Differential Forms of the Kramers-Krönig Dispersion Relations

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***Abstract*—Differential forms of the Kramers-Krönig dispersion relations provide an alternative to the integral Kramers-Krönig dispersion relations for comparison with finite-bandwidth experimental data. The differential forms of the Kramers-Krönig relations are developed in the context of tempered distributions. Results are illustrated for media with attenuation obeying an arbitrary frequency power law ($\alpha(\omega) = \alpha_0 + \alpha_1|\omega|^y$). Dispersion predictions using the differential dispersion relations are compared to the measured dispersion for a series of specimens (two polymers, an egg yolk, and two liquids) exhibiting attenuation obeying a frequency power law ($1.00 \leq y \leq 1.99$), with very good agreement found. For this form of ultrasonic attenuation, the differential Kramers-Krönig dispersion prediction is found to be identical to the (integral) Kramers-Krönig dispersion prediction.**

I. Introduction

It has been demonstrated that (integral) Kramers-Krönig (K-K) dispersion relations in the context of point functions [1] and tempered distributions [2] are available for media with attenuation obeying a frequency power law. The development of the K-K dispersion relations for tempered distributions was prompted in part by the work of Szabo [3], [4], which overcame a weakness of the nearly local approximation to the K-K dispersion relations [5], [6] through the use of a form of tempered distributions.

The agreement of the K-K dispersion predictions with measured dispersions hinges upon the appropriateness of the chosen ultrasonic propagation model. Under typical experimental conditions, knowledge of the propagation properties of interest across the entire (i.e., infinite) frequency spectrum are never available. Consequently, compromises must be made regarding the K-K analysis of the experimentally measured data. One compromise, which has been successful for certain media with attenuation obeying a frequency power law, is the extrapolation of the measured ultrasonic parameters outside the experimentally available bandwidth for use with integral K-K dispersion predictions [1], [4], [7]. An alternative approach that recently was explored for the case of resonant suspensions of ultrasonic contrast agents is the truncation of the K-K integrals [8]. In contrast to these two approaches, we explore the development of differential (sometimes referred to as nearly local) forms of the K-K dispersion relations appropriate for power law attenuation.

We begin with a review of derivative analyticity relations that originated in the particle scattering literature [9]. We extend the mathematical formalism of the differential dispersion relation technique to the generalized K-K dispersion relations, placing the nearly local approximation to the ultrasonic K-K dispersion relations developed by O'Donnell *et al.* [5], [6] in the broader context of the derivative analyticity relations. In Section III, we discuss differential dispersion relations for the case of the ultrasonic complex wave number and consider the specific case of the attenuation coefficient obeying a frequency power law. We compare predictions of dispersion using the developed differential dispersion relations to experimentally measured dispersions in Section IV. In Section V we discuss the equivalence of the differential dispersion relations and the (integral) K-K relations for media with frequency power law attenuations.

II. Review of Differential Dispersion Relations

The differential approximations of O'Donnell *et al.* [6] provided a simple and useful approach to the study of media in which the derivative of the attenuation coefficient and phase velocity did not vary rapidly over the frequency bandwidth of interest. Subsequent to the introduction of the differential approximations known as the nearly local approximations to the K-K relations, a variety of researchers have found them useful for measurements in biological media [10], [11], man-made composites [12], [13], and polymeric materials [7], [14], namely, those media with attenuation that increases approximately proportional to frequency. The following mathematical development places the nearly local approximations to the K-K relations in the broader context of differential K-K dispersion relations.

A. Derivative Analyticity Relations (or Differential Dispersion Relations)

The use of differential forms of dispersion relations, or derivative analyticity relations, has not been restricted to the field of ultrasonics. A mathematically equivalent ap-

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proach by Bronzan *et al.* [9] has been proposed for use in high-energy particle scattering measurements, with a similar restriction that the scattering amplitude not vary rapidly over the (energy) domain of interest. Although the relations have been found useful [15], [16], they have received some criticism with regard to their mathematical foundation [17], [18]. Nevertheless, derivative analyticity relations have been extended to the study of scattering amplitudes for which the smoothness restriction has been relaxed [19]. More recently, differential or quasilocal dispersion relations, as derivative analyticity relations also have been called, have been applied in high energy hadron scattering measurements [20], [21].

The derivative analyticity relation originally proposed by Bronzan *et al.* [9] that relates the real and imaginary parts of a scattering amplitude is given by:

$$\mathrm{Re}f(s,t) = s^{\alpha} \tan\left[\frac{\pi}{2}\left(\alpha - 1 + \frac{d}{d\ln s}\right)\right] \frac{\mathrm{Im}f(s,t)}{s^{\alpha}}, \tag{1}$$

where s is the energy, t is the four-momentum transfer, and α is a free parameter. By convergence restrictions, $0 < \alpha < 2$ is required, with α typically near 1. The derivative analyticity relation (1) is conventionally utilized by keeping only the first term of the tangent expanded in series form.

