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The beam-beam interaction of finite length bunches in hadron colliders

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The influence of finite bunch lengths on the dynamics of head-on beam-beam interactions is studied analytically and by simulation. Compared to infinitesimally short bunches, the resonance widths of bunches in the Tevatron are an order of magnitude smaller. With finite length bunches we find that the strengths of beam-beam effects oscillate with decreasing amplitude as a function of the bunch length. The results suggest that it may be possible to increase the integrated luminosity delivered by a collider with a careful choice of bunch length.

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In the next round of collider experiments at the Tevatron (Run II), the number of proton and anti-proton bunches will each be increased by a factor of six to thirty six [1]. Each bunch will experience two head-on collisions and seventy long-range collisions with the counter rotating beam. Collider operation will be much more sensitive to the effects of the beam-beam interactions. This motivates a detailed study of these interactions with the goal of selecting beam parameters that optimize accelerator operation. Typically beam-beam effects are analysed by assuming that bunches are infinitesimally short (see e.g. [2]). A first attempt to include the effect of finite bunch lengths on the strengths of beam-beam driven resonances was made by Krishnagopal and Siemann [3]. Several assumptions were made, the most important of which was that the amplitude function (β function) stays constant over the length of the interaction and therefore the phase advances linearly with distance. This rapid change in phase leads to resonance strengths decreasing exponentially with bunch length. These authors had also remarked that this exponential form factor overestimates the impact of the bunch length. In this letter I revisit the study with finite bunch lengths without many of the simplifying assumptions made in the earlier analysis. Contrary to the earlier results I find that beam-beam effects oscillate with bunch length. This suggests that it may be possible to maximize the integrated luminosity in a hadron collider by a proper choice of bunch length.

When the ratio of the bunch length σ_s to the beta function β^* at the interaction point (IP) ~ 1, the change in the transverse beam size along the bunch length is significant (the "hour-glass effect") and reduces the effective luminosity. At the Tevatron where $\sigma_s \simeq 36$cm, $\beta^* = 35$cm, the reduction is about 25%. We expect that in addition to this purely geometrical effect, dynamical effects of the bunch length are also significant at the Tevatron.

We begin our analysis with the Hamiltonian of a charged particle(say an anti-proton in the Tevatron)

$$H = \frac{\nu_x}{R}J_x + \frac{\nu_y}{R}J_y + H_s + V(x,y,s) \ . \tag{1}$$

Here (ν_x, J_x), (ν_y, J_y) are the linear tunes and actions in the horizontal and vertical planes respectively, R is the radius of the ring, H_s is the Hamiltonian describing the purely longitudinal motion. The beam-beam potential is $V(x,y,s) = (N_b r_p/\gamma_p)\rho_L(s+ct)U(x,y)$ where N_b is the bunch intensity, r_p the classical proton radius, γ_p the relativistic factor and ϵ is the beam emittance. The beam-beam interaction is usually characterized by a parameter ξ which for round beams equals $N_b r_p/(4\pi\gamma_p\epsilon)$. ρ_L is the longitudinal density of the opposing bunch, $s+ct$ is the distance between the center of the opposing bunch and the particle at time t. The transverse part U of the beam-beam potential due to a transverse Gaussian density distribution of rms beam sizes $\sigma_{x,op}, \sigma_{y,op}$ is given by

$$U(x,y) = \int_0^\infty \frac{dq}{[(2\sigma_{x,op}^2+q)(2\sigma_{y,op}^2+q)]^{1/2}} \left\{ 1 - \exp[-\frac{x^2}{2\sigma_{x,op}^2+q} - \frac{y^2}{2\sigma_{y,op}^2+q}] \right\} \tag{2}$$

The variables $x, y, c\tau$ describing the motion of the particle can be expressed as $x = \sqrt{2\beta_x J_x}\cos[\phi_x + \psi_{x,0}]$, $y = \sqrt{2\beta_y J_y}\cos[\phi_y + \psi_{y,0}]$, $c\tau = a_s\sigma_s\cos[\psi_s + \psi_{s,0}]$, where ϕ_x, ϕ_y are the transverse phases, a_s is the dimensionless synchrotron oscillation amplitude and ψ_s is the longitudinal phase. $\psi_{x,0}, \psi_{y,0}, \psi_{s,0}$ are the initial phases in the respective degrees of freedom. In general the Fourier transform of U depends on s via the dependence of $\sigma_{x,op}$, $\sigma_{y,op}$ on s for beams of arbitrary aspect ratio. However when the beams are round over the length of the interaction, as is true at the Tevatron, we can write $\sigma_{x,op}(s) = \sigma_{y,op}(s) = \sqrt{\beta(s)\epsilon_{op}}$ and we find

