\begin{align*}
& \delta \Omega=R_{z}^{*}  \tag{190}\\
& \delta U=M_{x}^{*}  \tag{191}\\
& \delta V=M_{y}^{*} \tag{192}
\end{align*}
$$

The stretching analogue is a plate with a dislocation whose characteristics $\delta u, \delta v$ and $\delta \omega$ at point $A_{0}$ are dual of $\delta U, \delta v$, and $\delta \Omega$, respectively. This problem was treated in paragraph 6. The position of point $A_{0}$ and the shape of the dislocation curve are arbitrary (10).

If stress boundary conditions are given on the boundary curve, $R_{z}^{*}, M_{x}^{*}$ and $M_{y}^{*}$ are known. They are statically equivalent to the difference between the specified boundary forces and the boundary forces due to the particular solution of the equilibrium equations. If $R_{z}^{*}, M_{x}^{*}$ and $M_{y}^{*}$ are unknown the extremities of the dislocation curve may be placed at two points where $w, w, x$, and $w, y$ are known. Applying Eqs. 187 to 189 , in the present case along one face of the dislocation, yields the required three additional equations for the determination of $R_{z}^{*}, M_{x}^{*}$ and $M *$
e. Mixed Boundary Conditions

Two cases of mixed boundary conditions are considered.
In the first case $w$ and $M_{n n}$ are specified on the boundary. In the dual
stretching problem Airy's stress function and the extensional strain $\varepsilon_{s s}$ are known at the boundary. For each boundary side i one equation expressing $M_{i}^{*}$ in terms of nodal values of $U$ and $V$ is written using Eç. 181. Another equation is obtained by using Eq. 189 where now $A$ is an arbitrary fixed point at the middle of a side and $B$ is $a$ variable mid-side point. For a simply connected domain $w, x$ and ${ }_{\mathrm{w}}^{\mathrm{w}, \mathrm{y}} \mathrm{A}$ may be chosen arbitrarily. The left hand side of Eq. 189 is thus known and the right hand side is expressed in terms of $U$ and $V$. The last equation, where $B$ coincides with $A$, is a condition of singlevaluedness for $w$. The above procedure yields two equations per boundary node. Three of these equations are omitted, however, in the case of a simply connected domain. Instead the three elements dual of a rigid body displacement are arbitrarily specified.

In the case of a multiply-connected domain $w^{A}$ and $w^{A}$, cannot be chosen arbitrarily. Two corresponding additional equations are obtained by equating to zero the resultants on the boundary curve of $F_{x k}^{\prime}$ and $F_{y k}^{\prime}$, respectively. These equations are singlevaluedness conditions for $w, x$ and $w, y$ as shown by Eqs. 187 and 188.

An alternate method consists in using instead of Eq. 189 the three equations 187 to 189 where $A$ and $B$ are at the mid-points of two consecutive sides and $w, x$ and $w, y$ are two additional unknowns per side. The sums on the right hand sides of Eqs. 187 to 189 involve then one node only. One of Equations 187 and 188 may be replaced by the following simpler relation obtained from them.

$$
\begin{equation*}
\left(x_{B}-x_{A}\right) w^{B}, x^{B}+\left(y_{B}-y_{A}\right) w^{B}=y^{B}-w^{A} \tag{193}
\end{equation*}
$$

The alternate method involves more equations than the first method but these equations have the possible advantage of following the same pattern as other equations as regards the nodes they involve. The method also includes implicitly the singlevaluedness conditions for $w, w, x^{\text {and }} w, y$.

It is interesting to note that the boundary conditions $w=0$ and $M_{n n}^{*}=0$ have as dual conditions $\Psi=0$ and $\varepsilon_{s s}=0$ i. e. the boundary curve is inextensible and is the funicular of the boundary stress resultants due only to Airy's stress function. The cases of a rectangular boundary and of a circular boundary under axisym netrical conditicns are particularly simple to treat.