B. Differential Forms of the Generalized Kramers-Krönig Dispersion Relations

We now develop differential forms of the generalized K-K dispersion relations in order to extend the range of applicability to media with attenuation obeying an arbitrary frequency power law. The mathematical steps of the O'Donnell *et al.* [6] and the Bronzan *et al.* [9] approaches are nearly identical. However, we follow the Bronzan *et al.* [9] approach which avoids an approximation used by O'Donnell *et al.* [6] and has a more elegant form. Furthermore, we consider the complex susceptibility of interest (e.g., the complex wave number) as a tempered distribution [22], which is a type of generalized function (or functional). Note, however, that it also is possible to develop similar differential forms if one treats the complex susceptibility as a (point) function and uses the method of subtractions [23]. For point functions, the susceptibility must converge to zero at arbitrarily large frequencies. In contrast, tempered distributions need only be bounded by a power law. (See [2] for further details.) Consequently, tempered distributions are ideal for the study of media with attenuation obeying a frequency power law.

We begin with the generalized K-K dispersion relation of order n (where n is an even integer):

$$\frac{\mathrm{Re}G(\omega)}{(i\omega)^n} = \frac{2}{\pi}\mathcal{P}\int_0^{\infty} \frac{\omega' \mathrm{Im}G(\omega')}{(i\omega')^n} \frac{d\omega'}{\omega'^2 - \omega^2}, \tag{2}$$

where $\mathcal{P}$ signifies the Cauchy principal value of the integral. Eq. (2) is equivalent to the generalized Hilbert transform, having been modified such that the integral is explicitly over positive frequencies. (The inner product notation of the generalized K-K dispersion relation is implicit here. (See (8a) of [2] and (3.16) of [24] for further details.) Here, n is chosen such that the generalized Hilbert transform is convergent. We discuss the order n in more detail below. The generalized Hilbert transform is operationally equivalent to normalizing the complex susceptibility by some power of frequency. It should be noted, however, that the theory of generalized functions provides a well developed formalism for such operations. In addition, the present use of generalized functions follows that of the time-causal theory of Szabo [3], [4]. (Szabo used the generalized functions in the form of good and fairly good functions developed by Lighthill [25], which is based on the development of Schwartz [22].)

We assume $G(\omega)$ to be some susceptibility distribution with the restriction that $\mathrm{Re}G(\omega)$ has even symmetry and $\mathrm{Im}G(\omega)$ has odd symmetry. Upon using a simple variable substitution ($x' = \ln(\omega'/\omega)$) and integration by parts, we find:

$$\mathrm{Re}G(\omega) = \frac{1}{\pi}\omega^{n-1} \int_{-\infty}^{\infty} dx' \left[\ln\coth\left|\frac{x'}{2}\right|\right] \frac{d}{dx'} \frac{\mathrm{Im}G(\omega')}{\omega'^{n-1}}. \tag{3}$$

Considering the term $\dfrac{\mathrm{Im}G(\omega')}{\omega'^{n-1}}$ as a uniformly convergent Taylor series and expanding about $x' = 0$ (or $\omega' = \omega$), we find:

$$\mathrm{Re}G(\omega) = \frac{1}{\pi}\omega^{n-1} \int_{-\infty}^{\infty} dx' \left[\ln\coth\left|\frac{x'}{2}\right|\right] \frac{d}{dx'} \sum_{m=0}^{\infty} \frac{x'^m}{m!} \left(\frac{d^m}{dx'^m} \frac{\mathrm{Im}G(\omega')}{\omega'^{n-1}}\bigg|_{x'=0}\right). \tag{4}$$

Subsequent to interchanging the order of summation and integration, one can evaluate the integral (see [9]) such that:

$$\mathrm{Re}G(\omega) = \omega^{n-1} \sum_{l=0}^{\infty} \left(\frac{\pi}{2}\right)^{2l+1} \frac{2^{2l+2}\left(2^{2l+2}-1\right)}{(2l+2)!} B_{l+1} \frac{d^{2l+1}}{dx'^{2l+1}} \frac{\mathrm{Im}G(\omega')}{\omega'^{n-1}}\bigg|_{\substack{x'=0\\(\omega'=\omega)}}, \tag{5}$$

where B_l represents Bernoulli's numbers [26]. We also note that the sum in (5) is a series expansion of the tangent function. We can write the differential form of the generalized K-K dispersion relation (for n even) going from $\mathrm{Im}G(\omega)$ to $\mathrm{Re}G(\omega)$ as:

$$\mathrm{Re}G(\omega) = \omega^{n-1} \tan\left(\frac{\pi}{2}\frac{d}{d\ln\omega}\right) \frac{\mathrm{Im}G(\omega)}{\omega^{n-1}}. \tag{6}$$

We may also consider the second of the pair of generalized K-K dispersion relations of order n (for n even):

$$\frac{\mathrm{Im}G(\omega)}{(i\omega)^n} = -\frac{2\omega}{\pi}\mathcal{P}\int_0^{\infty} \frac{\mathrm{Re}G(\omega')}{(i\omega')^n} \frac{d\omega'}{\omega'^2 - \omega^2}. \tag{7}$$