$$U_{0,0} = \int_0^1 \frac{du}{u}\left\{1 - \exp[-\frac{(J_x+J_y)u}{2\epsilon_{op}}]\right\} I_0[\frac{J_x u}{2\epsilon_{op}}] I_0[\frac{J_y u}{2\epsilon_{op}}] \tag{3}$$

$$U_{2m_x,2m_y} = 2(-1)^{(m_x+m_y+1)}[2 - \delta_{m_x,0} - \delta_{m_y,0}] \int_0^1 \frac{du}{u}\exp[-\frac{(J_x+J_y)u}{2\epsilon_{op}}] I_{m_x}[\frac{J_x u}{2\epsilon_{op}}] I_{m_y}[\frac{J_y u}{2\epsilon_{op}}] \ ,$$
$$|m_x| + |m_y| \neq 0 \tag{4}$$

I_{m_x}, I_{m_y} are the modified Bessel functions. Only the even harmonics of U are non-zero due to the fact that it is an even function of the transverse coordinates (x, y). The fact that $U_{0,0}, U_{2m_x,2m_y}$ and consequently the tune shifts with with amplitude for example depend on the emittance but are independent of the beam sizes is valid only for round beams. The Fourier harmonics of the potential $V(x, y, s)$ for round beams factorize as a product

$$V_{\vec{m},n} = \frac{N_b r_p}{\gamma_p} U_{2m_x,2m_y} L_{\vec{m}} \tag{5}$$

where $\vec{m} \equiv (2m_x, 2m_y, m_s)$. When the tunes are close to the resonance $2m_x\nu_x+2m_y\nu_y+m_s\nu_s = n$, $V_{\vec{m},n}$ is associated with a slowly varying phase and will be the dominant harmonic. Let $\Delta = 2m_x\nu_x + 2m_y\nu_y + m_s\nu_s - n$ represent the nearness of the resonance. The longitudinal part of the harmonic is

$$L_{\vec{m}} = \int\int \frac{ds d\psi_s}{(2\pi)^2} \rho_L(2s - a_s\sigma_s \cos(\psi_s + \psi_{s,0})) \exp[-i(\Delta + m_s\Delta\nu_s)s/R] \exp[i2m_+\chi] \tag{6}$$

where $m_+ = m_x+m_y$, $\chi = \arctan(s/\beta^*)$, $\Delta\nu_s(a_s)$ is the change in synchrotron tune with a_s. The phase χ accumulated over the rms length of the bunch is about 53° when $\sigma_s = \beta^*$. If we regard each kick as a phasor then clearly the net effect of the beam-beam kicks will depend strongly on the phases between the kicks delivered over the length of the bunch and also on the phase between kicks over successive turns.

We consider the case of a Gaussian longitudinal density distribution $\rho_L(s) = \exp[-s^2/(2\sigma_{s,op}^2)]/(\sqrt{2\pi}\sigma_{s,op})$. In what follows we approximate the phase factor $\exp[-i(\Delta + m_s\Delta\nu_s)s/R]$ by unity. This is a good approximation for several reasons - close to a resonance $\Delta \ll 1$, while both the synchrotron tune ν_s and the spread in synchrotron tune $\Delta\nu_s < 10^{-3}$ in the Tevatron and hadron colliders generally. Under these conditions the synchrotron sideband number m_s plays no further role in the beam-beam harmonics. The synchrotron sidebands would be important however if there were other sources of synchro-betatron coupling not considered here such as a crossing angle at the IP (envisaged for the later stages of Run II in the Tevatron) or dispersion at the IP. Under these conditions

$$L_{\vec{m}} = \frac{1}{(2\pi)^{3/2}} \exp[-\frac{a_s^2}{4}] \sum_{j=-\infty}^{\infty} (-1)^j I_j(\frac{a_s^2}{4}) F_j(a_s, \sigma_s/\beta^*; m_+) \tag{7}$$

$$F_j = \int_0^\infty du \exp[-2u^2] \cos[2m_+ \mathrm{arctan}(\sigma_s u/\beta^*)] I_{2j}(2a_s u) \tag{8}$$

