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An Overview of Quasiconcavity and its Applications in Economics

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AN OVERVIEW OF QUASICONCAVITY AND ITS APPLICATIONS IN ECONOMICS

Introduction

The concept of "quasiconcavity," and its equally well known subcategory of "strict quasiconcavity," is more comprehensive than it appears. Quasiconcavity is the most inclusive class of generalized concave functions, broad enough to include both concave and convex shapes. Moreover, since quasiconcavity is the mirror image of quasiconvexity, the analysis of quasiconcavity may be interpreted as the analysis of curvature characteristics in general. The majority of economics-oriented mathematical works treat curvature characteristics in the framework of quasiconcavity (strict quasiconcavity).[1]

The varying definitions of quasiconcavity are not completely equivalent. They are used interchangeably, and the extent of overlap between each definition and the definitions of closely-related concepts eludes those who do not deal with these issues on a regular basis. Over the past decades, several works have appeared under the rubric of "generalized concavity" to standardize the criteria for quasiconcavity and for concepts closely related to it. An article by Arrow and Enthoven, entitled "Quasi-Concave Programming" may have been the first such work (Arrow and Enthoven, 1961). Mangasarian's book "Nonlinear Programming" (Mangasarian, 1969) and Ginsberg's article "Concavity and Quasiconcavity in Economics" (Ginsberg, 1973), which also benefitted from the advice of Arrow, represented further major advances in this field. Authored by Crouzeix, the entry under quasiconcavity in the "The New Palgrave, A Dictionary of Economics," is perhaps the best short summary of the subject (Crouzeix, 1987). The book, "Generalized Concavity," jointly authored by Avriel, Diewert, Schaibe, and Zang (Avriel et al, 1988), appears to be the most extensive, in-depth treatment of quasiconcavity and related issues to date. Besides providing excellent definitions and discussions, Huang and Crooke use Wolfram's software, MATHEMATICA 3.0, in their book "Mathematics and Mathematica for Economists" (Huang and Crooke, 1997) to analyze and provide visual illustrations of quasiconcavity. Several other highly valuable books and articles on the subject will be mentioned in the text.

By telescoping existing conventions, this paper provides a unified framework for the definitions of

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[1] "...quasiconvex and strictly quasiconvex functions, which are in a natural sense the opposites of quasiconcave and strictly quasiconcave functions, are less commonly encountered in economics." (Cornes, 1994, p. 8.)

quasiconcavity. It also summarizes and illustrates uses of the concept. The paper does not provide descriptions of topological properties and skips generalizations into hyper-spaces. However, it assumes familiarity with "econ-math," more specifically geometric and algebraic skills involving 2-and 3-dimensional graphs. The paper may be particularly helpful to readers of economic modeling literature, because some of the standard conditions for optimization in partial and general equilibrium models are formulated in terms of quasiconcavity. (See, Arrow and Debreu, 1954, and Ginsburgh and Keyzer, 1997).

Quasiconcavity

Arrow and Debreu used the terms quasiconcavity and strict quasiconcavity in their epoch-making article on modeling general equilibrium (Arrow and Debreu, 1954). The term "strictly quasiconcave" may have gained widespread use as a result of publications by Arrow and Enthoven, 1961, Elkin, 1968, Mangasarian, 1969, Ortega and Rheinboldt, 1970, and Ginsberg, 1973.[2]

In most economic applications, quasiconcave curves are allowed to have only a single peak; therefore, they may also be characterized as "single-humped" or "unimodal" functions. Such quasiconcave curves cannot have pits or valleys. This paper deals exclusively with unimodal quasiconcavity. Taking this restriction into consideration, but using all its definition-sanctioned possibilities, a quasiconcave curve may trek upwards along various curved and straight-line paths, with or without flat resting segments (but with the exclusion of vertical segments). It may peak once in a point, or a dome, or a flat line, and trek down along various curved and straight-line paths, with or without flat resting places (but, again with the exclusion of vertical segments). Any portion of this most comprehensive version is also quasiconcave. However, if flat segments, as well as ascending and descending straight lines are disallowed along its path, the quasiconcave curve becomes strictly quasiconcave.

Quasiconcavity (strict quasiconcavity) is defined in geometric,[3] set-theoric, and algebraic terms. Three

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[2] DeFinnetti and Fenchel are considered the early pioneers of quasiconcavity. Their publications on the subject date to the late 1940s and early 1950s (Crouzeix, 1987).

[3] Those definitions will be considered "geometric" that involve either comparisons between the arc of a curve and the chord drawn between its end points, or comparisons between the estimated and actual direction of the curve. The word "geometric" is used as an abbreviation of the more correct (but longer) expression of

geometric definitions are commonly used: the function-value comparison, the minimum function value test, and the differential-based approach.[4] A quasiconcave curve may be generated by continuous or discontinuous functions. Continuous twice differentiability (and hence, continuousness) is a *sine quo non* only under the algebraic definition.

Testing the differentiability of a function at a given point on it is complete only if this property is explored in every possible direction. If the rate of change can be determined from every point on the function in every possible direction, the function is called directionally differentiable. The literature on curvature characteristics provides definitions of quasiconcavity and its kindred concepts in terms of directional derivatives. (See, for example, Diewert, Avriel, and Zang, 1981). However, for the purpose of this paper it seems adequate to characterize curvatures only in terms of differentiation along the axes of independent variables, a special subset of directional derivatives. Continuous, twice differentiable functions are assumed to be also twice continuously directionally differentiable. Furthermore, the domains of all the functions considered in this paper will be convex sets containing only positive values and zero. (For example, in the two-dimensional case, the positive segment of the horizontal axis and the origin will be considered.) The domain will always be considered open, leaving the function all the "space" it requires to reveal its curvature characteristics. These simplifications also may not diminish the elementary grasping of the subject at hand.

The literature on quasiconcavity appears to be written exclusively in terms of local properties, allowing for only an indirect exploration of global properties. That is, the satisfaction of certain criteria in a suitable local neighborhood along the curve is the focus of the analysis, followed by generalizations using the word "any" or "every" in the various formulations, derivations, and proofs.[5]

Before summarizing the various approaches in defining quasiconcavity, it may be useful to contrast it with concavity.

"geometrically suggestive," used by Ginsberg (Ginsberg, 1973).

[4] This classification is new; therefore, references under these headings cannot be found in the subject indices of books dealing with generalized concavity.

[5] The direct exploration of global properties draws mainly from a combination of surface theory and differential geometry, that is, differential and integral calculus applied to planes and spaces. Rather than moving along the curve, as in the analysis of local properties, global property analysis involves reference points "inside" the curve. Using these reference points, such as the center of the curvature, the analysis establishes the nature of bending. The most popular global properties known to students of mathematics are the geodesic surface and the minimal curvature. Global property analysis is frequently applied in technology and in the natural sciences.

Concavity and Quasiconcavity

Takayama provides the generally used geometric definition of concavity (Takayama, 1994).[6] Accordingly, a real valued function f (x) defined over a convex set X is concave if

(1) $$f[tx_0 + (1-t)x] \geq tf(x_0) + (1-t)f(x)$$

for all x_0, $x \in X$, such that $x \neq x_0$, and $0 \leq t \leq 1$. In words, a function is concave if its value at the linear combination between two points in its domain is greater than or equal to the weighted average of the function's values at each of the points considered. A function is strictly concave if strict inequality is required in equation 1:

(2) $$f[tx_0 + (1-t)x] > tf(x_0) + (1-t)f(x)$$

In practical terms, the difference is that concavity allows for linear segments, but strict concavity does not. Concavity allows for ascending and descending linear segments. Vertical segments are excluded because of $x \neq x_0$. Horizontal segments are excluded, because such lines would allow chords to be drawn above the curve, violating the requirements of equation (1).

The definitions shown under equations (1) and (2) correspond to the function-value comparison among the geometric definitions of quasiconcavity. The minimum function value test does not exist for concavity. The (rough) equivalent of the differential-based approach may be summarized as follows. If the curve is strictly concave, all its points, with the exception of the point of tangency, lie below the tangent line. If it is concave, a tangent line may overlap a segment of the curve. (The similarity between the application of this type of definition of concave and quasiconcave curves is that both are based on the relationship between the differential of the curve at a single point and the curve itself.)[7]

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[6] All definitions subsume that the domain of the real and single-valued function (f) is a convex set (X) in the n-dimensional, Euclidean space, R^n.

[7] Naturally, the lines become planes in 3 dimensions and hyperplanes in "n" dimensions.