Performing the same steps that led to (6), we find the differential form (for n even) going from $\mathrm{Re}G(\omega)$ to $\mathrm{Im}G(\omega)$ to be:

$$\mathrm{Im}G(\omega) = -\omega^n \tan\left(\frac{\pi}{2}\frac{d}{d\ln\omega}\right)\frac{\mathrm{Re}G(\omega)}{\omega^n}. \tag{8}$$

If instead we assume the order n of the generalized Hilbert transform to be odd, which affects the symmetry of the K-K kernel, we then find the differential forms of the generalized K-K dispersion relations to be:

$$\mathrm{Re}G(\omega) = \omega^n \tan\left(\frac{\pi}{2}\frac{d}{d\ln\omega}\right)\frac{\mathrm{Im}G(\omega)}{\omega^n}, \text{ and} \tag{9}$$

$$\mathrm{Im}G(\omega) = -\omega^{n-1} \tan\left(\frac{\pi}{2}\frac{d}{d\ln\omega}\right)\frac{\mathrm{Re}G(\omega)}{\omega^{n-1}}. \tag{10}$$

The order of n reflects the symmetry of the integrand of the Hilbert transform. In addition, the value of n is chosen such that the generalized Hilbert transform is convergent, which, for the case of the ultrasonic wave number, depends upon the growth properties of both the attenuation coefficient and phase velocity. It is sufficient to choose the lowest value of n that satisfies the Hilbert transforms, although one may in principle choose larger values of n [23]. For example, one can use (8) to solve for the dispersion for the case $n = 2$ and $1 < y < 3$ but could also use (10) for the case $n = 3$ and $1 < y < 3$. Operationally, one may view the use of the generalized Hilbert transform with tempered distributions as dividing the wave number by some power of frequency. However, as previously mentioned, it is the theory of distributions that provides a rigorous mathematical foundation which permits one to do so. For the case $n = 1$, the first order expansion of the derivative analyticity relation (1) with $\alpha = 1$ or (9) can be shown to be equivalent to the (original nearly local) differential approximation:

$$\alpha(\omega) - \alpha(0) \approx -\frac{\pi}{2}\omega^2 \frac{d}{d\omega}\left(\frac{1}{c(\omega)}\right), \tag{11}$$

as we shall see in Section III-B.

We now have available differential dispersion relations for both even and odd orders of the generalized Hilbert transform. However, we will specifically consider those differential forms of the K-K relations for comparison with experimental data, namely, (8) and (9) for which the lowest order n is sufficient to satisfy the convergence of the generalized Hilbert transforms. In Section V we briefly discuss the issue of a preferred directionality to the K-K dispersion relations (e.g., predicting dispersion from attenuation or vice versa) for comparison with experimental data. We remark also that the differential dispersion relations are accurate to within a constant value. To avoid such an ambiguity, this constant value is often accounted for by subtracting the value of the susceptibility function at a reference frequency.

III. Application of the Differential Forms of the Kramers-Krönig Dispersion Relations

As mentioned above, the nearly local approximation has been successfully applied in several types of ultrasonic measurements. Despite this past success, it recently was shown that the nearly local approximation (11) may not be adequate for some media [3], [7]. Specifically, it appears to break down for media with attenuation obeying a frequency power law $|\omega|^y$ where $y \geq 1.5$ (roughly), as was considered in [3] and [7]. This stems from the fact that the Hilbert transform for point functions is not strictly convergent for such attenuation coefficients. More generally, we would expect that the nearly local approximation would break down for $|y - 1| > 0.5$. Consequently, the nearly local approximation becomes less accurate as the frequency power law diverges from 1. We now, however, consider the differential dispersion relations (which are not susceptible to this approximation) applied to the particular case of media with attenuation obeying a frequency power law. We first consider the general case of an arbitrary power law, then restrict our attention to a specific range of power laws ($1 \leq y < 3$).

For all the subsequent cases considered, we investigate the propagation term $\gamma(\omega)$ as given by:

$$\gamma(\omega) = iK(\omega) = -\alpha(\omega) + i\frac{\omega}{c(\omega)}, \tag{12}$$

where $K(\omega)$ is the complex wave number and $c(\omega)$ is the phase velocity. Furthermore, we assume the model for the attenuation coefficient is:

$$\alpha(\omega) = \alpha_0 + \alpha_1 |\omega|^y. \tag{13}$$

We will find, however, that a nonzero, frequency-independent term α_0 appears to be necessary only for the polymers. Although this model has been used to fit the ultrasonic measurements, we hesitate to attach any physical significance to the frequency-independent term α_0, which is far outside the experimental bandwidth. However, one could speculate that a viscoelastic material such as a polymer could have a nonzero loss at very low frequencies (i.e., those frequencies approaching direct current values). Consequently, it might be more appropriate to fit a model to the experimental data, such as $\alpha(\omega) = \alpha(\omega_c) + \alpha_1(|\omega|^y - |\omega_c|^y)$, that does not explicitly refer to values outside the experimentally available bandwidth. Regardless of which form of the model we use, the K-K relations will predict the same dispersion. Furthermore, the predicted dispersion is independent of the frequency-independent contribution of the attenuation coefficient.