The function F_j has the general form of a Hankel transform [4]. We note that the only dependence on the betatron harmonics m_x, m_y and bunch length σ_s occur in the phase factor in F_j. At large m_+, the integrands have a faster variation and cancel more effectively so the beam-beam resonance strengths decrease even faster with increasing $m_x + m_y$ than would be predicted by the transverse harmonics $U_{2m_x,2m_y}$ alone. In the limit $\sigma_s/\beta^* \to 0$, F_j attains its maximum value $F_j^{max} = (\sqrt{\pi}/[2\sqrt{2}]) \exp[a_s^2/4] I_j(a_s^2/4)$. Using the Bessel function identity $\sum_{j=-\infty}^{\infty}(-1)^j I_j^2 = 1$ [6] we obtain in this limit $L_{\vec{m}} = 1/(8\pi)$, a constant independent of the harmonic numbers. This however is not the upper bound on $L_{\vec{m}}$ as it involves the sum over alternatingly positive and negative values. It is therefore possible for $L_{\vec{m}}$ to be larger with non-vanishing bunch length and synchrotron oscillation amplitude a_s.

To estimate the behaviour of the harmonics for arbitrary bunch length we proceed by replacing the Bessel function I_{2j} in F_j with its integral representation. The integrand in the integral over u vanishes almost everywhere except in a region around $u = a_s$. We therefore use the saddle point approximation to pick out the dominant contribution and obtain

$$F_j \simeq \frac{1}{\sqrt{2\pi}} e^{a_s^2/4} \int_0^\pi d\theta \cos 2j\theta \cos[2m_+\chi_0] \exp[\frac{a_s^2}{4} \cos 2\theta] \tag{9}$$

where $\chi_0 = \arctan(\lambda \cos\theta)$. $\lambda = a_s\sigma_s/(2\beta^*)$ is the dimensionless parameter that characterizes the longitudinal motion. F_j can be evaluated by considering the complex integral

$$\mathcal{F}_j = \frac{1}{2\sqrt{2\pi}} e^{a_s^2/4} \int_0^{2\pi} d\theta \exp[i2j\theta] \left[\frac{1 + i\lambda\cos\theta}{1 - i\lambda\cos\theta}\right]^{m_+} \exp[\frac{a_s^2}{4} \cos 2\theta] \tag{10}$$

It can be easily shown that $F_j = \mathrm{Re}(\mathcal{F}_j)$. Here we have used the fact that $\exp[i2m_+\chi_0] = [(1 + i\lambda\cos\theta)/(1 - i\lambda\cos\theta)]^{m_+}$. This integral can be evaluated by a contour integration over the unit circle. Substituting $z = e^{i\theta}$, the singularities of the integrand within the contour are an essential singularity at the origin and a pole of order m_+ on the positive imaginary axis at $z = z_+ = i(\sqrt{\lambda^2 + 1} - 1)/\lambda$. The other singularity is along the negative imaginary axis at $z = z_- = -i(\sqrt{\lambda^2+1}+1)/\lambda = 1/z_+$. The residue at the origin can be found by a Laurent series expansion of the integrand valid in the range $|z| < |z_+|$ while the residue at z_+ can be evaluated by standard techniques. Here we consider some particular cases of m_+.

The tunes at most accelerators are chosen to avoid resonances of order lower than or equal to ten so the realistic value of $|m_x|+|m_y|$ is typically greater than five. However $m_+ = m_x + m_y$ can be small close to difference resonances where m_x, m_y have opposite signs. When $m_+ = 1$, we obtain

$$L_{\vec{m}} = \frac{1}{4\pi}\left\{\frac{2\exp[-\frac{a_s^2}{4}(1+\frac{2}{\lambda^2})]}{\sqrt{\lambda^2+1}}\sum_j(-1)^j z_+^{2j} I_j(\frac{a_s^2}{4}) - \left(1+\sum_{k=1}^{\infty}c_{2k}\sum_{j=-\infty}^{\infty}(-1)^j I_j(\frac{a_s^2}{4})I_{j+k}(\frac{a_s^2}{4})\right)\right\}, \quad m_+ = 1 \tag{11}$$

where $c_k = 2(z_-+z_+)(z_-^k - z_+^k)/(z_- - z_+)$. This harmonic is dominated by the terms in parenthesis (), the contribution from the residue at the origin. As a function of a_s, $|L_{\vec{m}}|$ with $m_+ = 1$ decreases with increasing a_s, crosses zero (at $a_s \approx 3.4$ for Tevatron parameters) and then increases again. $|L_{\vec{m}}|$ with $m_+ = 2, 3$ has similar behaviour except that the zero crossings occur at smaller values of a_s. This suggests that for the majority of particles in the RF bucket, the beam-beam force will decrease with increasing synchrotron amplitude close to these resonances.