In set-theoric terms, a function is concave if its hypograph (the area below the curve) is convex. If the curve is strictly concave (has no linear segments), then its hypograph will be strictly convex. The same rule applies to uppercontour sets, which may be regarded as hypographs, truncated at a function value.[8]

The algebraic definition of concavity uses the Hessian matrix of second derivatives; thus, it assumes that the function in question is at least twice differentiable. (The same will be true for quasiconcavity.) The fundamental rule is that a function is strongly concave if its Hessian (that is, the matrix of second partial derivatives) is a negative definite at each point. (A matrix is a negative definite if its successive--or leading--principal minors alternate their signs.) The function is concave (sometimes "weakly" concave), if its Hessian is a negative semidefinite. A matrix is a negative semidefinite if occasional zeros are allowed to show up among the successive principal minors of alternating signs. (Of course, when this happens the principal minor has no sign.) For details, see Arrow and Enthoven 1961, Ginsberg, 1973, Takayama, 1994, and Huang and Crooke, 1997.[9]

Concerning the relationship between strong and strict concavity, the following rule applies. If a function is strongly concave, then it is strictly concave, but the reverse is not necessarily true. (For details and proof, see Ginsberg, 1973). Ginsberg uses the single-valued function $Y = - X^4$ as an example of the irreversibility of the above rule. This function is strictly concave because its hypograph is strictly convex. However, the function's second derivative ($Y'' = - 12 X^2$) assumes zero when X is zero. Thus, it is a "negative semidefinite" rather than a "negative definite," indicating that the function is weakly rather than strongly concave. The algebraic conditions of both strict and strong concavity demand that a continuous and at least twice continuously differentiable function rise, peak, and descend over an unrestricted domain. Terminological and definitional conventions regarding concavity are better established than those regarding quasiconcavity. This can be explained in part by the fact that quasiconcavity allows for a greater variety of curvatures than concavity. Therefore, in the case of quasiconcave

[8] See more on uppercontour sets and on the convexity (strict convexity) of sets, in general, under the section entitled "Quasiconcavity in Set-Theoric Terms."

[9] References to the Hessian bring to mind multivariate functions, but the definitions based on the Hessian may also be applied to singe-variable functions. In the single-variable case, the "negative definite" of the "Hessian" is simply the negative sign of the second derivative for every point in the domain. The "negative semidefinite" for the "Hessian" is a negative second derivative that may be equal to zero at a point. In the next paragraph, a single-variable function will be used to illustrate an important rule in the relationship between strong and strict concavity.

functions, the task of characterization at every point of the domain is much more complex than in the case of concave functions. Indeed, the difference between concavity and quasiconcavity is in kind, rather than in degree.

Geometric Definitions of Quasiconcavity

As mentioned above, there are three widely-known geometric definitions of quasiconcavity: the function-value comparison, the minimum function value test, and the differential-based approach. Under all three definitions, which also serve as tests to examine the presence of quasiconcavity, if the curvature in question is strictly quasiconcave then it must be quasiconcave, but not vice versa.

Function value comparison.--According to Takayama, 1994, a real-valued function f (x) is quasiconcave if for every $x \neq x_0$:

(3) $$f(x) \geq f(x_0) \Rightarrow f[tx + (1-t)x_0] \geq f(x_0)$$

In words, the function is quasiconcave if $f(x) \geq f(x_0)$ implies that its value at a linear combination between two points in its domain is greater than or equal to the function value at the smaller of the domain value (x_0). Strict inequality distinguishes strict quasiconcavity from quasiconcavity:

(4) $$f(x) \geq f(x_0) \Rightarrow f[tx + (1-t)x_0] > f(x_0)$$

Hence, as mentioned earlier, quasiconcavity allows for flat segments whereas strict quasiconcavity does not.[10] It may be pointed out that as far as the three geometric definitions are concerned, only flat segments (that is, horizontal to the independent variable axis) distinguish quasicocavity from strict quasiconcavity. Ascending or descending lines would pass the function-value comparison, as well as the two other geometric tests. (Vertical lines are excluded by the mandatory inequality between x_0 and x.) It is the set-theoric definition that excludes ascending and descending straight lines. (See the section entitled "Quasiconcavity in Set-Theoric Terms.")

[10] To simplify notation, some authors designate $[t x + (1-t) x_0]$ by a separate symbol, representing all possible linear combinations in the open interval (x_0, x). See Elkin, 1968.

The equality in the initial condition $f(x) \geq f(x_0)$, that is, weak inequality, assures that a strictly quasiconcave function may reach a unique global maximum, in the form of a single peak and an associated descending leg. An example of the strictly quasiconcave curve under this definition is the traditional production function (Beattie and Taylor, 1985). The function demonstrates the comprehensiveness of quasiconcavity as far as curve shapes are concerned. In the 2-dimensional single input-single output case, the first phase is strongly convex (increasing returns) the second phase is approximately homogeneous (constant returns), the third phase (decreasing returns) is strictly concave, and the fourth phase (negative returns) is downward sloping.

The weak inequality in the initial condition also assures that the descending leg cannot go below the starting point of the ascending leg. If the curve turns upward again after reaching its lowest point, that is, it forms a valley, it no longer passes this test. This may be seen by drawing a chord parallel with the horizontal axis above the valley that has been created by ascension after reaching the deepest point of the descending leg. Such a chord would form two intersections on the curve above the valley. These intersections may be identified as the ordered pairs $(x_0, f(x_0))$ on the "left" side and $(x, f(x))$ on the "right" side. It is obvious that the function values between x_0 and x will be smaller than $f(x_0) = f(x)$, rather than greater than or equal to them, as stipulated by equations (3) and (4).

In general, under-the-peak-parallel-chord tests[11] play a critical role in testing quasiconcave functions for unimodality. Probing quasiconcave curves with such tests will also reveal the weakness of the concept in identifying the global maximum. As the chord moves upwards and reaches the maximum, $f(x)$ virtually coincides with $f(x_0)$, making linear combinations between the two also virtually equal to them. However, they cannot all be equal and at the same time greater than $f(x_0)$. Thus, when maximum is reached the test will break down.

The minimum function value test.--Quasiconcavity may also be defined as follows:

[11] "Under-the-peak-chord" appears to be a necessary adjective to describe the word "test," since all the tests specified under the first two definitions of quasiconcavity may be regarded as chord tests.

(5) $$f[tx + (1-t)x_0] \geq \min[f(x), f(x_0)]$$

(See, for example, Arrow and Enthoven, 1961, Ortega and Rheinboldt, 1970; Ginsberg ,1973; and Avriel et al, 1988.) Based on equation (5), two approaches may be used to define strict quasiconcavity. When the definition specifies that it is valid for *every* x_0 and x, as in Avriel et al, 1988, the following equation applies.

(6) $$f[tx + (1-t)x_0] > \min[f(x), f(x_0)]$$

However, when the definition specifies that the criterion must hold for *any* x_0 and x, as for example in Ginsberg, 1973, the following condition is specified:

(7) $$f(x) = f(x_0) \Rightarrow f[tx + (1-t)x_0] > tf(x) + (1-t)f(x_0) = f(x) = f(x_0)$$

To appreciate the difference, draw a chord under the peak of a unimodal curve, parallel with the horizontal axis, creating the previously mentioned intersections of (x_0 , f (x_0)) on the "left" and (x, f (x)) on the "right." Assume that a flat resting place occurs between the chord and the peak, as allowed under quasiconcavity, but disallowed under strict quasiconcavity. If the definition specified by equation (6) had said *any* instead of *every*, this selection of the two ordered pairs would have allowed the curve to pass as strictly quasiconcave when in reality it is not. However, by specifying *every*, equation (6) excludes this possibility, since the cited x_0 and x must also be selected in such a way that the two function values be equal. Identification of the flat line between these two points along the curve would disqualify the curve from being strictly quasiconcave. On the other hand, when the qualifier *any* is used, the condition specified by equation (7) tests all the cases when the function values are equal, that is when f (x) = f (x_0). Evidently, no function value between these could have the same value if the curve is to be considered strictly quasiconcave. Thus, flat lines are properly excluded also under this approach.

Based on the minimum function value test, under-the-peak-parallel-chord tests can verify a quasiconcave curve for unimodality.[12] Similar to the case shown under the function value comparison test, the minimum value test will also lose meaning at the global maximum.