The differential forms of the K-K relations are accurate to within a constant, as mentioned previously. We then consider relative changes in the propagation term, $\gamma(\omega) - \gamma(\omega_{ref})$, where we conveniently assume the reference frequency $\omega_{ref} = 0$ (i.e., $G(\omega) = \gamma(\omega) - \gamma(0)$). Because the differential forms of the K-K relations are formulated

in the context of positive frequencies, it is not necessary to carry the absolute value sense of the frequency. Where it is possible, we evaluate the differential dispersion relations exactly. In the cases that this does not appear possible, we perform a first-order expansion of the differential form of the K-K relations. It is this first-order expansion that has been termed a nearly local approximation to the K-K dispersion relation. We remark that naming this approximation nearly local was somewhat unfortunate. It is still necessary to integrate over all frequencies to arrive at the differential form of the K-K relations, as described in Section II-B.

A. Differential Dispersion Relations of Even Order $n(n-1<y<n+1)$

We first consider the differential dispersion relation (8) (n even) for an arbitrary frequency power law, in which the power law y is a positive, real number that is not an odd integer. That is, we will restrict the range of power laws for a given order of the differential dispersion relations to $n-1<y<n+1$, for some given even integer n (e.g., $n=2$ and $1<y<3$). This ensures that the generalized Hilbert transform will converge. The differential dispersion relation that expresses the phase velocity in terms of the attenuation coefficient (noted as $\alpha \to c$) is given by:

$$\frac{\omega}{c(\omega)} = +\omega^n \tan\left(\frac{\pi}{2}\frac{d}{d\ln\omega}\right)\frac{\alpha(\omega)-\alpha(0)}{\omega^n}. \tag{14}$$

Assuming the attenuation coefficient is given by (13), this differential form of the K-K relations can be evaluated exactly. Using the relation:

$$\left(x\frac{d}{dx}\right)^m x^p = p^m x^p, \tag{15}$$

one can show:

$$\tan\left(\frac{\pi}{2}\frac{d}{d\ln\omega}\right)\omega^{y-n} = \tan\left(\frac{\pi}{2}y\right)\omega^{y-n}, \tag{16}$$

where we have used the property $\tan\left(\frac{\pi}{2}(y-n)\right)\omega^{y-n} = \tan\left(\frac{\pi}{2}y\right)\omega^{y-n}$ for n an even integer. Considering the dispersion between the two frequencies ω and ω_0 within the experimentally useful bandwidth, we find the dispersion is given by (14) and (16):

$$\frac{1}{c(\omega)} - \frac{1}{c(\omega_0)} = +\alpha_1 \tan\left(\frac{\pi}{2}y\right)\left(\omega^{y-1}-\omega_0^{y-1}\right). \tag{17}$$

We note that the differential dispersion relation for even order n (17) is equivalent to the integral K-K dispersion relation of even order n. (See (6) and (10) of [1] for the cases $n=0$ and $n=2$ and [27] for the more general case.)

B. Differential Dispersion Relations of Odd Order n $(y \approx n)$

We now consider the differential dispersion relations (n odd) for an arbitrary frequency power law, where the exponent y is near an odd integer. We begin with the differential dispersion relation of odd order n (9) for which we consider the attenuation coefficient in terms of the phase velocity ($c \to \alpha$):

$$\alpha(\omega)-\alpha(0) = -\omega^n \tan\left(\frac{\pi}{2}\frac{d}{d\ln\omega}\right)\frac{\omega}{\omega^n c(\omega)}. \tag{18}$$

Our goal is to develop a dispersion relation with the dispersion expressed in terms of the attenuation coefficient. The simplest approach is to consider a first-order expansion of the tangent differential operator. We expand the tangent term in a series and retain only the first term:

$$\alpha(\omega)-\alpha(0) \approx -\frac{\pi}{2}\omega^{n+1}\frac{d}{d\omega}\frac{1}{\omega^{n-1}c(\omega)}, \tag{19}$$

noting that $d/d\ln\omega = \omega d/d\omega$. Although the approximation is useful, it is not clear what precise restrictions are placed on the complex wave number. However, we can say that the approximation implies that higher-order derivatives of $1/\omega^{n-1}$ are negligible.

For the case $n=1$, (19) corresponds to the original (nearly local) differential approximation (11), [5]. For the case $y=n$, we find the dispersion given by:

$$\frac{1}{\omega^{n-1}c(\omega)} - \frac{1}{\omega_0^{n-1}c(\omega_0)} \approx -\frac{2}{\pi}\alpha_1 \ln\left|\frac{\omega}{\omega_0}\right|, \tag{20}$$

for integration over an experimentally available bandwidth from ω_0 to ω. For the case $y=n=1$, the expected logarithmic dispersion is given by:

$$\frac{1}{c(\omega)} - \frac{1}{c(\omega_0)} \approx -\frac{2}{\pi}\alpha_1 \ln\left|\frac{\omega}{\omega_0}\right|. \tag{21}$$

The differential dispersion relations of (20) and (21) are equivalent to the result from the (integral) K-K dispersion relation. (See (13) of [1] for the case $n=1$ and [27] for the more general case.) Also note that one can arrive at the odd order $n=1$ dispersion relation (21) by taking the limit of the even order n dispersion relation (17) as the power law $y \to 1$, such that no approximation is necessary. This is done in (22) (see next page) using the Taylor series expansion for ω^ε and $\tan(\varepsilon\pi/2)$, and the substitution $y = 1+\varepsilon$. We remark that the nearly local approximation to the K-K dispersion relations appears to fail for some media with attenuation obeying a frequency power-law for which the power law is $y>1$, as was discussed at the beginning of Section III. However, in order to appropriately apply the 'nearly-local' approximation one must insure that the original form of the Hilbert transform (upon which the approximation is based) is convergent.