Close to sum resonances m_+ is large and a useful expression for the harmonics can be extracted by an asymptotic evaluation of F_j using the stationary phase approximation [7] with the expression for F_j in Equation (9). The only stationary points of the phase factor $\cos[2m_+\chi_0]$ are at the two endpoints 0 and π, neither of which is a point of inflection. Summing the contributions from these stationary points we find that

$$\lim_{m_+\to\infty} L_{\vec{m}} = \frac{1}{2(2\pi)^{3/2}}\frac{1}{\sqrt{m_+\lambda}}\cos[2m_+\arctan(\lambda)] + O(\frac{1}{m_+}) \tag{12}$$

This simple expression shows that to leading order when m_+ is large, the beam-beam harmonics have a damped ($\propto (m_+\lambda)^{-1/2}$) oscillatory behaviour in the parameter λ. One may define a quasi-wavelength of these oscillations as $(\pi/m_+)(\lambda/\arctan(\lambda))$ which in the limit $\lambda \ll 1$ is π/m_+ while in the opposite limit $\lambda \gg 1$ is $2\lambda/m_+$. Figure 1 shows the behaviour of $L_{\vec{m}}$ in the asymptotic limit for $m_+ = 8, 9, 10$ as a function of λ. At small λ the quasi-periods of the oscillations are short while at large λ, $L_{\vec{m}}$ approaches zero asymptotically. Thus at short bunchlengths (or wavelengths) observables such as beam lifetime (due to the beam-beam interactions) are likely to change quickly with bunchlength while at long bunchlengths the lifetime may be somewhat insensitive to the choice of bunch length. The zero crossings of $L_{\vec{m}}$ correspond to the cases where the beam-beam kicks add up destructively. In practice, the phase cancellation can never be perfect in an accelerator. Nevertheless some remnant of the oscillatory behaviour predicted by the theory may survive and be observable in one of the existing hadron colliders (Tevatron, RHIC).

The Fourier harmonics of the potential can now be used to calculate the resonance widths. The resonance half widths in action J_x, J_y are

$$\Delta J_{x,w} = m_x\left[2|\frac{U_{2m_x,2m_y}(J_{x,s},J_{y,s})L_{\vec{m}}}{[m_x^2 U_{0,0;J_xJ_x} + 2m_xm_yU_{0,0;J_xJ_y} + m_y^2U_{0,0;J_yJ_y}]L_{\vec{0}}}|\right]^{1/2} \tag{13}$$

$$\Delta J_{y,w} = m_y\left[2|\frac{U_{2m_x,2m_y}(J_{x,s},J_{y,s})L_{\vec{m}}}{[m_x^2 U_{0,0;J_xJ_x} + 2m_xm_yU_{0,0;J_xJ_y} + m_y^2U_{0,0;J_yJ_y}]L_{\vec{0}}}|\right]^{1/2} \tag{14}$$

Here $U_{2m_x,2m_y}$ is evaluated at the locii of stable fixed points $(J_{x,s}, J_{y,s})$ corresponding to the resonance, $U_{0,0;J_xJ_x} \equiv \partial^2 U_{0,0}/\partial J_x^2$ etc. At the operating tunes in the Tevatron $\nu_x = 0.581, \nu_y = 0.575$, the beam straddles the sum twelfth order resonances with fifth and seventh order resonances just outside the beam distribution. We have calculated the twelfth order resonance widths numerically using Equations (3), (4), (7) and (8). Both $U_{2m_x,2m_y}$ and $L_{\vec{m}}$ decrease with increasing resonance orders m_x, m_y for long bunches. For infinitesimally short bunches $L_{\vec{m}}/L_0$ is replaced by unity. Figure 2 shows the resonant amplitudes, the amplitudes where the resonance condition is exactly satisfied, and widths corresponding to the sum twelfth order resonances as functions of $r = a_y/(a_x+a_y)$. a_x, a_y are the dimensionless transverse amplitudes in units of the rms size. A necessary but not sufficient condition for resonances to overlap is that the blue and green curves in Figure 2 corresponding to neighbouring resonance islands touch or intersect. The top figure in Figure 2 shows that it is possible for the $10\nu_x + 2\nu_y$ and $8\nu_x + 4\nu_y$ resonances to overlap in the region ($\sim 0.15 < r < 0.3$) for infinitesimally short bunches. The widths of these same resonances calculated with $\sigma_s = 36$cm and synchrotron oscillation amplitude $a_s = 1.0$ are about an order of magnitude smaller. Now the resonance islands are well isolated and there is no possibility of overlap as seen in the lower part of Figure 2. This implies that chaotic motion is unlikely and particle orbits should be stable in this region of phase space when the bunch is long.