The second case of mixed boundary conditions is one in which $w_{, n}$ and $Q_{n e}$ are specified on the boundary. Eqs. 183 and 182 provide at each node $k$ one equation expressing $Q_{k}^{*}$ in terms of nodal values of $U$ and V. A second equation may be associated with each side $i$ by expressing $w, n$ in terms of $w, x$ and $w, y$ and then using Eqs. 187 and 188 to express these in terms of the stress functions.
f. Elastic Boundary Conditions

Consider a plate supported along its boundary by distributed elastic springs. The boundary conditions are taken in the form

$$
\begin{gather*}
w^{s}-w=f_{z z} Q_{n e}  \tag{194}\\
-w_{, n}^{s}+w, n=f_{s s} M_{n n} \tag{195}
\end{gather*}
$$

where $w^{s}$ is the displacement and $w^{s}{ }^{s}$ the rotation of the spring support, and $f_{z Z}$ and $f_{s s}$ are flexibility coefficients. $z$ and $s$ refer to the directions normal to the plate and tangent to the boundary, respectively. It will be convenient to write Eqs. 194 and 195 in the form

$$
\begin{gather*}
w=f_{z z}\left(Q^{s}-Q_{n e}\right)  \tag{196}\\
-w, n=f_{s s}\left(M^{s}-M_{n n}\right) \tag{197}
\end{gather*}
$$

where

$$
\begin{align*}
& Q^{s}=\frac{w^{s}}{f_{z z}}  \tag{198}\\
& M^{s}=\frac{-w, n}{f_{s s}} \tag{199}
\end{align*}
$$

The complementary potential energy of the spring forces per unit length of boundary is obtained from Eqs. 196 and 197 in the form

$$
\begin{equation*}
B^{\prime}=\frac{1}{2} f_{z z}\left(Q^{s}-Q_{n e}\right)^{2}+\frac{1}{2} f_{s s}\left(M^{s}-M_{n n}\right)^{2} \tag{200}
\end{equation*}
$$

Let $Q_{n e}$ and $M_{n n}$ be separated into the parts due to the stress functions and the parts due to the particular solution of the equilibrium equation i.e. let

$$
\begin{align*}
& Q_{n e}=Q_{n e}^{*}+Q_{n e}^{p}  \tag{201}\\
& M_{n n}=M_{n n}^{*}+M_{n n}^{p} \tag{201}
\end{align*}
$$

$B^{\prime}$ takes the form

$$
\begin{equation*}
B^{\prime}=\frac{1}{2} f_{z z}\left(Q^{\circ}-Q_{n e}^{*}\right)^{2}+\frac{1}{2} f_{s s}\left(M^{\circ}-M_{n n}^{*}\right)^{2} \tag{202}
\end{equation*}
$$

where

$$
\begin{align*}
& Q^{\circ}=Q^{s}-Q_{n e}^{p}  \tag{203}\\
& M^{\circ}=M^{s}-M_{n n}^{p} \tag{204}
\end{align*}
$$

The above expression of $B^{\prime}$ replaces that given in Eq. 121 for use in the functional $\Pi^{\prime}$, Eq. 115. As was done in paragraph $7, \pi^{\prime}$ may be put in the form

$$
\begin{equation*}
\pi^{\prime \prime}=\iint\left(W^{\prime \prime}+P^{\prime \prime}\right) d A+\oint B^{\prime \prime} d s \tag{205}
\end{equation*}
$$

where $W^{\prime \prime}$ and $P^{\prime \prime}$ are given as before through Eqs. 147 and 151 and B" takes now the form

$$
\begin{equation*}
B^{\prime \prime}=-K_{x} y, s+K_{y} x_{, s} V+B^{\prime} \tag{206}
\end{equation*}
$$

The dual of $-T^{\prime \prime}$ is the total potential energy of a plate bounded by an edge beam. Letting $\bar{N}_{n x}$ and $\bar{N}_{n y}$ be the force intensity components applied externally on the edge beam and using the notation of paragraph 5 the duality is summarized in the table below.

Table 2
EA, EI

$$
\varepsilon, \chi
$$

$$
\begin{aligned}
& \varepsilon^{\circ}, x^{\circ} \\
& \bar{N}_{n x}, \bar{N}_{n y}
\end{aligned}
$$

$$
\begin{aligned}
& -\mathrm{f}_{\mathrm{ss}},-\mathrm{f}_{\mathrm{zz}} \\
& \mathrm{M}_{\mathrm{nn}}^{*}, \mathrm{Q}_{\mathrm{ne}}^{*} \\
& \mathrm{M}^{\circ}, \mathrm{Q}^{\circ} \\
& -\mathrm{K}_{\mathrm{x}}^{\mathrm{y}}, \mathrm{~s}, \mathrm{~K}_{\mathrm{y}} \mathrm{x}, \mathrm{~s}
\end{aligned}
$$