The differential dispersion relations of odd and even order are summarized in Table I. As noted above, the choice of which dispersion relation to apply to experimental data depends upon the growth property of the attenuation over the available bandwidth, which determines the order n of the differential dispersion relations.

$$\begin{aligned}\lim_{y\to 1}\tan\left(y\frac{\pi}{2}\right)\left(\omega^{y-1}-\omega_0^{y-1}\right) &= -\lim_{\varepsilon\to 0}\frac{\omega^{\varepsilon}-\omega_0^{\varepsilon}}{\tan\left(\varepsilon\frac{\pi}{2}\right)}\\ &= -\lim_{\varepsilon\to 0}\frac{(1+\varepsilon\ln\omega+\dots)-(1+\varepsilon\ln\omega_0+\dots)}{\varepsilon\frac{\pi}{2}+\dots}\\ &= -\frac{2}{\pi}\ln\frac{\omega}{\omega_0}\end{aligned} \tag{22}$$

TABLE I
DIFFERENTIAL DISPERSION RELATIONS OF ODD AND EVEN ORDER.*

Order	Power law	Direction	Differential Kramers-Krönig dispersion relations
n odd	$y=n$	$c\to\alpha$	$\frac{1}{\omega^{n-1}c(\omega)}-\frac{1}{\omega_0^{n-1}c(\omega_0)}\approx-\frac{2}{\pi}\alpha_0\ln\left\lvert\frac{\omega}{\omega_0}\right\rvert$
n even	$n-1<y<n+1$	$\alpha\to c$	$\frac{1}{c(\omega)}-\frac{1}{c(\omega_0)}=+\alpha_0\tan\left(\frac{\pi}{2}y\right)\left(\omega^{y-1}-\omega_0^{y-1}\right)$

*Summary of the differential dispersion relations for different power law attenuations that have been found to empirically agree with experimental measurement. The order and direction of application of the dispersion relation also is noted.

IV. Experimental Results

A series of five specimens are considered: two polymers, an egg yolk, and two liquids. These specimens offer a range of power laws ($1.00 \le y \le 1.99$) to test the dispersion predictions of the differential K-K relations.

A. Polymer Specimens

The first set of measurements we consider are of a clear polystyrene and a polymethyl methacrylate (Lucite^{TM}, Lucite International, Southampton, England). The measurements were performed in an immersion through-transmission setup. Analyses were implemented in both Igor Pro (Wavemetrics, Inc., Lake Oswego, OR) and Mathematica (Wolfram Research, Inc., Champaign, IL) using a C MathLink module (see Waters *et al.* [1] for further details). Both polymer specimens were 7 mm in thickness. To compensate for the transmission losses, we used the densities $\rho_{cs} = 1040$ kg/m^3 and $\rho_L = 1180$ kg/m^3 for the clear polystyrene and Lucite, respectively, and the measured phase velocities at 5 MHz. The temperature of the water bath was 24.1°C.

Figs. 1 and 2 show the experimental measurements of the attenuation coefficient and phase velocity with standard deviations for the Lucite and clear polystyrene polymers, respectively. In addition, the prediction for phase velocity of the clear polystyrene using the differential dispersion relation of order 2 (17) is provided. The prediction of the phase velocity of the Lucite polymer used the differential dispersion relation of order 1 (21). Predictions for phase velocity were determined by fitting the attenuation model (13) to the experimentally measured attenu-