The differential-based approach.--A continuous and continuously differentiable function in all its variables f (x) is quasiconcave, if for any x_0 and x belonging to the set,

(8) $$f(x) \geq f(x_0) \Rightarrow \nabla f(x_0)(x - x_0) \geq 0$$

where ▿ f (x) (read as "del" f (x)) is the gradient vector of f (x). (For details, see Arrow and Enthoven, 1961, Ginsberg, 1973, and Huang and Crooke, 1997.) If there are n variables, the gradient vector is denoted as the column vector ▿ f (x) = $[f_1, f_2, \ldots, f_n]$. If the function has only one independent variable, the gradient vector of the function is simply its first derivative. (Some authors write $(x - x_0)^T$ instead of $(x - x_0)$ to emphasize that it is a row vector.) Thus, a column vector, the gradient, is multiplied by a row vector of equal elements to obtain a scalar. The function is strictly quasiconcave if for any x_0 , x , belonging to the set

(9) $$f(x) \geq f(x_0) \Rightarrow \nabla f(x_0)(x - x_0) > 0$$

The differential-based definition easily accommodates ascension under either the quasiconcave or the strictly quasiconcave definition. Difference in the independent variables is multiplied by a first derivative which is either positive or zero. To account for the descending leg (where the first derivative is negative), the test accommodates the definition by subtracting the measure of the larger point in the domain from the smaller one. Thus, two negative numbers (or one negative number and zero, in case of a flat segment) are multiplied to obtain a positive number (or zero, in case of a flat segment). A convenient way of standardizing the determination of signs is to mark the value of the independent variable at the peak, for example as x_1. All differences in the domain will be positive "left" from this reference point and all differences will be negative "right" from it. The definition properly

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[12] The "under-the-peak-chord" test may have played a role in eliminating the use of the maximum version of the minimum function value test. In some older texts, equation (5) had an alternative version: $f[tx + (1-t)x_0] \leq \max[f(x), f(x_0)]$. When $f(x) = f(x_0)$, the test based on this definition fails to recognize the peak above the under-the-peak chord formed by the line that connects the two function values. See, for example, Avriel, 1976.

rejects valleys, because any ascension linked to the descending leg would fail the test by virtue of multiplying a negative domain difference (negative since it is located "right" to x_1) with a positive derivative. A new convention to determine signs is required for under-the-peak-chord tests, since under-the-peak chords extend to both sides of x_1. Considering the length of the chord to be positive when the curve ascends and to be negative when it descends seems appropriate.

If the function has only piece-wisely continuous derivatives, *every*, instead of *any*, should be used in the preambles to equations (8) and (9). In other words, the closest checking of the function is required, because its continuousness and continuous differentiability are no longer present to exclude impermissible (that is, valley or pit-forming) twists and turns along the curve. If the function is piecewise differentiable, but the differential is zero everywhere, this test is inapplicable.

As in the previous two tests, the differential-based approach will also become meaningless at the global maximum, where $\nabla f(x_0)$ will be zero in equation (9).

Quasiconcavity in Set-Theoric Terms

The definition of quasiconcavity in set-theoric terms plays an important role in describing equilibrium conditions. The definition centers around the concept of the uppercontour set, which, in turn, is defined as $V = \{x \mid f(x) \geq \alpha\}$. For example, if f (x) is the utility function, then $W = \{x \mid f(x) = \alpha\}$ is the indifference curve (Takayama, 1994). To make the level "α" more comparable with actual choice, that is, with the set of commodities included in choice vectors, the uppercontour set may also be written as

$V = \{x \mid f(x) \geq f(x^*)\}$. When expressed in such a way, the uppercontour set may also be called the no-worse-than-x* set (Cornes, 1994). The uppercontour set of function "f " associated with a constant "α" is usually denoted as $U(f, \alpha)$.

The commodity choice x* may be regarded as an arbitrary consumption vector that the utility function "f " would assign to the appropriate indifference curve. Since f (x*) is a constant, choices representing the same level of satisfaction as x* would be on the same indifference curve. Larger levels of consumption in all goods would lie on higher indifference curves, positioned in a northeasterly direction from f (x*), as may be seen in any standard microeconomics textbook. Thus, the uppercontour set of an indifference curve may also be defined as the loci of all commodity combinations from which the consumer derives a level of satisfaction identical to, or greater than, the level associated with the indifference curve in question.

Concerning quasiconcavity, the following definition applies. A function f (x) is quasiconcave if the uppercontour set associated with every point in it is convex. A set is convex if it contains all the possible linear combinations of its members. This definition allows for linear segments in the boundary. Linear combinations between two boundary points are also in the boundary. However, the location of the linear combinations may be further restricted by the requirement that they lie in the interior of the set. In such a case, linear combinations cannot be on the boundary, because it curves. Such a set is called strictly convex. A function f (x) is strictly quasiconcave if its uppercontour set associated with every point "x" is strictly convex. This explains the exclusion of all straight lines, regarding whether they are ascending, flat, or descending, from the admissible shapes of a strictly quasiconcave curve. The requirement in economic models (including general equilibrium models) that the preferences must be "convex" or "strictly convex" refers precisely to this definition. Evidently, only strict quasiconcavity (strict convexity of the associated uppercontour set) allows for a unique, global optimal solution when such solution is sought through fitting the budget line to the highest possible curve in the indifference map (Cornes, 1994).

Texts sometimes do not clarify whether the uppercontour set to which they refer is a level set, determined by a truncating surface, or whether it is the space above the truncating surface. The difference, which is crucial, may be demonstrated easily with the help of the 2-product, 3-dimensional model where each product lies along a horizontal axis and utility or production is measured along the vertical axis. The result is the familiar, 3-dimensional, bell-shaped hillock. Uppercontour level sets that are formed from such 3-dimensional spaces by slicing them horizontally may be convex (strictly convex) even if the space above the horizontal slice is not convex (strictly convex). For example, if the total product surface is truncated below the inflection plane (that is, where returns are increasing), the uppercontour set in terms of the space above the truncating plane will not be convex. A chord may be drawn between a point that lies below the inflection surface and another point that lies above it (that is, where the returns are decreasing). This chord will be left to the inflection surface and outside of the 3-dimensional space determined by the truncation. Consequently, linear combinations between these two points will also lie outside the uppercontour set, indicating its lack of convexity. (For details, see Koopmans, 1957.[13]) Generally, if the text does not specify the nature of the uppercontour set, it is likely that it refers to level sets. Some authors use the expression "upper level sets" to make clear that they are talking about uppercontour level sets.

A crucial precondition for the applicability of the uppercontour set to determine quasiconcavity and/or strict quasiconcavity is that the function "f" that creates it must be continuous (Elkin, 1968).[14]

Quasiconcavity in Algebraic Terms

The algebraic definition of quasiconcavity uses the bordered Hessian matrix, that is, the second derivatives bordered by the first derivatives.[15] Thus, the definition that follows assumes that the function in question is at least twice differentiable. A function is (weakly) quasiconcave if its bordered Hessian is a negative semidefinite at each point. It is strongly quasiconcave if its bordered Hessian is a negative definite at each point. For details, see Arrow and Enthoven,[16] 1961, Ginsberg, 1973, Takayama, 1994, and Huang and Crooke, 1997.[17] For an application of weak and strong quasiconcavity, see Barten and Bohm, 1982.

If a function is strongly quasiconcave, then it is strictly quasiconcave. Again the rule is not reversible. (For details and proof, see Ginsberg, 1973). Here too, the single-valued function $Y = - X^4$ may serve as an

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[13] Koopmans' work also contains interesting historical notes regarding research on this subject.

[14] Curves analyzed under the geometric definitions can be, but do not have to be, twice differentiable.

[15] Some algebraic definitions use eigenvalues to characterize quasiconcavity and related concepts. See references to this approach in Diewert, Avriel, and Zang, 1981.

[16] Arrow and Enthoven provided the original proof for this theorem. The Appendix presents a simplified interpretation of their proof. The authors used the expression "quasiconcavity" and "strict quasiconcavity" for the concepts that later were called "weak quasiconcavity" and "strong quasiconcavity."

[17] Proofs that strong concavity requires that the Hessian be a negative definite can be found in several mathematical economic textbooks. However, books and articles consulted for this paper all refer to Arrow and Enthoven, 1961, for the proof of the above-indicated relationship between quasiconcavity and the bordered Hessian. For background information on this subject, it may be useful to consult Appendix A, section V in Samuelson, P.A., *Foundations of Economic Analysis*, Cambridge University Press, 1948.

example for the rule's irreversibility. This function is not only strictly concave, but also strictly quasiconcave. At any point in the domain, the generated uppercontour set is strictly convex. However, for the reason seen before, it does not have a negative definite at each point. The value of the second derivative will be zero at the point where the curve touches the horizontal axis. Thus, the equivalent of the bordered Hessian in this single-variable case is only a negative semidefinite, disqualifying the function from being strongly quasiconcave.