The predictions of the analytical calculations can be tested with particle tracking using the equations of motion resulting from the Hamiltonian in Equation (1). We have calculated the amplitude growth of particles in the "weak" bunch (anti-protons at the Tevatron) due to interactions with the strong bunch (protons) with extensive multi-particle simulations. The strong bunch is sliced into several disks of charge (9 are found to suffice) of equal length. The longitudinal charge density of each disk falls off as a Gaussian from the center of the bunch. Particles in the weak bunch are subject to impulsive kicks from the center of each disk followed by a drift to the center of the next disk. We model the Tevatron by including two interaction points in the ring to simulate the collisions at CDF and D0. In the simulations reported here, the time evolution of 1000 particles in the weak bunch was followed for 100,000 turns (more than 70 synchrotron periods) at each setting of the parameters. The maximum and minimum amplitude reached by each particle during the 100,000 turns is recorded and the ratio of these limits (the swing) is taken as a measure of the resonance strength experienced by the particle (see e.g. [8]). The maximum swing over the distribution of 1000 particles is chosen as a measure of the beam-beam resonance strength experienced by the bunch. In one set of simulations all particles are chosen to have $a_s = 1$ while in the second set we choose a Gaussian distribution in a_s with a cutoff at $a_s = 3$. The calculations are repeated with different values of the bunch length with other parameters staying constant. Table I shows the values of the relevant parameters used in the simulations.

Figure 3 shows the maximum relative swing recorded amongst the 1000 particles when the tunes are set to the Tevatron values $\nu_x = 0.581, \nu_y = 0.575$. Several features are noteworthy. The amplitude swing is very low at these well chosen tunes - the swing would be unity in the absence of the beam-beam interactions. However the maximum swings do show clear evidence of oscillation with increasing bunch length. The height of these peaks decrease as the bunch length increases. Both these feature are in qualitative agreement with the results predicted by the asymptotic analysis seen in Figure 1. The maximum changes in the emittance of the distribution shows a similar dependence on the bunch length as the maximum amplitude swing. These simulations have been repeated at tune values closer to low order resonances. Figure 4 shows the results obtained near fourth order resonances. Now the maximum amplitude swings are large, between 5 to 13, and again oscillate as the bunch length increases. Similar results are observed near 6th order resonances. The numerical evidence at three different tunes confirms the prediction of the oscillatory dependence on bunch length.

In summary we have shown that the choice of bunch length has a significant impact on the dynamics of the beam-beam interaction. Analytical calculations and multi-particle tracking show that amplitude growth of particles is a decreasing oscillatory function of the bunch length. This suggests that if the lifetime in a collider is limited by the beam-beam interactions then the integrated luminosity could be improved by a proper choice of bunch length. It may be possible to study the bunch length dependence during an upcoming series of beam-beam experiments planned at the Tevatron and RHIC [9]. We also find that beam-beam resonance strengths with bunch lengths comparable to β^* are significantly weaker than those for infinitesimally short bunches. We expect that this might reduce the impact of synchro-betatron resonances when crossing angles are introduced in the second stage of Run II.

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[1] Run II handbook, Fermilab report (unpublished)
[2] S. Peggs, *Part. Accel.*, **17**, 11 (1985); D. Neuffer, *Part. Accel.*, **20**, 97 (1986)
[3] S. Krishnagopal and R. Siemann, *Phys. Rev. D*, **41**, 2312 (1990)
[4] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press (1966)
[5] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover Publications (1965)
[6] This relation does not appear in the standard mathematical literature [4], [5] but it is easily proven.
[7] E.T. Copson, *Asymptotic Expansions*, Cambridge University Press (1965)
[8] A. Piwinski, IEEE Trans. Nuc. Science, **NS-32**, 2240 (1985)
[9] Workshop at Brookhaven National Laboratory on *Beam experiments in hadron colliders*, February 2000 (unpublished).