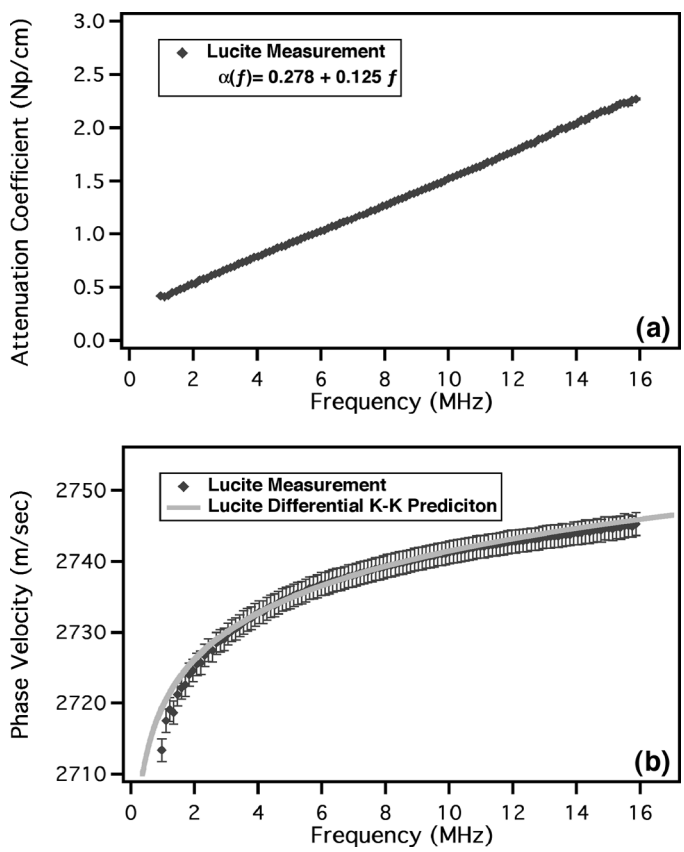

Fig. 1. (a) Attenuation coefficient and (b) phase velocity measurements with standard deviation bars for polymethyl methacrylate (Lucite^{TM}) polymer. The phase velocity predicted using the differential Kramers-Krönig relations of order 1 (21) also is shown.

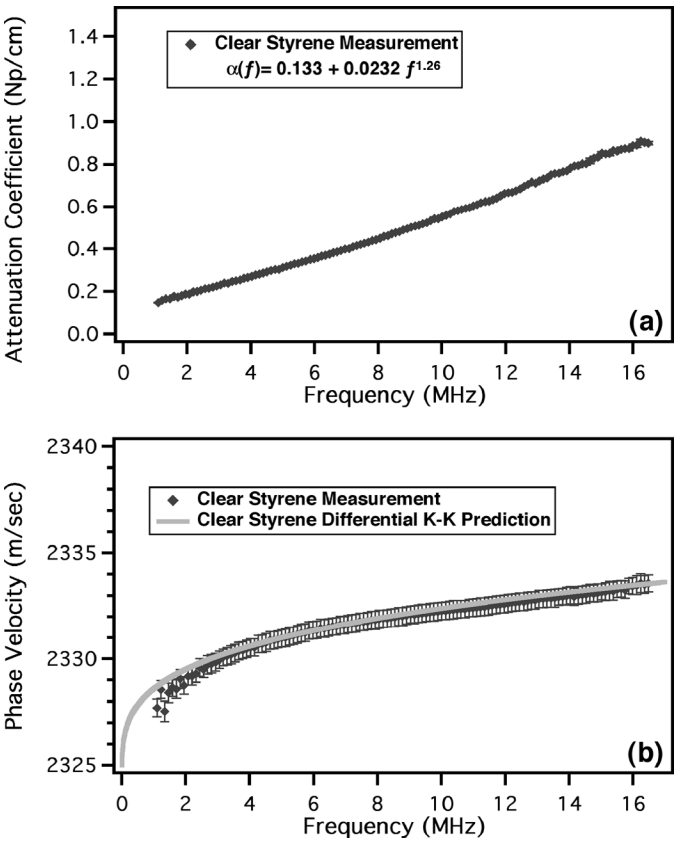

Fig. 2. (a) Attenuation coefficient and (b) phase velocity measurements with standard deviation bars for a clear styrene polymer. The phase velocity predicted using the differential Kramers-Krönig relations of order 2 (16) also is shown.

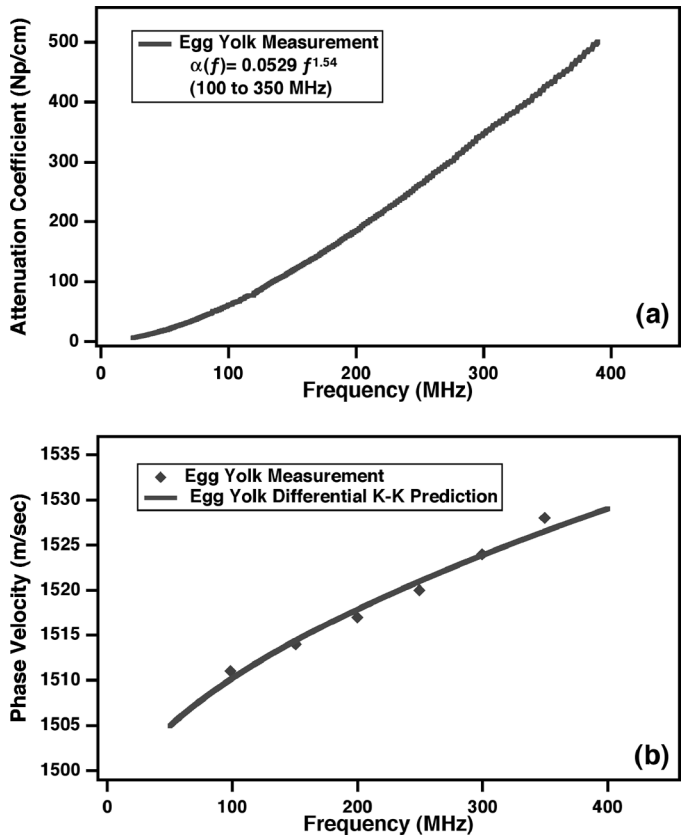

Fig. 3. (a) Attenuation coefficient and (b) phase velocity measurements for white leghorn egg yolk as measured by Akashi *et al.* [10]. The phase velocity predicted using the differential Kramers-Krönig relation of order 2 (17) also is shown.

ation coefficients. We find very good agreement between the measured and predicted dispersions.