## 10. Plate Bounded by an Edge Beam

Consider the beam isolated as a free body and subjected to the load $-Q_{n e},-M_{n n}$ at its junction with the plate and to the external load $\bar{Q}_{n e}$ and $\bar{M}_{n n}$. The resultant load is formed of

$$
\begin{align*}
& \Delta Q_{n e}=\bar{Q}_{n e}-Q_{n e}=\bar{Q}_{n e}-Q_{n e}^{p}-Q_{n e}^{*}  \tag{207}\\
& \Delta M_{n n}=\bar{M}_{n n}-M_{n n}=\bar{M}_{n n}-M_{n n}^{p}-M_{n n}^{*} \tag{208}
\end{align*}
$$

Let $P_{z}^{b}$ be the transverse shear and $C_{x}^{b}$ and $C_{y}^{b}$ be the $x$ and $y$ components of the moment vector acting on a positive cross section. Using Eqs. 178 to 180 we can write

$$
\begin{align*}
& P_{z}^{b}=\Omega-\left(\Omega_{o}+P_{z o}^{b}\right)-P_{z}^{\prime}  \tag{209}\\
& C_{x}^{b}=U-\left(U_{o}+C_{x o}^{b}\right)+\left(y-y_{o}\right)\left(\Omega_{o}+P_{z o}^{b}\right)-C_{x}^{\prime}  \tag{210}\\
& C_{y}^{b}=V-\left(V_{o}+C_{y o}^{b}\right)-\left(x-x_{o}\right)\left(\Omega_{o}+P_{z o}^{b}\right)-C_{y}^{\prime} \tag{211}
\end{align*}
$$

where subscript o refers to an arbitrary cross section o and $P_{z}^{\prime}$, $C_{x}^{\prime}$ and $C_{y}^{\prime}$ are the resultant and the moments, respectively, at a cross section(s) of the forces ( $\bar{Q}_{n e}-Q_{n e}^{p}$ ) and moments ( $\bar{M}_{n n}-M_{n n}^{p}$ ) acting on the portion of the beam oriented positively between 0 and (s), Fig. 21. Let $M_{b}^{b}$ and $M_{t}^{b}$ be the normal and tangential components, respectively, of the moment vector at a positive cross section, Fig. 22. $M_{b}^{b}$ is the bending moment and $M_{t}^{b}$ is the torsional moment. They are related to $C_{x}^{b}$ and $C_{y}^{b}$ through the vector transformation formulas

$$
\begin{align*}
& M_{b}^{b}=-C_{x}^{b} y, s+C_{y}^{b} x, s  \tag{212}\\
& M_{t}^{b}=C_{x}^{b} x, s+C_{y}^{b} y, s \tag{213}
\end{align*}
$$

The complementary strain energy of the beam is taken in the form

$$
\begin{equation*}
w^{\prime b}=\frac{1}{2} \frac{\left(M_{b}^{b}\right)^{2}}{E I}+\frac{1}{2} \frac{\left(M_{t}^{b}\right)^{2}}{G J} \tag{214}
\end{equation*}
$$

where EI and GJ are the bending and torsional rigidities, respectively. In terms of $\mathrm{C}_{\mathrm{x}}^{\mathrm{b}}$ and $\mathrm{C}_{\mathrm{y}}^{\mathrm{b}}$ obtain

$$
\begin{align*}
W^{\prime} & =\left[\frac{(y, s)^{2}}{2 E I}+\frac{(x, s)^{2}}{2 G J}\right]\left(C_{x}^{b}\right)^{2}+\left[\frac{x, s y, s}{G J}-\frac{x, s, y, s}{E I}\right] C_{x}^{b} C_{y}^{b} \\
& +\left[\frac{\left(x, s^{2}\right.}{2 E I}+\frac{(y, s)^{2}}{2 G J}\right]\left(C_{y}^{b}\right)^{2} \tag{215}
\end{align*}
$$

Introducing the notation

$$
\begin{align*}
& \Omega^{s}=\Omega_{0}+P_{z o}^{b}+P_{z}^{\prime}  \tag{216}\\
& U^{s}=\left(U_{o}+C_{x o}^{b}\right)-\left(y-y_{o}\right)\left(\Omega_{o}+P_{z o}^{b}\right)+C_{x}^{\prime}  \tag{217}\\
& V^{s}=\left(V_{o}+C_{y o}^{b}\right)+\left(x-x_{o}\right)\left(\Omega_{o}+P_{z o}^{b}\right)+C_{y}^{\prime} \tag{218}
\end{align*}
$$

$$
\begin{align*}
& f_{x x}=\frac{(y, s)^{2}}{E I}+\frac{(x, s)^{2}}{G J}  \tag{219}\\
& f_{x y}=x_{, s} y, s\left(\frac{1}{G J}-\frac{1}{E I}\right)  \tag{220}\\
& f_{y y}=\frac{(x, s)^{2}}{E I}+\frac{(y, s)^{2}}{G J} \tag{221}
\end{align*}
$$

W ${ }^{\text {b }}$ takes the form

$$
\begin{equation*}
W^{\prime}{ }^{b}=\frac{1}{2} f_{x x}\left(U-U^{s}\right)^{2}+f_{x y}\left(U-U^{s}\right)\left(V-V^{s}\right)+\frac{1}{2} f_{y y}\left(V-V^{s}\right)^{2} \tag{222}
\end{equation*}
$$

the dual of $-W^{\prime b}$ is the strain energy, $B$, of the springs in the case of a plate with elastic boundary conditions, Eq. 16. The duality is summarized in the table below.