As mentioned before, some authors call strictly quasiconcave functions "strongly quasiconcave" (Avriel et al, 1988). This is evidently incorrect, since, as the above example shows, there are strictly quasiconcave functions that are not strongly quasiconcave. The two definitions are not interchangeable. Moreover, some authors call weak quasiconcavity what others call quasiconcavity (Beattie and Taylor, 1985) to distinguish it from strict quasiconcavity. That is, "weak" quasiconcavity is used both to designate quasiconcavity defined under a geometric definition and as the opposite of strong quasiconcavity. Again, the two definitions do not cover identical functions.

Operations with Quasiconcave Functions

Any monotonic nondecreasing function of a quasiconcave function is quasiconcave. In other words, any monotonic nondecreasing transformation of a quasiconcave function is quasiconcave. For example, if q = f (x) is a quasiconcave production function, then multiplying every "q" with another quasiconcave function, for example, ln q, will also be quasiconcave. That is, y = q . ln q will also be quasiconcave. Such monotonic transformations also preserve strict quasiconcavity.[18]

In contrast, there is no guarantee that the sum of quasiconcave functions is quasiconcave. For example, adding twice its negative value to any quasiconcave function "f " results in f - 2 f = - f, which, by definition is quasiconvex. Or, consider the functions sin [xy] and cos[x]. When $x \in [0, \Pi/4]$ and $y \in [0, \Pi/4]$, both functions are quasiconcave. However, adding them up violates quasiconcavity, because the new curve's descending leg will dip below the starting point of its ascending leg. It is also possible that the sum of two nonquasiconcave functions

[18] In general, a transformation "T" is characterized as nondecreasing monotonic, if $x_1 \rangle x_0$ implies $f(x_1) \rangle f (x_0)$. In the above example, $q_1 \rangle q_0$ implies $q_1 \ln q_1 \rangle q_0 \ln q_0$.

adds to a quasiconcave function. (For details on operations with quasiconcave functions, see Arrow and Enthoven, 1961, Crouzeix, 1987, and Avriel et al, 1988.)

Closely-related, Partially Overlapping Concepts

Economic literature also refers to "explicit," "semistrict," and "pseudo" concavity. As mentioned in the introduction, these concepts, along with concavity and quasiconcavity, come under the common heading of generalized concavity. Excluding certain small classes of exception that are not seen in general economic texts, these concepts form a fairly transparent order (Thomson and Parke, 1973). Nevertheless, the overlaps and quasi overlaps represent a non-negligible source of confusion. They are a consequence of independent research in separate fields, such as nonlinear programming, consumer theory, and general equilibrium modeling.

***Explicit quasiconcavity or semistrict quasiconcavity*.--**Takayama defines explicit quasiconcavity in the same way as Avriel et al define semistrict quasiconcavity. Hence, the two concepts are treated as identical. Nevertheless, this section will indicate which concept an author defines.

According to Takayama, 1994, a real-valued function f (x) defined over a convex set X in R^n is explicitly quasiconcave if (1) it is quasiconcave and (2) it satisfies the following condition:

(10) $$f(x) > f(x_0) \Rightarrow f[tx + (1-t)x_0] > f(x_0)$$

This definition corresponds to the function value comparison method. Except for the strict inequality in the precondition, it is identical to equation (4). Avriel et al, 1988, define semistrict quasiconcavity exactly the same way.[19]

Huang and Crooke, 1997, define explicit quasiconcavity, using the minimum function value test:

[19] Diewert, Avriel, and Zang (1981) confirm the identity of the definitions for semistrict quasiconcavity and explicit quasiconcavity.

(11) $$f[tx + (1-t)x_0] > \min[f(x_0), f(x)]$$

This equation differs from equation (6), the definition of strict quasiconcavity under the minimum function value test, only in preconditions. Whereas equation (6) requires $x_0 \neq x$, the definition in equation (11) requires $f(x_0) \neq f(x)$.

Ginsberg, 1973, uses the differential-based approach to define semistrictly quasiconcave functions for any x_0 and x belonging to the domain set as:

(12) $$f(x) > f(x_0) \Rightarrow \nabla f(x_0)(x - x_0) > 0$$

This equation differs from equation (9), which defines strict quasiconcavity with the differential- based approach, by using $f(x) \rangle f(x_0)$ instead of $f(x) \geq f(x_0)$ as a precondition.

In essence, the definitions of the explicit/semistrict quasiconcavity complete the requirement of quasiconcavity with conditions that guarantee strict quasiconcavity. Splitting the definition of strict quasiconcavity into two separate criteria has created an interesting maze of technical exceptions from their joint occurrence. For example, a function consisting of a straight line parallel with the horizontal axis, interrupted by a single point below it on the vertical axis, is evidently not strictly increasing, yet it passes the criteria specified under both equations (10) and (11).[20] (The differential-based approach is not applicable, because the "function" is only piece-wisely differentiable and the differential is zero everywhere.) As remarked by Avriel et al, 1988, proved by Ginsberg, 1973, and demonstrated by Huang and Crooke, 1997, there are entire families of explicitly/semistrictly quasiconcave nondifferentiable functions that are no longer quasiconcave or strictly quasiconcave.[21] Nevertheless, explicit/semistrict quasiconcavity appears to be useful as an intermediary concept in generalized curvature analysis.

[20] See more on this "curve" under the section entitled "Surprises and Doubts: Two Examples."

[21] The same example as the one cited above, with the horizontal line interrupted by a single point on the vertical axis, will illustrate this point, as well. See under "Surprises and Doubts: Two Examples."

***Pseudoconcavity*.--**The concept is used mainly in connection with continuous and at least twice-continuously differentiable functions. Pseudoconcavity, which, like quasiconcavity, also has a “strict” version, is defined in two steps. The first step deals with the curve in general and the second step with the neighborhood of the maximum, in particular. Equation (13) show the first part of the definition for pseudoconcavity and strict pseudoconcavity, respectively:

(13) $$f(x) > f(x_0) \Rightarrow \nabla f(x_0)(x - x_0) > 0 \quad ; \quad f(x) \geq f(x_0) \Rightarrow \nabla f(x_0)(x - x_0) > 0$$

The general condition for pseudoconcavity is identical to the condition specified for explicit/semistrict quasiconcavity, equation (12), and the general condition for strict pseudoconcavity is identical to the condition specified for strict quasiconcavity, equation (9). (Equation (13) merely repeats the two equations cited.)

The second step of the definition, dealing with the neighborhood of the maximum, specifies maximum conditions as $\nabla f(x) = 0$ and $\nabla^2 f(x) \leq 0$. The second expression is equivalent to the requirement that the Hessian be a negative semidefinite. The two conditions together are standard for local maxima. The difference between a pseudoconcave and a strictly pseudoconcave function with respect to the conditions specified in the second step of the definition is that the first has a local maximum and the second a strict local maximum.[22] However, since the function is unimodal, the local maximum is global maximum for both cases. (See Diewert et al, 1981, Avriel et al, 1988, and Crouzeix, 1987). It is important to note that the neighborhood of the maximum value is sufficiently large to reveal curvature characteristics. That is, in the analysis of pseudoconcave functions, the “neighborhood” that econ-math text books often characterize with the topological concept of “open ball” is not suitable. Open ball neighborhoods usually are of an infinitesimal magnitude.

Major works on generalized concavity begin the discussion of pseudoconcavity with references to O.L. Mangasarian, whose work in the mid-1960s is credited with initiating the concept’s modern usage. (See, for example, Diewert, et al, 1981 and Avriel et al 1988.) Mangasarian did not use the distinguishing category of strict

[22] A function that has a local maximum at point c if $f(c) \geq f(x)$ in a neighborhood sufficiently close to c. The local maximum is strict, if in the same neighborhood $f(c) \rangle f(x)$.

pseudoconcavity. However, from his 1969 work (Mangasarian, 1969) it appears that his definition (that is, the first part of defining this type of function) corresponds to the later developed category of strict pseudoconcavity:

(14) $$f(x) > f(x_0) \Rightarrow \nabla f(x_0)(x - x_0) > 0; \quad 0 \geq \nabla f(x_0)(x - x_0) \Rightarrow f(x) \leq f(x_0)$$

The first condition of this two-part definition states that the gradient vector (first derivative of a single-variable function) times a distance in the domain will be positive when the function increases. However, when the function decreases, its gradient vector times a distance in the domain will be negative because a negative and a positive number are multiplied together. The equality sign, $f(x) = f(x_0)$, is approached infinitesimally as x approaches x_0 at the maximum. This makes the condition specified in equation (14) similar to the requirement specified under strict quasiconcavity, that is , the second part of equation (13).