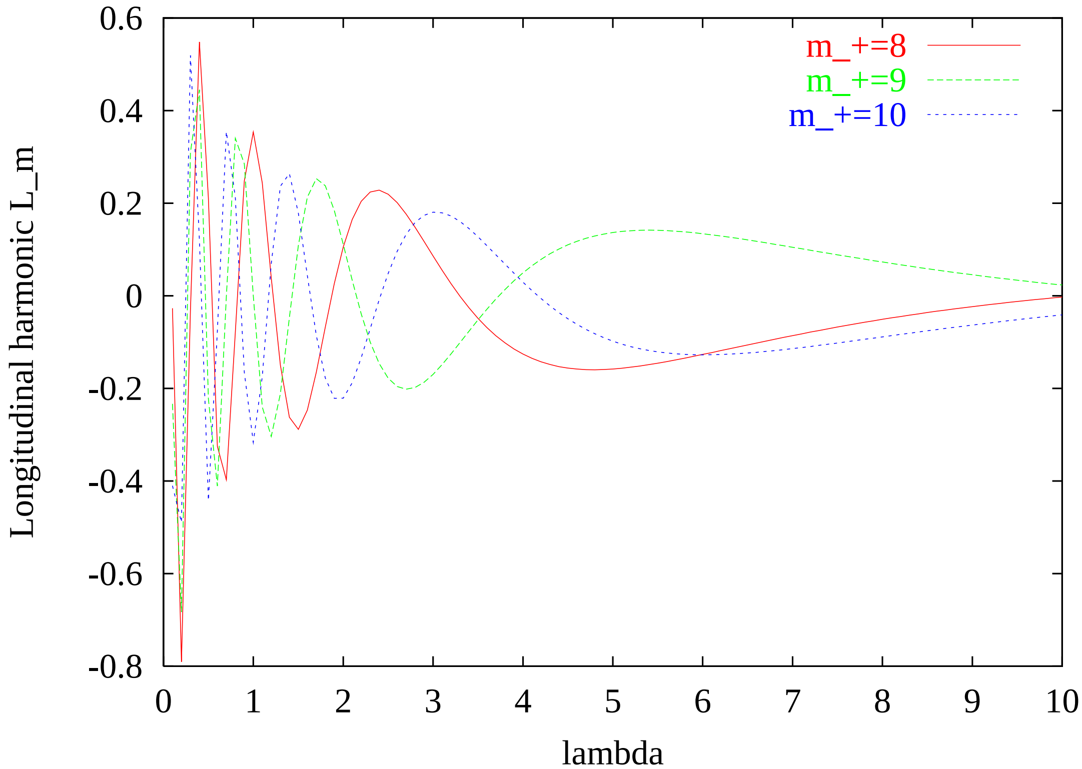

FIG. 1. (Color) Asymptotic behaviour of the longitudinal part of the beam-beam harmonics, $L_{\vec{m}}$ for large m_+ at $m_+ = 8, 9, 10$ as a function of $\lambda = a_s\sigma_s/(2\beta^*)$.