Table 3

$$
\begin{array}{ll}
k_{x x}, k_{x y}, k_{y y} & -f_{x x^{\prime}}-f_{x y},-f_{y y} \\
u, v & U, V \\
u^{s}, v^{s} & U^{s}, v^{s} \\
N_{n x}=-\frac{\partial B}{\partial u} & -x_{s y}^{*}=\frac{\partial W^{\prime}{ }^{b}}{\partial U} \\
N_{n y}=-\frac{\partial B}{\partial v} & x_{s x}^{*}=\frac{\partial W^{b}}{\partial V}
\end{array}
$$

In order to be able to use the duality established above the quantities $\left(\Omega_{o}+P_{z o}^{b}\right),\left(U_{o}+C_{x o}^{b}\right)$ and $\left(V_{o}+C_{y o}^{b}\right)$ in Eqs. 216 to 218 must be determined. Their dual quantities characterize a rigid body displacement of the spring support. For a simply connected domain this rigid body displacement is arbitrary. $U_{o}, V_{o}$ and $\Omega_{o}$ may therefore be chosen such that

$$
\begin{align*}
& \Omega_{o}+P_{z o}^{b}=0  \tag{223}\\
& U_{o}+C_{x o}^{b}=0  \tag{224}\\
& V_{o}+C_{y o}^{b}=0 \tag{225}
\end{align*}
$$

$C_{x}^{\prime}$ and $C_{y}^{\prime}$ are then dual of the support displacements $u$ and $v$. For a multiply-connected domain Eqs. 223 to 225 may be kept for one of the boundary curves. For each of the remaining boundary curves three equations are needed for the determination of $\left(\Omega_{o}+P_{z o}^{b}\right),\left(U_{o}+C_{x o}^{b}\right)$ and $\left(V_{o}+C_{y o}^{b}\right)$. These characterize in the dual stretching problem a rigid body displacement of the boundary curve and are therefore related in that problem to the resultants and moment of the spring forces. If the parts of the spring forces that are attributable to Airy's stress function $\psi$ are not self equilibrating, $\psi$ and its first derivatives are multivalued. Returning to the bending problem where the plate is assumed to be free of dislocations, the three equations per boundary curve that are necessary for the determination of $\left(\Omega_{0}+P_{z o}^{b}\right)$, $\left(U_{0}+C_{x o}^{b}\right)$ and ( $\left.V_{o}+C_{y o}^{b}\right)$ are therefore singlevaluedness conditions for $w, w, x$ and $w, y$. These are expressed in terms of the stress functions by equating to zero the resultants and the moment with regard to some point of the generalized nodal rotations $F_{x k}^{\prime}$ and $F_{y k}^{\prime}$.

## 11. Formulation of a Stretching Problem Dual of a Given Bending

 Problem. Choice of a Particular Solution.The stretching analogue of a bending problem is uniquely defined once adefinite particular solution of the bending equilibrium equation is obtained.

It is recalled that the transverse shears and the stress couples of the particular solution, as defined through Eqs. 126 to 130, are the internal forces that would occur in two families of strips parallel to the coordinate axes and supporting together but independently of each other the load q. In order to obtain a definite particular solution the boundary conditions of the strips and the part of the load q car ried by one family of strips must be specified. If $c(x, y)$ is the proportion of the load carried by the strips parallel to the x axis, Eq. 131 may be replaced by the two equations

$$
\begin{align*}
& {\left[D_{x}\left(K_{y}+\nu_{x} K_{x}\right)\right], x_{x}=c q}  \tag{226}\\
& {\left[D_{y}\left(K_{x}+\nu_{y} K_{y}\right)\right]_{, y y}=(1-c) q} \tag{227}
\end{align*}
$$

Solving the above equations is the same as determining the shear and bending moment in the two families of strips. With $K_{x}$ and $K_{y}$ determined the dual stretching problem is well defined.