Mangasarian's definition, just as the strict quasiconcavity-based definition of strict pseudoconcavity, mandates that the function ascend, peak, and descend. This requirement is inherent in the equality of $f(x)$ and $f(x_0)$, a condition that can be achieved along a continuously differentiable curve only if the two points on the function can be found on either side of the maximum point. That is, if they could be connected with an under-the-peak chord. Similar to the geometric definitions of quasiconcavity and strict quasiconcavity, equation (14) also breaks down at the maximum. (Since $x \neq x_0$ by definition in equation (14), when the curve peaks and the gradient vector is zero, a positive number is multiplied by zero, violating the equation's requirement.) Hence the requirement to have a second step in defining pseudoconcavity.

Development of the concept of pseudoconcavity has grown out of the need to strengthen the conditions of global maximum. The concept of strong pseudoconcavity that is built on this need results in a more exacting extremum property than the one represented by strict pseudoconcavity. Strongly pseudoconcave functions are defined as strictly pseudoconcave functions that fulfill an additional criterion that states that the Hessian of the second derivatives is a negative definite in the neighborhood where all the elements of the gradient vector are zero. This makes strong pseudoconcavity in the neighborhood of the global maximum point very similar to strong concavity, that is, more exacting than strong quasiconcavity.[23] Sometimes authors make further restrictions to disqualify strictly pseudoconcave functions with very flat curvatures around the maximum from being also strongly pseudoconcave. Such a restriction may be that the curve is required to ascend and descend at least at a quadratic rate around the maximum (Avriel et al, 1988).

Thinking of pseudoconcave curves as two curves with different characteristics put together may be a useful simplification in distinguishing pseudoconcave functions from other types of curves (surfaces) used in generalized concavity. A pseudoconcave function is like an explicitly/semistrictly quasiconcave curve (surface) with a dome containing the maximum point placed on it. This dome could be virtually flat. The strictly pseudoconcave curve is like a strictly quasiconcave curve (surface) with a dome containing the maximum on top of it. However, this dome must have a discernible ascent and descent. It cannot be virtually flat. A strong pseudoconcave curve (surface) is like a strictly pseudoconcave curve (surface) with an even more pronounced ascent and descent around the maximum. This imagery may also explain the name pseudoconcavity: A concave top on the rest of the curve that may not be concave. For applications of strong pseudoconcavity in consumer theory, see Blackorby and Diewert, 1979, and Donaldson and Eaton, 1981.

Surprises and Doubts: Two Examples

The application of the criteria for shapes created by smooth, differentiable functions, such as the equation of the normal curve, is straightforward and without controversy. But rule-breaking equations and associated shapes are common in generalized curvature analysis. The following examples demonstrate this assertion.

Example 1: The presence of quasiconcavity.

Consider equation (15) and figure (1).

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[23] Strong quasiconcavity is formulated in terms of the bordered Hessian, whereas the conditions of strong pseudoconcavity are formulated in terms of the Hessian. At the global maximum, the directional derivatives must be uniformly zero in the cases of strong concavity and strong pseudoconcavity. However, when the function is strongly quasiconcave, some of the first derivatives bordering the submatrix of second derivatives may not be zeros. For details, see Crouzeix, 1987, and Donaldson and Eaton, 1981. Nevertheless, for continuous, twice continuously differentiable, perfectly well-behaved functions this distinction is superfluous. Therefore, economic literature often equates strong quasiconcavity with strong pseudoconcavity (Avriel et al, 1988).

(15) $$y = 2x + 2,\ if\ x \in [-1;0];\ y = -2x + 2,\ if\ x = \in [1;0]$$

The shape of this continuous, but only piece-wisely differentiable function is apparently quasiconcave but not strictly quasiconcave. The upper contour sets, U(y, 0), for example, will be convex when strict quasiconcavity requires that they be strictly convex. (Linear combinations along the line determined by the function will remain on the same line.) However, the shape fulfills the requirements for strict quasiconcavity under the function value comparison and the minimum function value tests (equations (4), (6), and (7)). Since the "curve" is only piece-wisely differentiable, one needs to consider the two lines separately when applying the differential-based approach. Under these circumstances, equation (9) will also indicate strict quasiconcavity.

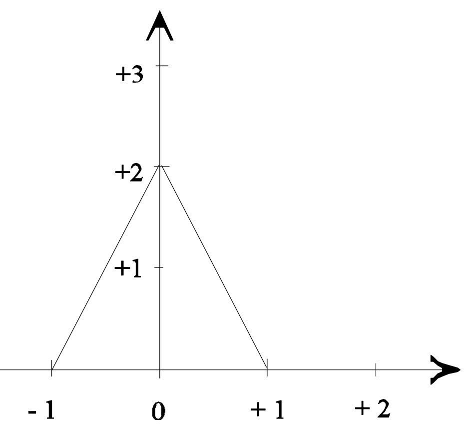


Figure 1

Example 2: The absence of quasiconcavity.

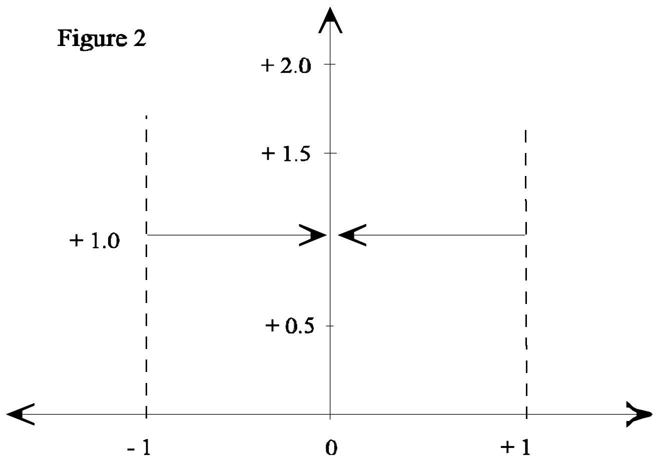
Consider equation (16) and figure (2).[24]

[24] The example was taken from Avriel et al, 1988, p. 81, where it was used to illustrate the lack of perfect overlap between the categories of quasiconcavity and explicit/semistrict quasiconcavity. This point will be reiterated after using the example to show rejection of quasiconcavity.

(16) $$y = 1 \;\; if \;\; x \in [-1;1], x \neq 0 \; ; \; y = 1/2 \;\; if \;\; x = 0$$

.

Figure 2



In words, the value of this piecewise, continuous function ("y") is 1 between the points of -1 and 1 in the domain, and it is ½ at zero. The set-theoretic test rejects quasiconcavity, because a nonconvex uppercontour set may be found. Indeed, U(f, 1), which includes only the line y = 1, is not convex. (Since ½ is below the horizontal line y = 1, linear combinations between points above that line and ½ will be outside the uppercontour set.)

The "curve" also fails the geometric tests. In the function value comparison test, equation (3), choosing -.5 for x_0 and 0.5 for x; 0.5 for t , hence 0.5 for (1-t), yields f [tx + (1-t) x_0] = 0. To pass the test, this value would have to be greater than or equal to the value of the function at f (x_0). But the value of "y" at -.5 is 1, greater than zero.

The result of the minimum function value test will be the same. Based on equation (5), the min [f (x), f (x_0)] will be ½. (The choice is between 1 and ½ .) Using the same weights as under the function value comparison test, zero may be reproduced on the left-hand side of equation (5). Zero is evidently smaller than ½ when these values should be ½ or greater. The differential-based approach is not applicable, because the derivative of the function is zero everywhere.

Although the "curve" is clearly not quasiconcave, it is explicitly or semistrictly quasiconcave. The reader can easily verify this by using equations (10) and (11). Thus, this example, as shown in Avriel et al, 1988, violates the general rule that explicitly or semistrictly quasiconcave functions form a subset of strictly quasiconcave and, hence, quasiconcave functions.

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The imperfection in the definitions of curvature characteristics, and in the tests implied by them, cannot be eradicated. The classification of variants of general concavity is like a grid superimposed on the universe of eligible functions. Since the grid consists of countable cells, but the number of eligible functions is infinite, exceptions requiring modifications in the grid will always be found. This thought can be expressed in another way: The match of general and permissive definitions with concrete and often irregular applications provides an inexhaustible source of exceptions. The constancy of surprises keeps doubts alive about the blanket applicability of the rules. However, as will be shown, the nature of applications of quasiconcavity in economics is such that these doubts do not diminish the usefulness of the concept.

The Minimum Order and the Wild Beyond

As has been seen from references thus far, some of the categories of generalized concavity completely include some others. All strictly quasiconcave functions are also quasiconcave, because they satisfy stricter criteria than required for quasiconcavity. Similarly, all strictly pseudoconcave functions are also pseudoconcave, and all strictly concave functions are also concave.