B. Egg Yolk Specimen

The attenuation coefficient of (white leghorn) egg yolk was measured by Akashi *et al.* [10] over a frequency bandwidth of 25 to 400 MHz at 23.9°C. They also reported phase velocity measurements from 100 to 350 MHz at 50-MHz intervals over a temperature range of 23.4°C to 23.7°C. Fig. 3 reproduces the measurements (which we have digitized from the published paper) of the attenuation coefficient and phase velocity of [10]. The authors found that the attenuation data grew as a power law of $y = 1.65$ over most of the spectrum (<300 MHz), but increased somewhat slower ($y \cong 1.5$) near the high end of the spectrum (400 MHz). For our purposes, we fit a single power law ($y = 1.54$) over the bandwidth (100 to 350 MHz) for which both attenuation coefficient and phase velocity measurements were performed. This power law ($y = 1.54$) falls between those often found in liquids ($y > 1.6$) and polymers ($y < 1.3$), and thus represents an interesting test. In Fig. 3(b) the dispersion predictions are compared to the measurements. We find good agreement between the measured and predicted phase velocities.

C. Liquid Specimens

Previously we reported on ultrasonic measurements of liquid specimens that exhibited attenuation obeying a frequency power law over the experimentally available bandwidth [1]. The experimental details and data reduction techniques are similar to the measurements of the polymer specimens described in Section IV-A. We found excellent agreement between measured dispersions and dispersions predicted using the integral K-K relations for (point) functions and the method of subtractions (order 2), or equivalently, the generalized K-K relations of order 2. Further details regarding the liquid specimens and the experimental techniques can be found in [1].

Figs. 4 and 5 show experimental measurements of the attenuation coefficient and phase velocity with standard deviations for castor oil and a silicone fluid (DC 705, Dow Corning, Midland, MI), respectively. In addition, predictions for phase velocity using the differential dispersion relation of order 2 (17) are provided. The prediction for phase velocity was determined by using a fit of the attenuation model (13) to the experimentally measured attenuation coefficient. As expected, we find very good agreement between the measured and predicted dispersions for these liquids that exhibit attenuation coefficients with a power-law frequency dependence ($1.76 \leq y \leq 1.99$).

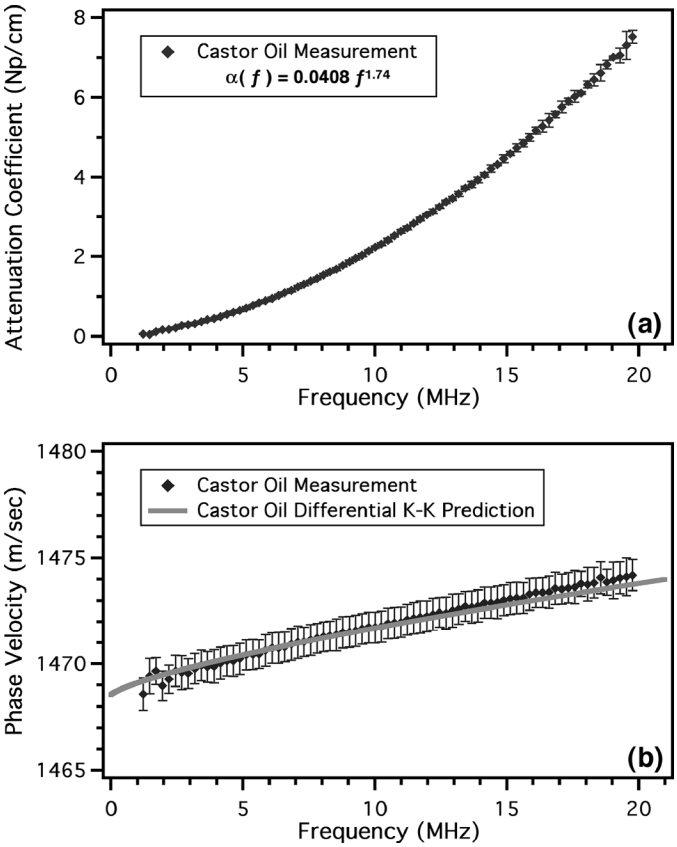

Fig. 4. (a) Attenuation coefficient and (b) phase velocity measurements with standard deviation bars for castor oil. (These measurements were originally reported in Waters *et al.*[1].) The phase velocity predicted using the differential Kramers-Krönig relations of order 2 (17) also is shown.