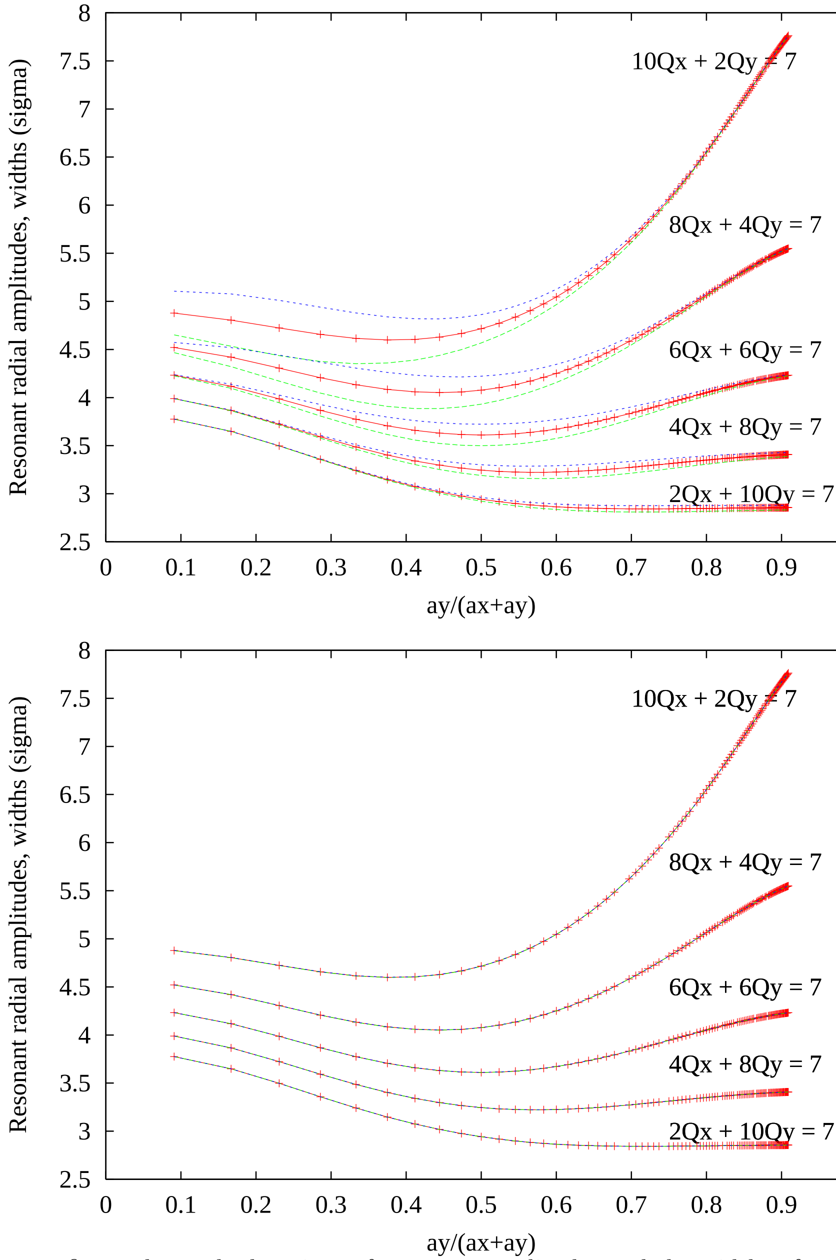

FIG. 2. (Color) The top figure shows the locations of resonant amplitudes and the widths of sum twelfth order resonances calculated for infinitesimally short bunches, β^*=0.35m as a function of $r = a_y/(a_x + a_y)$. Q_x, Q_y denote the horizontal and vertical tunes respectively. The curves in red show the locations of the resonant amplitude while the curves in blue and green on either side show the width of the resonance. We see that there is the possibility of overlap between the $10Q_x + 2Q_y$ and $8Q_x + 4Q_y$ resonances for $\sim 0.15 < r < 0.3$. At the bottom we show the same resonances and widths calculated with a bunch length of 36cm and $a_s = 1.0$. The resonance widths are all reduced by an order of magnitude. Now there is no possibility of overlap between any of these resonances.