The simplest choice for $c$ is a constant. The cases $c=0$ and $c=1$ correspond to one family of strips only in the $y$ and $x$ directions, respectively. The case $c=1 / 2$ corresponds to two families of strips, each family in equilibrium under the load $q / 2$. Although the exact solution does not depend on the particular choice of $c$, it may be expected that more accurate results are obtainedif, everything else being equal, the behavior of the strips is nearer to that of the plate. The influence of the choice of $c$ on the accuracy of the analysis is discussed in the next section.

As an example consider a homogeneous and isotropic rectangular plate with a uniform load such as shown in Fig. 23a. It will be assumed, for simplicity of presentation of the boundary forces in the dual problem, that the plate is clamped along its edges. Considering the case $c=1$

Eqs. 226 and 227 are satisfied by letting

$$
\begin{align*}
& K_{x}=-\nu K_{y}  \tag{228}\\
& K_{y, x x}=\frac{q}{D\left(1-\nu^{2}\right)} \tag{229}
\end{align*}
$$

The strips are taken as simply supported. Whence

$$
\begin{equation*}
K_{y}=\frac{q}{2 D\left(1-\nu^{2}\right)}\left(x^{2}-a^{2}\right) \tag{230}
\end{equation*}
$$

The dual of the load components $p_{y}$ and $p_{x}$ in the stretching problem are $K_{y, y}$ and $K_{x, x}$, respectively, i.e.

$$
\begin{equation*}
K_{y, y}=0 \tag{231}
\end{equation*}
$$

and

$$
\begin{equation*}
K_{x, x}=-\frac{\nu g x}{2 D\left(1-\nu^{2}\right)} \tag{232}
\end{equation*}
$$

Because q is independent of y Eqs. 226 and 227 may also be satisfied by letting

$$
\begin{align*}
& K_{x}=0  \tag{233}\\
& K_{y, x x}=\frac{q}{D} \tag{234}
\end{align*}
$$

whence

$$
\begin{equation*}
K_{y}=\frac{q\left(x^{2}-a^{2}\right)}{2 D} \tag{235}
\end{equation*}
$$

This particular solution is that of cylindrical bending in the $\mathbf{x}$ direction. There are only boundary forces in the dual stretching problem as shown in Fig. 23b.

In order to illustrate the case where the function $q$ is discontinuous, consider the plate of the above example loaded partially with a uniform load over the shaded area, Fig. 24a. Taking one family of
strips in the x direction Eqs. 228 and 229 remain valid if $q$ is interpreted as the loading function of the present case. It is seen that $K_{y}$ is discontinuous across lines $A B$ and $A^{\prime} B^{\prime}$. This discontinuity is dual of the intensity of a line load along $A B$ and $A^{\prime} B^{\prime}$ as shown in Fig. 24b.

If q is a line load a particular solution may be obtained as in the case above by means of one family of strips crossing the line load.

It is noted that in the two examples above the stress functions are continuous as the displacements are in the dual stretching problem. If the load function $q$ is a concentrated load it is treated as dual of a characteristic of dislocation.

The computation of the generalized nodal rotations at an interior node due to $K_{x, x}$ and $K_{y, y}$ is dual of the computation of the generalized nodal forces due to $\mathrm{P}_{\mathrm{x}}$ and $\mathrm{p}_{\mathrm{y}}$. It is however possible to use directly $\mathrm{K}_{\mathrm{x}}$ and $K_{y}$ in this computation. The contribution of one element to the generalized nodal rotations $P_{x k}^{\prime}$ and $P_{y k}^{\prime}$ at node $k$ is

$$
\begin{align*}
& P_{x k}^{\prime}=\iint K_{x, x} \xi_{k} d A  \tag{236}\\
& P_{y k}^{\prime}=\iint K_{y, y} \xi_{k} d A \tag{237}
\end{align*}
$$

Using Green's theorem and the expression, Eq. 25 , of $\boldsymbol{\xi}_{k}$ in terms of $x$ and $y$, Eqs. 236 and 237 take the form

$$
\begin{align*}
& P_{x k}^{\prime}=\frac{b_{k}}{2 A} \iint K_{x} d A+\oint K_{x} \xi_{k} d y  \tag{238}\\
& P_{y k}^{\prime}=-\frac{a_{k}}{2 A} \iint K_{y} d A-\oint K_{y} \xi_{k} d x \tag{239}
\end{align*}
$$