These concepts may be easily linked by the minimum requirement for concavity when the curves stand for continuous and continuously differentiable functions. This criterion links the above-mentioned concepts in such a way as to move from the one that has the least extensive minimum demand for the presence of concavity (quasiconcavity) to the one that has the most extensive minimum demand (strict concavity). Quasiconcavity qualifies as the weakest, most permissive among these curves. As mentioned before, the simplest one-variable case may contain convex and straight line segments, it may have a flat maximum neighborhood with an infinite number of maxima. The strictly quasiconcave curve moves closer to strict concavity, because it eliminates straight line segments and requires a unique maximum. However, the maximum could be far from the ideal seen in strictly concave functions. For instance, ascendence to the maximum point may proceed along a convex line segment and the descendence may move along a concave one. The pseudoconcave curve must have a maximum neighborhood that reminds one of concavity, although, in line with the rules that apply to concavity, it may still contain linear segments, including a "flat top." The strictly pseudoconcave curve resembles strictly concave curves in the maximum neighborhood. The concave curve is similar in its entirety to the maximum neighborhood of a pseudoconcave curve. Finally, the strictly concave curve is similar in its entirely to the maximum area of the pseudoconcave curve. This simple relationship among the categories mentioned may be called the "minimum order."

J. Ponstein, whose work has had a significant impact on the generalized concavity literature, has created the most quoted formal linkage among these concepts in the generalized concavity literature.[25] According to Ponstein (1967), strict concavity implies concavity, concavity implies pseudoconcavity, pseudoconcavity implies strict quasiconcavity, and strict quasiconcavity implies quasiconcavity. The expression "A implies B" may be interpreted as "A has characteristics that may be applicable to B.[26]" Using the information associated with the verb "implies" as the test for linking concepts, Ponstein developed new categories of quasiconcavity. In other words, his test results in the intersection among the concepts mentioned, creating new ones, such as X-concavity.

Some classes of functions satisfy all definitions in the minimum order over closed intervals. For example,

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[25] The Ponstein article analyzes curvature characteristics from the point of view of convexity. His statements and examples used here are contrapositive, that is, they have been converted to make them applicable to concavity.

[26] Ponstein arrives at this method by first demonstrating that violation of the requirements for a given category also violates the next stronger one. For example, he demonstrates that violation of monotonic ascendence and descendence in strictly quasiconcave functions will result in the inapplicability of criteria designed to characterize the next strongest category, pseudoconcavity. From this follows that two adjacent categories must have some requirements that are common to both. Going from the stronger category to the weaker one is likely to be accompanied by some loss of rigor according to one or more criteria that characterize both. Going from the weaker to the stronger category may require the formulation of criteria that the weaker one did not have. Therefore, in using the passive logical link inherent in the verb "imply," it is better to go from the stronger to the weaker. For example, as Ponstein stated, "pseudoconcavity implies strict quasiconcavity."

when x is between 0 and +1, - x^2 is quasiconcave, strictly quasiconcave, pseudoconcave, strictly pseudoconcave, concave, and strictly concave all at once. Other classes of functions satisfy only one or more, but not all the five categories. (See Ponstein, 1967.)

Transparency for the nonspecialist suffers when explicit/semistrict concavity, strong concavity, strong quasiconcavity, strict and strong pseudoconcavity are combined with the six kinds of concavity that form the minimum order. These additions have created partial overlaps, especially when functions are not continuously differentiable. Research has unveiled many interesting and useful relationships between the concepts included in the minimum order and those that are not. For example, differentiable, strictly concave functions pass the test of strict pseudoconcavity, thereby satisfying the criterion for explicit (or semistrict) quasiconcavity (Thomson and Parke, 1973). For quadratic functions, which are frequently used in descriptive microeconomics and economic models, quasiconcave and semistrictly quasiconcave functions are identical, as are strictly and strongly pseudoconcave functions (Avriel et al, 1988). The possibilities of proving and disproving equivalence between categories of general concavity for specific classes of functions are virtually inexhaustible.

A concern for the general reader of the modeling literature is that different concepts are often used as synonyms (Avriel et al, 1988). For example, strong quasiconcavity and pseudo quasiconcavity are often used interchangeably. Moreover, some call strictly quasiconcave functions “strongly quasiconcave," thereby inadvertently equating strict quasiconcavity and strong pseudoconcavity, which, as mentioned in the section entitled “Concavity and Quasiconcavity,” is a half-truth. The strongly quasiconcave function is strictly quasiconcave, but the rule has no blanket validity when reversed. Unusual, rarely-used names, particular to an author, also surface in the literature (Ponstein, 1967).

A saving grace amidst this potential for confusion is that books and articles that do not have general concavity as their main subject would rarely venture beyond the concepts included in the minimal order. On the other hand, works that do venture beyond it tend to delimit and define their own systems for the purpose of exposition in their particular analytical contexts. This may be seen in the works of Blackorby and Diewert, 1979, and Donaldson and Eaton, 1981, that use strong pseudoconcavity.

A further circumstance that extenuates the potential harm from confusion is that once a function is completely determined, its classification into a category of generalized concavity becomes relatively easy and, perhaps, of secondary importance. Moreover, as will be shown in the next two sections, the concept's main use is to reference broadly understood information about functions and to invoke visual images.

The Usefulness of Quasiconcavity

Quasiconcavity helps formulate theorems and establish quantitative relationships parsimoniously. The usefulness of the concept may be summarized under six points:

(1) A quasiconcave function can express increasing, constant, zero, or decreasing returns to scale. Hence, quasiconcavity is the most convenient reference to describe a function that incorporates at least two of these characteristics. The same is true for strict quasiconcavity, which can describe all of the above types of return to scale, excepting zero return.

The Cobb-Douglas function is one of the most frequently used functions in economic theory and modeling:

(17) $$f(q_1, q_2, \ldots, q_n) = A(q_1)^{\alpha_1}(q_2)^{\alpha_2}\ldots(q_n)^{\alpha_n}$$

This function is quasiconcave and it illustrates well the concept's adaptability to express various returns to scale. It shows increasing returns to scale when the exponents add to more than 1, constant returns to scale when they add to 1, decreasing returns to scale when they add to less than 1, but more than zero; zero returns to scale when they add to zero, and negative returns when they add to a negative number. It is useful to remember that when the Cobb-Douglas shows constant returns to scale, it is no longer strictly concave, only concave.[27] As

[27] The following is a simple proof that strict concavity disallows linear homogeneity. It closely follows

$$f(0.5x_0 + 0.5x) > 0.5f(x_0) + 0.5f(x)$$

mentioned earlier, the traditional production function is the most frequently-cited example of strict quasiconcavity.

(2) Ascending strictly quasiconcave functions, such as the generic utility function known from microeconomic textbooks, generate level sets (indifference curves in the case of utility functions) that are perfectly convex to the origin. The indifference curve will show flawlessly diminishing marginal rates of substitution. Similarly, descending strictly quasiconcave functions, such as certain cost surfaces, [28] generate concave-to-the-origin production possibility or transformation curves. Such curves will show flawlessly increasing marginal rates of transformation.[29] Both the curves that generate convex-to-the-origin and concave-to-the-origin shapes may come from surfaces that imply varying returns to scale. In both instances, strictly quasiconcave functions generate strictly convex uppercontour (level) sets. This is a vital requirement in optimization problems where the budget line (in the simplest two-dimensional case) can touch these contours at one single point only.[30]

(3) For strictly quasiconcave functions, the local maximum is also global maximum. Thus, the strictly quasiconcave function has the same property as the strictly concave function, but with the added advantage of

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Takayama (Takayama, 1995, endnote 11, page 132). Choosing t = 0.5 in equation (2), strict concavity requires:

If "f" were linearly homogeneous, then a scalar α could be found, such that $x = \alpha x_0$. Substituting this expression into the above equation, and rearranging it with a presumption of linear homogeneity, results in an identity:

$$(1 + \alpha) f(x_0) = (1 + \alpha) f(x_0)$$

This contradicts strict concavity. The proof that linear homogeneity also contradicts strict convexity is analogous.

[28] For the simplest case of two products and a single allocable factor, such curves may be represented by the implicit function, $F(q_1, q_2, Y) = 0$. Let the two products, q_1 and q_2, be represented along the two horizontal axes, and the amount of resource not used, Y, along the vertical axis. At zero production, all Y is unused, and, therefore, the surface is at its maximum. As production increases to the maximum, using up the entirety of the resource, the surface will decelerate to zero. The deceleration implies increasing marginal costs. The curves generated by this process will be the familiar family of product transformation or production possibility curves.

[29] Arrow and Enthoven, 1961, defined quasiconcave functions as ones that generate level sets bearing the characteristic of diminishing marginal rates of substitution if the function ascends or the characteristic of increasing marginal rates of transformation if the function descends.