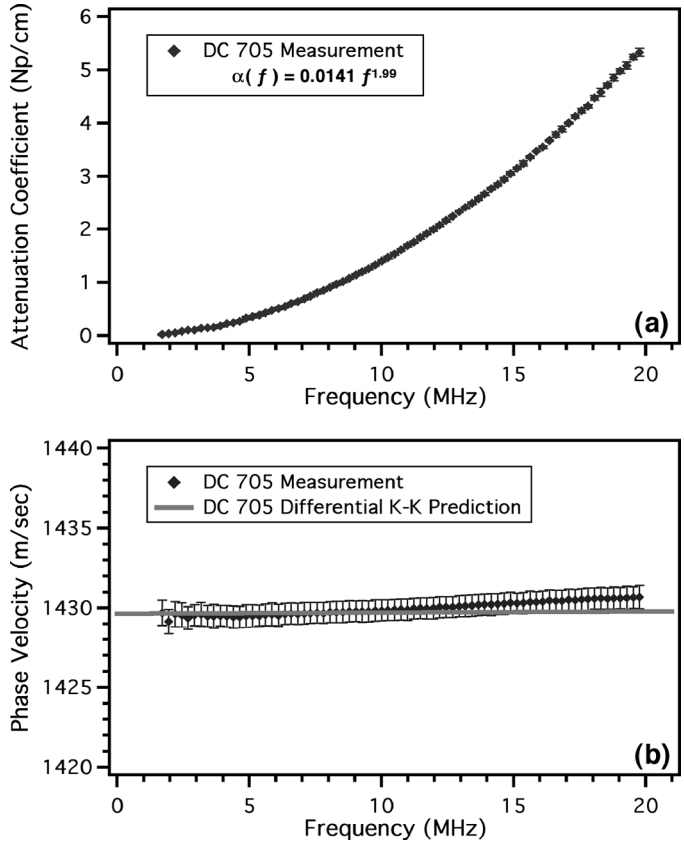

Fig. 5. (a) Attenuation coefficient and (b) phase velocity measurements with standard deviation bars for a silicone fluid (DC 705). (These measurements were originally reported in Waters *et al.* [1].) The phase velocity predicted using the differential approximation to the Kramers-Krönig relations order 2 (17) also is shown.

V. Discussion

Use of the differential forms of the K-K relations requires explicit evaluation only over a limited experimentally-available bandwidth. This is in contrast to the application of integral K-K dispersion relations in which one must explicitly extrapolate the behavior of the attenuation coefficient over the entire frequency spectrum from measurements over a limited bandwidth for evaluation of the integrals. However, we must remark that this evaluation over all frequencies is implicit in the differential dispersion relations. Through use of the approach of Bronzan *et al.* [9], we were able to demonstrate that the integral and differential dispersion relations are equivalent for media with frequency power-law attenuation. For the case of differential relations of odd order n (20), we found that an approximation (i.e., a first-order expansion) was necessary, whereas for the case of differential relations of even order n (17), no approximation was necessary.

The differential dispersion relations are accurate for media with attenuation obeying a frequency power law, but it does not appear that this may necessarily be the case for other forms of attenuation (e.g., a resonant loss). This may be due, in part, to the power-law attenuation ($\alpha(\omega) = \alpha_0 + \alpha_1|\omega|^y$) being a special case in which a closed-form solution is available (i.e., the power law may be considered an eigenfunction of the differential operator $d/d\ln\omega$). For other forms of attenuation, it is not clear that accurate differential dispersion relations are available. In principal, one could attempt a power series fit to a given attenuation measurement in order to predict dispersion. One would then find that only the odd integral powers of the series would contribute to the predicted dispersions, as all even powers of attenuation lead to predictions of zero dispersion. At present, however, we have not been successful in applying this approach.

We close with a brief discussion of the apparent preferred directionality of the K-K dispersion relations. One goal of this research has been to provide simple relations for the prediction of dispersion from the measured attenuation coefficient that compares well with experimental data. For the case $1 < y \leq 2$, we used an even-ordered ($n = 2$) differential dispersion relation (14) to predict dispersion directly from the attenuation ($\alpha \rightarrow c$). In contrast, for the case $y = 1$ of the odd-ordered ($n = 1$) differential dispersion relations, we used (19), which provides the attenuation in terms of a derivative of the phase velocity ($c \rightarrow \alpha$) that could then be integrated to predict dispersion. If for the case $y = 1$ we instead try to use the dispersion relation that predicts dispersion directly from the at-

tenuation ($\alpha \to c$), we find that we predict a dispersionless system that does not agree with the standard logarithmic dispersion found for such a system. At present, the issue of preferred directionality ($\alpha \to c$ or $c \to \alpha$) to the K-K dispersion relations is not completely understood and is a subject of continuing research.

VI. Conclusions

We have developed differential forms of the generalized Kramers-Krönig relations that accurately predict the dispersion in media with attenuation obeying a frequency power ($1.00 \leq y \leq 1.99$). Although these relations are differential in form and may imply a need to know only a small range (i.e., nearly local) of frequencies, they may be applicable only for those materials with an attenuation coefficient that has the same functional form over a wide spectrum. For the case of media with attenuation obeying a frequency power law, the prediction of dispersion using the differential Kramers-Krönig relations has been shown to provide identical predictions of dispersion using the (integral) Kramers-Krönig relations.

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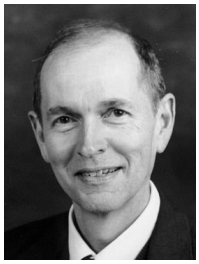

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