The total generalized nodal rotations $G_{x k}^{\prime}$ and $G_{y k}^{\prime}$ at node $k$ are ob-

node the boundary integrals in Eqs. 238 and 239 add up to zero because $\xi_{k}=0$ on the sides opposite to node $k$ and the integrands take opposite values on sides common to the triangular elements. Thus

$$
\begin{align*}
& G_{x k}^{\prime}=\sum \frac{b_{k}}{2 A} \iint K_{x} d A  \tag{240}\\
& G_{y k}^{\prime}=\sum-\frac{a_{k}}{2 A} \iint K_{y} d A \tag{241}
\end{align*}
$$

where the summation extends over the elements having node k in common.
At a boundary node the superposition of the boundary integrals in Eqs. 238 and 239 yields the same values as $F_{x k}^{\prime p}$ and $F_{y k}^{\prime p}$, respectively, Eqs. 166 and 167. From Eqs. 162 and 163 it appears that the use of Eqs. 240 and 241 at a boundary node takes account completely of the contribution of the particular solution to the generalized nodal rotations.

The type of particular solution of the equations of equilibrium considered earlier is restricted by the requirement $M_{x y}^{p}=0$. This is not necessary and a more general form of the particular solution is obtained by letting instead of Eq. 134

$$
\begin{equation*}
M_{x y}=M_{x y}^{*}+M_{x y}^{p} \tag{242}
\end{equation*}
$$

where

$$
\begin{equation*}
M_{x y}^{p}=M_{y x}^{p}=-D_{x y} K_{x y} \tag{243}
\end{equation*}
$$

and

$$
\begin{equation*}
D_{x y}=\frac{G h^{3}}{6} \tag{244}
\end{equation*}
$$

Instead of Eq. $131 \mathrm{~K}_{\mathrm{x}}, \mathrm{K}_{\mathrm{y}}$ and $\mathrm{K}_{\mathrm{xy}}$ must now satisfy

$$
\begin{equation*}
\left[D_{x}\left(K_{y}+\nu_{x} K_{x}\right)\right]_{, x x}+2\left(D_{x y} K_{x y}\right), x y+\left[D_{y}\left(K_{x}+\nu_{y} K_{y}\right)\right], y y-q=0 \tag{245}
\end{equation*}
$$

The form of the functional $\pi$ " remains unchanged but Eqs. 151, 153 and 154 are replaced by

$$
\begin{equation*}
P^{\prime \prime}=\left(K_{x, x}-K_{x y, y}\right) U+\left(K_{y, y}-K_{x y, x}\right) V \tag{246}
\end{equation*}
$$

$$
\begin{align*}
& x_{s y}^{*}=-w_{, y s}+K_{x, s} y+K_{x y} x, s  \tag{247}\\
& x_{s x}^{*}=-w, x s+K_{x y} y, s+K_{y} x, s \tag{248}
\end{align*}
$$

The change to the duality is that ( $\mathrm{K}_{\mathrm{x}, \mathrm{x}}-\mathrm{K}_{\mathrm{xy}, \mathrm{y}}$ ) instead of $\mathrm{K}_{\mathrm{x}, \mathrm{x}}$ and $\left(K_{y, y}-K_{x y, x}\right)$ instead of $K_{y, y}$ are now dual of the load components $p_{x}$ and $p_{y}$, respectively. Also $\chi_{x y}^{*}$ instead of $\chi_{x y}$ is dual of $N_{x y}$, where

$$
\begin{equation*}
x_{\mathrm{xy}}^{*}=\frac{12}{\mathrm{Gh}^{3}} \mathrm{M}_{\mathrm{xy}}^{*}=-\mathrm{w}_{, \mathrm{xy}}+\mathrm{K}_{\mathrm{xy}} \tag{249}
\end{equation*}
$$

The generalized nodal rotations at node $k$ due to one element are now given through the relations

$$
\begin{align*}
& P_{x k}^{\prime}=\iint\left(K_{x, x}-K_{x y, y}\right) \xi_{k} d A  \tag{250}\\
& P_{y k}^{\prime}=\iint\left(K_{y, y}-K_{x y, x}\right) \xi_{k} d A \tag{251}
\end{align*}
$$

As was done before in obtaining Eqs. 238 and 239 from Eqs. 236 and 237, Eqs. 250 and 251 may be transformed into the form

$$
\begin{align*}
& P_{x k}^{\prime}=\frac{b_{k}}{2 A} \iint K_{x} d A+\frac{a_{k}}{2 A} \iint K_{x y} d A+\oint \xi_{k}\left(K_{x} d y+K_{x y} d x\right)  \tag{252}\\
& P_{y k}^{\prime}=-\frac{a_{k}}{2 A} \iint K_{y} d A-\frac{b_{k}}{2 A} \iint K_{x y} d A-\oint \xi_{k}\left(K_{y} d x+K_{x y} d y\right) \tag{253}
\end{align*}
$$