[30] MATHEMATICA 3.0 can generate indifference or transformation curves from specific functions. See Huang and Crooke, 1997. A simple function that generates convex-to-the origin level sets may be V=sin (x) + + (sin (y) when both x and y are between 0 and Π/2. The level curves will form an indifference map. In contrast, the function W=cos (x) + cos (y) , also when both x and y are between 0 and Π/2, will generate a map of concave-to-the-origin curves. Note that both V and W show varying rates of return along their respective paths.

allowing the curve more variation. Combining this characteristic with the one described under point (1), strict quasiconcavity allows a function to reach and pass its global maximum in a variety of ways, while still yielding indifference curves/isoquants strictly convex to the origin.

(4) Quasiconcavity allows characterization of the time paths of variables in dynamic models. Orbit diagrams generated by dynamic models, showing a historically driven or projected upward drift are more likely to fit the definition of quasiconcavity or strict quasiconcavity than concavity or strict concavity.

(5) Whether the curve is quasiconcave or strictly quasiconcave, the descending segment cannot dip below the initial value on the ascending segment. Since this requirement can accommodate a curve that starts from the horizontal axis and returns there, the concept of quasiconcavity is also often used to characterize unimodal probability density functions (such as the normal curve or the gamma function).

(6) Quasiconcave functions combine minimum restrictions regarding curvature configurations while maintaining algebraic characteristics required for the numerical realization of optimization models.[31] In their seminal, 1961 article, Arrow and Enthoven defined a strand of nonlinear programming, called *quasiconcave programming* as a constrained maximization problem where the maximand and the constraints are quasiconcave (Arrow and Enthoven, 1961). Quasiconcave programming extends the applicability of the concepts and methods of nonlinear maximization from more or less restrictive shapes (for example concave maximands and constraints) to the least restrictive functions, thereby enlarging significantly the type and number of functions that may be used to model producer and consumer behavior. The proofs provided by Arrow and Enthoven could be used to define quasiconvex programming and could be applied to models featuring minimization. (For a description of nonlinear programming, in general, see Intriligator, 1987.)

Quasiconcavity in Specific Contexts

The quotes and references included in this section have been collected from publications that do not

[31] One may even attempt to describe quasiconcavity as the concept that combines the maximum relaxation of curvature configurations with the maintenance of algebraic properties consistent with the basic conditions and procedures of optimization.

expressly deal with the subject of quasiconcavity or generalized concavity. They show the use of quasiconcavity in special analytical contexts.

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Increasing returns to scale can be represented by strictly convex functions and decreasing returns to scale can be represented by strictly concave functions; however, no separate term can tie a specific curvature characteristic to constant returns to scale. Therefore, authors may invoke quasiconcavity exclusively to express constant returns to scale. For example,

> "...the CES function satisfies the condition for quasiconcavity..." (Huang and Crooke, 1997, p. 389.)

The CES function is linearly homogeneous and, as such, it undoubtedly satisfies the definition of quasiconcavity. Of course, convex or concave functions would also satisfy the definition. Or,

> "Sometimes we relax strict concavity and replace it by strict quasiconcavity (e.g., when utility is homogeneous of degree one) or by concavity (when we approximate utility by linear segments)." (Ginsburgh and Keyzer, 1997, p. 63.)

The next quote uses the concept of strict quasiconcavity to imply that the function is always increasing but could assume any shape:

> "... log h(x) is strictly concave. Since e^{x} is strictly increasing, $e^{\log h(x)} = h(x)$ is strictly quasiconcave." (Fusselman and Mirman in Becker et al, 1993, p. 381.)

By using the increasing values of the function log h(x) as exponents of the Napierian number, the authors perform a strictly convex transformation on a strictly concave function. Thus, they combine two contrary influences on the function "h(x)." The stronger one will dominate or they may neutralize one another, resulting in a curvature that implies constant returns to scale. The outcome will depend entirely on the characteristics of h(x). For example, if "h" happens to be raising the x to the power of 0.01, that is, $h(x) = x^{0.01}$, the function will be strictly concave between the values 1 and 50 and, by turning into a straight line, it would approach constant returns to scale for domain values over 50.[32]

Sometimes quasiconcavity is presented as a premise, rather than a conclusion:

> "For example, if f is a utility function that has convex upper preference set, then f is quasiconcave." (Green and Heller, 1982.)

Ginsburgh and Keyzer use the concept to show the uniqueness of an independent variable. They define the input demand function as $v = v(p_v, q)$, where v is the input demand, p_v is the input price and q is the output level and then they state:

> "Uniqueness of $v(p_v, q)$ follows from the strict quasiconcavity of f (v)." (Ginsburgh and Keyzer, 1997, p. 49.)

Non-uniqueness, that is, multiple values for a given input "v," are excluded under the definition of quasiconcavity by specifying that, in this application, $v_0 \neq v$. In other words, vertical segments are excluded. There is no input value to which both a higher and a lower output value would pertain.

The following quote, taken also from Ginsburgh and Keyzer, is a particularly good example of using quasiconcavity to help visualize the relationship of curves in optimization problems. After defining the input requirement set as $V(q) = \{v \mid f(v) \geq q\}$, that is, the set of all input bundles that produce at least "q" units of output,[33] they continue:

> "If, for example, the production function f (v) is strictly quasi-concave, ensuring convexity of V (q), one could use cost functions, and most of the properties of the case where f (v) is strictly concave will carry over...If f (v) is quasi-concave, $V(q) = \{v \mid f(v) \geq q\}$ is a convex set, by definition of quasi-concavity. V (q) is precisely the input requirements set." (Ginsburgh and Keyzer, 1997, p. 53.)

The strictly quasiconcave production function ensures the convexity of the input requirement set, because the uppercontour set will be the area above one specific isoquant that stands for the production level "q." (The

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[32] Experiment performed with MATHEMATICA 3.0.

[33] For a description of the input requirements, see elsewhere in Ginsburgh and Keyzer, 1997, as well as in Varian, 1992.

authors could have said “strict convexity of V (q)” instead of the “convexity of V (q),” to set off the case of strict quasiconcavity-strict convexity from the case of quasiconcavity-convexity, mentioned in the second sentence.) Indeed, this makes possible the use of cost functions in optimization programs.[34] The expression “most of the properties of the case where f (v) is strictly concave will carry over” refers to the fundamental similarities between strictly quasiconcave and strictly concave functions. These are, as mentioned earlier, the uniqueness of the maximum and the strictly convex-to the origin characteristic of isoquants, and, hence isocost curves associated with the function. This compatibility makes equilibrium conditions tractable.

The possibility of maximizing a utility or a production function is often expressed in terms of quasiconcavity, leaving out the intermediate step of creating level sets. For example, when Diewert says that quasiconcavity guarantees that non-convex isoquants cannot occur (Diewert, 1982, p. 545), he refers to slices of production surfaces. These are best visualized in the traditional 2-dimensional framework, when the two products are shown along the two axes.

The strict quasiconcavity of a function may be used as a condition for the differentiability of the function’s dual. For example,

> “...strict quasiconcavity of the utility function implies differentiability of the cost function, while differentiability of the utility functions implies strict quasiconcavity of the cost function.” (Deaton and Muellbauer, 1986, p. 51.)

Although these correspondences are relatively complex, they may be grasped without mathematical proofs or consumer-theoric jargon under the assumption that equilibrium exists. If the utility function is strictly quasiconcave, then it has strictly convex indifference curves. This means that the demand functions are single-valued functions of prices and income. Thus, when income is fixed, commodity demand functions contain only

[34] The convexity of the production set is a precondition for solving the optimization problem by defining the input requirements, and, therefore, the technology. The expression “one could use cost functions” refers to the general preference of modelers to optimize production by taking the cost function as given and deriving the technology that could have resulted in that cost function. This approach is called “dual” to the “primal” formulation of deriving the cost function from a choice of technologies. The requirement to provide explicit representations of various technological solutions makes the primal approach more difficult, hence, it is less favored. For details, see Ginsburgh and Keyzer, 1997, and Varian, 1992.

prices as independent variables and, by Shepard's lemma, may be considered the first derivatives of a linearly homogenous indirect cost function.[35] Hence, differentiability of the (indirect) cost function is a precondition for the existence of smooth (zero-homogenous) demand curves, which, in turn, is implied by the strict quasiconcavity of the utility function. The second correspondence may be interpreted as follows: Since the utility function is differentiable, a continuum of distinct marginal utilities exists. If prices are normalized on the unit simplex (that is, their sum is 1), the price of each commodity can assume any value between 0 and 1. Therefore, under an indicated presumption of equilibrium, prices will be found that satisfy the equality of marginal utility derived from a good divided by its price across the spectrum of goods considered in a model. This assures the existence of rational consumer behavior, which, in turn, requires the linear homogeneity of the indirect cost curve. As shown before, linear homogeneity cannot exist under strict concavity; it can exist only under strict quasiconcavity. For example, let the indirect cost function be Cobb-Douglas, as in equation (17). If one writes income (Y) in place of one "q" and prices (p_j) for the rest of the "q's", then this function will be linearly homogenous, as long as the exponents add to 1. Since such a function is constantly increasing but cannot be strictly concave, as shown earlier, it is strictly quasiconcave as claimed in the quote. [36]

Appendix

The Arrow-Enthoven Proof

In their milestone work, Arrow and Enthoven, 1961, proved that, for a function (at least twice differentiable) to be quasiconcave, it is both sufficient and necessary that the relevant bordered Hessian be a negative semidefinite. The bordered Hessian is formed by adding the row and column of the function's first

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[35] The indirect cost function is defined as the loci of minimized cost requirements to satisfy consumption at a given level of prices and utility. Some authors, including Deaton and Muellbauer, call the indirect cost function simply cost function (Deaton and Muellbauer, 1986).