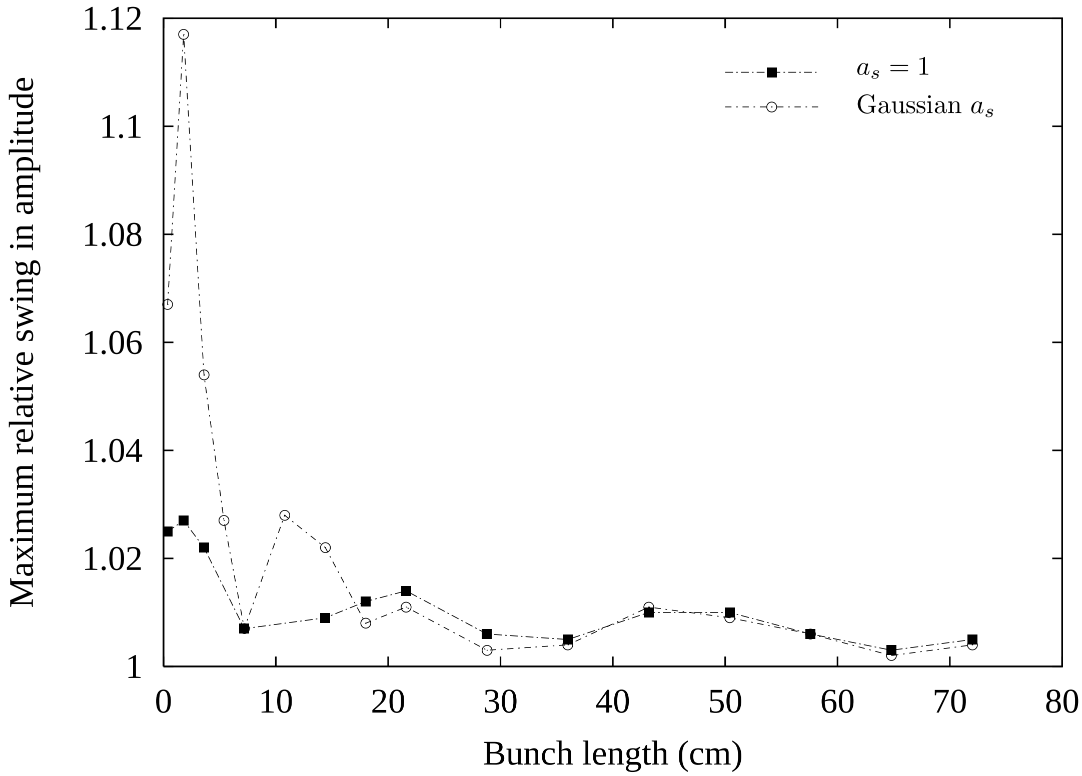

FIG. 3. Maximum relative amplitude swing amongst 1000 particles tracked for 100,000 turns as a function of bunch length. Tunes are the nominal Tevatron values $\nu_x = 0.581$, $\nu_y = 0.575$. The plot with solid squares had all particles with $a_s = 1$ while the plot with open circles had a Gaussian distribution in a_s. The oscillatory behaviour with bunch length is in qualitative agreement with the asymptotic theory developed here (see Figure 1).

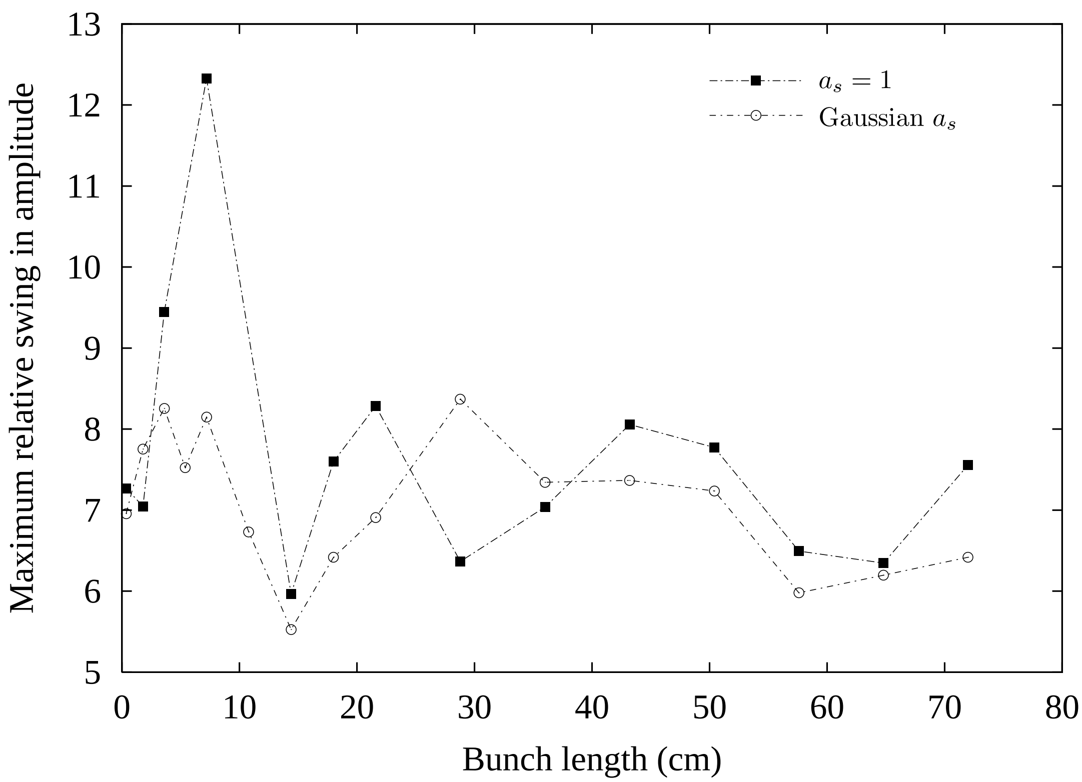

FIG. 4. Same as Figure 3 but at tunes close to the fourth order resonance $\nu_x = 0.257$, $\nu_y = 0.251$. Note that the maxima and minima in the amplitude swing now occur at different bunch lengths when the tune is changed.

Beam Energy [TeV]	1.0
Bunch Intensity $N_{b,op}$	2.7×10^{11}
Normalized emittance ϵ_N (95%) [mm-mrad]	20
Beta function at IP [m]	0.35
Bunch length σ_s [m]	0.36
Beam-beam parameter/IP ξ	9.9×10^{-3}
Synchrotron tune ν_s	7.2×10^{-4}
Revolution frequency [kHz]	47.713

TABLE I. Parameter values of the Tevatron used in the simulations.