The total generalized nodal rotations $G_{x k}^{\prime}$ and $G_{y k}^{\prime}$ at an interior node are obtained by superposition in the form

$$
\begin{align*}
& G_{x k}^{\prime}=\sum \frac{b_{k}}{2 A} \iint K_{x} d A+\sum \frac{a_{k}}{2 A} \iint K_{x y} d A  \tag{254}\\
& G_{y k}^{\prime}=-\sum \frac{a_{k}}{2 A} \iint K_{y} d A-\sum \frac{b_{k}}{2 A} \iint K_{x y} d A \tag{255}
\end{align*}
$$

where the summation extends over the elements having node $k$ in common.

At a boundary node the superposition of the boundary integrals in Eqs. 252 and 253 yields, as before, the contribution of the particular solution to the generalized nodal rotations arising from the boundary conditions.

The latter form of the particular solution may be used to advantage if a solution to the plate problem satisfying equilibrium and compatibility but not the boundary conditions is known. In such a case $K_{x}$, $K_{y}$, and $K_{x y}$ are compatible curvatures i. e. they are related to a deflection function $w^{\prime}$ through the relations

$$
\begin{align*}
& K_{x}=w^{\prime}, y y  \tag{256}\\
& K_{y}=w^{\prime}, x x  \tag{257}\\
& K_{x y}=w^{\prime}, x y \tag{258}
\end{align*}
$$

As may be expected in such a case, Eqs. 250 and 251 show that $P_{x k}^{\prime}$ and $P_{y k}^{\prime}$ are zero. The generalized nodal rotations are therefore zero at all interior nodes. At a boundary node Eqs. 247 and 248 yield

$$
\begin{align*}
& x_{s y}^{*}=w^{\prime}, y s-w, y s  \tag{259}\\
& x_{s x}^{*}=w^{\prime}, x s-w, x s \tag{260}
\end{align*}
$$

and the generalized nodal rotations $F^{\prime \prime} \times \mathrm{xk}$ and $\mathrm{F}_{\mathrm{yk}}^{\prime \mathrm{*}}$ are obtained from Eqs. 160 and 161 in a form similar to that of Eqs. 168 and 169.

$$
\begin{align*}
& F_{x k}^{\prime *}=\frac{1}{l_{n}} \int_{0}^{l_{n}}\left(w, y-w^{\prime}, y^{\prime}\right) d s-\frac{1}{l_{m}} \int_{0}^{\ell_{m}}\left(w, y-w^{\prime}, y\right) d s  \tag{261}\\
& F_{x k}^{\prime *}=-\frac{1}{l_{n}} \int_{0}^{l_{n}}\left(w, x-w^{\prime}, x^{\prime}\right) d s+\frac{1}{l_{m}} \int_{0}^{l_{m}}\left(w, x-w_{, x}^{\prime}\right) d s \tag{262}
\end{align*}
$$

If the particular solution to a problem is recycled by identifying $K_{x}, K_{y}$ and $K_{x y}$ in a second cycle with the curvatures $-\chi_{y y},-\chi_{x x}$, and $-\chi_{x y}$, respectively, obtained from a first cycle, the generalized nodai rotations should all be zero. This forms a compatibility check dual of an equilibrium check in the stretching problem.

## 12. Computation of the Stress Couples and of the Deflection

The use of piece-wise linear stress functions implies constant stress couples $M_{x x}^{*}, M_{y y}^{*}$, and $M_{x y}^{*}$ in each element. These values cannot be associated with any particular point of the element but represent global measures or averages for the element as a whole. It may, however, be necessary from a practical point of view to obtain values of the stress couples at definite points. The constant values for an element may then be associated, for lack of more precise information, with its centroid and the values of the stress couples at other points may be obtained through interpolation.

A different and preferable method based also on interpolation and whose application is simplest if the distribution of the nodes is regular consists in evaluating $M_{x x}^{*}, M_{y y}^{*}$, and $M_{x y}^{*}$ at a point by means of finite differences. The values of $M_{x x}, M_{y y}$ and $M_{x y}$ at a point are obtained by superimposing $M_{x x}^{p}, M_{y y}^{p}$, and $M_{x y}^{p}$ on $M_{x x}^{*}, M_{y y}^{*}$ and $M_{x y}^{*}$, respectively.