[36] Deaton and Muellbauer illustrate the validity of their assertion by showing graphically that the information "encoded" into a strictly convex indifference curve, which had to come from a strictly quasiconcave utility function, maps into a smooth, and hence differentiable, (indirect) cost curve and vice versa. That is, a strictly convex isocost curve maps into a smooth utility curve. Their method of proof hinges on showing that flat segments or bends in the indifference curve cause "kinks," that is nondifferentiability, in the cost curve, and vice versa. See, Deaton and Muellbauer, 1986, pp. 46-50.

derivatives. The following is a heuristic interpretation of the Arrow-Enthoven proof.

The authors demonstrate the theorem for a 2-variable function that is presumed to be quasiconcave. They proceed to show the conditions under which this function can be equated with another, n-variable function that preserves the 2-variable function's presumed quasiconcavity. This procedure is convenient, because the 2-variable (3-dimensional) graphs and their attendant algebra are familiar, formally proven work tools for economists.

In terms of notation, the 2-variable function used is g (u , v). Since the function is presumed to be quasiconcave, its uppercontour (level) sets will be convex to the origin along the ascending leg and they will be concave to the origin along the descending leg. The slope of such level curves is the well-known

(18)

$$du / dv = -g_v / g_u$$

where the ratio g_v / g_u is the marginal rate of substitution (for the ascending leg) or the marginal rate of transformation (for the descending leg). The downward slope when sliding along the horizontal axis (v) is the equally well-known formula:

(19)

$$\frac{d}{dv}\left(\frac{g_v}{g_u}\right) = -\frac{1}{g_u^3}\left[g_u^2 g_{vv} - 2 g_u g_v g_{uv} + g_v^2 g_{uu}\right]$$

When the strictly quasiconcave function is increasing, the level curves cut surfaces that look like indifference curves, convex to the origin. The indifference curve has a diminishing negative slope along the horizontal axis. This is the case of diminishing marginal rates of substitution. Therefore, the second derivative of the indifference curve, shown by equation (19) must be positive. Since g_u is positive, the second derivative will be positive if the parenthetical expression is negative.

When the strictly quasiconcave function is decreasing, the level curves cut surfaces that look like transformation curves (resembling quarter circles centered to the origin), concave to the origin. The transformation curve has an increasingly negative slope along the horizontal axis. This is the case of increasing marginal rates of transformation. The second derivative of the transformation curve, equation (19), must be negative. Since g_u is negative, the second derivative will be negative, again, only if the parenthetical expression is negative.

In both cases, strict quasiconcavity may be relaxed by allowing the parenthetical expression in equation (19) to be zero. When the first derivative, the marginal rate of substitution or the marginal rate of transformation, does not change slope, linear segments occur in the curve and it becomes quasiconcave instead of strictly quasiconcave.

Thus, the presence of quasiconcavity hinges upon the negativity or zeroness of the parenthetical expression in equation (19). In the words of Arrow and Enthoven, "*The twice differentiable function* $g(u, v)$ *with* $g_u > 0$ *and* $g_v > 0$ *everywhere, or* $g_u < 0$ *and* $g_v < 0$ *everywhere, is quasiconcave if and only if* $g_u^2 g_{vv} - 2 g_u g_v g_{uv} + g_v^2 g_{uu} \leq 0$." (Arrow and Enthoven, 1961, p. 796.)

The parenthetical expression happens to be the bordered Hessian matrix shown in equation (20), taken by the negative sign.

(20)
$$|\overline{D}| = \begin{vmatrix} 0 & g_u & g_v \\ g_u & g_{uu} & g_{uv} \\ g_v & g_{vu} & g_{vv} \end{vmatrix}$$

For the parenthetical expression in equation (19) to be negative, the bordered determinant, shown in equation (20) must be positive. By specifying that the bordered determinant be a negative semidefinite, this requirement is fulfilled.

As mentioned earlier, the requirement of negative semidefiniteness is that the successive principal minors alternate their signs. For the general case of the n x n matrix, the r-th successive bordered Hessian is the (r+1)-th successive principal minor:

(21)
$$|\overline{D_1}| \leq 0, |\overline{D_2}| \geq 0, |\overline{D_3}| \leq 0, \ldots, (-1)^n |\overline{D_n}| \geq 0$$

The first bordered Hessian is always negative or zero and the last one is always positive or zero. (If n is odd, the determinant is negative and so is $(-1)^n$, making their product positive. If n is positive, then two positive number are multiplied.) To simplify the condition, each bordered Hessian is raised to the power of its sequence number, thus making them all non-negative: $(-1)^r |D_r| \geq 0$ is the condition of quasiconcavity.

The use of negative semidefinitness assured that the value of the determinant shown in equation (20) will be positive and, consequently, the parenthetical expression in equation (19) will be negative. The requirement that the determinant be a positive semidefinite could have achieved the same goal. The choice of negative semidefinitness is mandated by another circumstance: the ascending, single-peaked, and descending shape of the quasiconcave curve. Negative definiteness (or semidefiniteness) is associated with curvatures that peak and descend, similarly to the second derivative of the single-valued continuous function that is required to be negative if the curve is to have such a shape. The condition specified achieves both goals, leaving none unsatisfied. Therefore, the requirement that the parenthetical expression stand for a negative semidefinite if the function is quasiconcave (negative definite, if it is to be strictly quasiconcave) is both necessary and sufficient.

Returning to the 2-variable case, the curve g (u, v) is quasiconcave, because it is associated with $(-1)^r |D_r| \geq 0$, where r = 1, 2. The rule applied to the 2-variable case can be extended to the general, n-variable case. If f (x) is such a function, then it is quasiconcave if the following condition holds:

(22) $$g(u,v) = f(ux^0 + vx^1)$$

If u and v are considered two products sold at prices p_u and p_v, then, given the budget constraint Y, $u\,p_u + v\,p_v = Y$ applies. Dividing through with Y results in the familiar distribution on the unit simplex, t and (1-t). The continuity in u and v guarantees continuity in t in the interval [0, 1]. Equation (22) may be rewritten as

(23) $$g(u,v) = g(t,1-t) = f(tx^0 + (1-t)x^1)$$

However, since g (t, 1-t) is quasiconcave, equation (22) may be developed further to look like

(24) $$g(t,1-t) \geq \min[g(0,1),g(1,0)] = \min f[x_0,x^1]$$

Applying the second equality of equation (23), equation (24) may be written as

(25) $$f(tx^0 + (1-t)x^1) \geq \min f[x_0, x^1]$$

which, of course, is the definition of quasiconcavity. Given a 2-variable quasiconcave function, another, n-variable function, which is equivalent to it in terms of function value, is also quasiconcave if this equality can be achieved by weighing two distinct independent variable vectors from the domain of the n-variable function with t and (t-1), where $t \leq 0$. (These weights correspond to the normalized values of u and v at a given level.) However, if the n-variable function is quasiconcave, then, similar to the 2-variable case, it must also have convex-to-the-origin hyperplane indifference curves when it ascends and concave to the origin transformation hyperplanes when it descends. Consequently, the expression corresponding to the parenthetical expression in equation (19) must be negative and the bordered Hessian determinant, the enlarged version of equation (20), must also be a negative definite. For the n-variable case, the negative definitiveness ensures that the principal minor corresponding to the full determinant is positive and that the curve itself (now as a hypersurface) has a maximum, that is, it ascends, peaks, and descends: "*. . . if f(x) is quasiconcave,* $(-1)^r D_r \geq 0$ *for r = 1, . . . , n and for all x, where* D_r *is the bordered determinant.*" (Arrow and Enthoven, 1961, p. 781.)

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