The determination of the alopes $w, x$ and $w, y$ and of the deflection $w$ may be made by means of Eqs. 187 to 189. In these $A$ and $B$ are two arbitrary mid-side points joined by a broken line $A i_{1} i_{2} \ldots i_{p} B$ along element sides as shown in Fig. 25. The plate may be imagined cut along $A i_{1} i_{2} \ldots i_{p} B$ and the right hand sides of Eqs. 187 to 189 may be evaluated once the solution for the nodal values of the stress functions is obtained. It may be noted that the above method of determining $w, x^{\prime} w, y$ and $w$ is a generalization of the so-called conjugate beam method.

## 13. Results and Conclusion

Results of bending analysis of isotropic rectangular and square plates under various loadings and boundary conditions are presented in Figs. 26 to 33. They are obtained by means of a computer program written for the analysis of plane stress and plane strain problems by the displacement method (13). The program was modified so as to compute the element stiffness matrix as established in this paper and to accept dislocations and arbitrarily specified boundary displacements.

Because of symmetry only a quarter of the plate is used in each analysis. The part of the stress couples due to the stress functions is computed at the nodes by means of finite differences. In all cases except that of the concentrated load the particular solution of the equilibrium equations is that of cylindrical bending in the $x$ direction. The case of the concentrated load at the center of a clamped square plate is treated as shown in Fig. 31 for the dual stretching problem.

It may be seen that in all cases with a uniform load a relatively coarse mesh of $4 \times 4$ is sufficient to determine with satisfactory accuracy the stress couples and therefore also the curvatures and the deflection. In the case of a concentrated load, Fig. 32, a finer mesh is needed in the vicinity of the point of application of the load in order to represent the singularity with sufficient accuracy.

The determination of the deflection in Fig. 33 is made by the conjugate beam method using linear inter polation for determining the curvature between nodes.

A comparison with results obtained by the displacement method using the so called (HCT) triangular element may be made through Figs. 26 and 33. The stiffness matrix of this triangular element is based on a displacement expansion satisfying compatibility of displacement and of normal slope between adjacent elements. It is established in (5) as the best available stiffness matrix for a triangular element.

A test case for displacement methods (5) not represented here is that of an isotropic square plate supported at three corners and acted upon by a force at the fourth corner. The stress function method solves this problem exactly with $M_{x y}$ constant and $M_{x x}=$ $M_{y y}=0$. Although more tests are needed for a thorough comparison, the results obtained here compare favorably with those of the displacement method as regards both accuracy and convergence. It should also be noted that there are three equations per node in the latter method but only two in the stress function method used here.

The availability of both methods for the solution of a given problem may provide a means of estimating the distance of the approximate solutions from the exact solution. This is possible if the displacement and the stress function methods provide minimizing sequences to the potential energy and to the complementary potential energy functionals, respectively. The (HCT) triangular element and the stress function method used here have this property. For example, the central deflection of the clamped plate under a unit concentrated load, Fig. 33, is equal to minus twice the potential energy and is therefore larger than the value obtained by the (HCT) displacement method. It is also equal to twice the strain energy and, by Castigliano's theorem, is smaller than the value obtained by the stress function method.

The stress function method may also be applied to plate stretching problems and to shell problems. In the former case the method uses Airy's stress function and is dual of the displacement method for plate bending. For shell problems, it is also possible to develop mixed methods using a formulation of shell theory in terms of both displacements and stress functions (14)(15). A basis for such a method could be the equations of shallow shells in terms of the normal deflection and of Airy's stress function. If the curvature of the shell element is neglected the method would reduce to a combination of the displacement method
for bending and of the dual stress function method for stretching of a flat element. Similarly the displacement method for stretching and the stress function method for bending developed here may be combined to treat the shell problem by means of flat triangular elements.


Figure 1


Figure 2


Figure 3


Figure 4


Figure 5


Figure 6


Figure 7


Figure 8


Figure 9


Figure 10


Figure 11 a


Figure llb


Figure 12


Figure 13


Figure 14


Figure 15a


Figure 15b


Figure 16


Figure 17


Figure 18


Figure 19


Figure 20


Figure 21


Figure 22


Figure 23a


Figure $24 a$


Figure 23b

Figure 24b


Figure 25


Figure 26

Uniform load 9 Clamped plate $\nu=0.3$

22



Figure 27


Figure 28

Uniform load 9
Simply supported plate

$$
\nu=0.3
$$




Figure 29

Uniform load 9 Clamped plate $\nu=0.3$



Figure 30


Figure 31


Figure 32


Central deflection $w$
Clamped square plate $2 a \times 2 a$
Concentrated load $P$

Figure 33

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