Chapter 1

Introduction

The main goal of this thesis is to propose and analyze a few numerical schemes for a simplified poroelastic model. The theories of quasi-static and dynamic poroelasticity presented by M.Biot [9, 10, 11, 12] have served as rigorous and formal foundations to numerous practical solutions concerning the stability of constructions on consolidating soils, stress diffusion around pressurized cavities [63], acoustic wave propagation in saturated sediments, biology [2, 52, 61, 77, 78], and mining and petroleum engineering [58, 79].

Several approaches have been used to derive varying forms of Biot's equations and determine their range of validity. We mention only a few of them. In [19] the Biot theory has been generalized for a porous elastic solid with a compressible viscous fluid. Its derivation is based on the equations of linear elasticity in the solid, and the linearized Navier-Stokes equations in the fluid. When the viscosity of the fluid is small, the resulting general equations become those of Biot. In [28] Biot's model has been coupled with a squirt-flow mechanism where the fluid is squeezed out of thin pores deformed by a passing wave. Finally, poroelastic models were proposed for a heart [78], tumors [61] and general soft tissues [52] (see [54] for references to other applications of poroelastic models).

Despite the great number of publications concerning numerical schemes for different poroelastic models (see e.g., [22] and references therein), there is a certain lack of efficient preconditioners for accelerating a convergence rate of iterative methods. In this work, we focus on the simplified Biot model to analyze a few preconditioning algorithms. The model couples two basic mechanical processes in porous media: fluid flow and rock deformation. The change in fluid pressure, which is caused by fluid injection or production (in the reservoir simulations), perturbs the stress state of the rock. The change in stress state triggers the rock deformation which, in turn, affects the fluid flow processes. Finally, the simplified model is formalized as a coupled system of the Lamé and diffusion equations.

Most popular iterative schemes for solving the coupled problem are based on an operator splitting methodology in which the Lamé and diffusion equations are solved separately. The main results of Chapters 3 and 4 are that the coupled problem may be solved very efficiently using only preconditioners for the Lamé and diffusion operators.

1.1 Review of discretization methods

In this thesis we consider two popular discretization schemes: a mixed finite element and a mimetic finite difference methods. The popularity of finite element (FE) methods is explained by their important features. The methods are essentially geometry-free and can be applied to domains of arbitrary shapes and with arbitrary boundary conditions. Therefore, the FE methods by their nature lead to unstructured meshes. The direct use of Galerkin or Petrov-Galerkin methods to derive FE methods leads to conservative and stable algorithms for most classes of problems in mathematical physics. The last, but not the least, is a solid and rich mathematical basis of FE methods. We shall use this basis in Chapter 5 to prove convergence estimates for the mimetic finite difference method. An excellent introduction to the theory of FE methods can be found in [23, 13, 64].

In many applications, the mathematical model is formulated initially as a system of first-order partial differential equations (PDEs), with each equation having a natural connection to physical aspects of the underlying problem. For instance, a diffusion part of the Biot model (2.9) consists of two first-order PDEs which describe the Darcy and mass conservation laws. A FE method involving simultaneous approximation of two or more physical variables is called a mixed finite element (MFE) method. Very often, a mixed formulation results in a saddle point problem. If so, the stability conditions [5, 16] impose some limitations on discrete FE spaces. In particular, the MFE spaces for the Biot model have to satisfy the Ladyshenskaya-Babuška-Brezzi condition. In Section 2.2, using the Brezzi's theory, we shall prove existence and uniqueness of a weak solution of the Biot model as well as solvability of the corresponding discrete FE problem.

The finite difference (FD) methods has a long history going back to Leonhard Euler (1707 - 1783) who began a study of the calculus of finite differences. Despite their simplicity for model problems, the classical FD methods are seldom used for solving complex problems. However, there are some notable exceptions to this, and one of these exceptions is considered in this thesis. To be more specific, we consider a class of FD methods primary used for solving a system of first-order PDEs. The increasing demand for accurate and robust numerical simulations has generated considerable interest in discretizations of this first-order form. One such methodology, the mimetic finite difference technique based on support-operators (SO), has been applied successfully to several applications including diffusion [37, 38, 57, 68], electromagnetics [36], and gas dynamics [21]. In particular, the numerical convergence of the SO discretization has been demonstrated in the presence of grid distortions and severe jumps in coefficients. However, a formal convergence proof of the SO discretization has been lacking, even for smooth coefficients on smooth grids.

The main idea of the SO method is to preserve important physical properties (symmetry, mass conservation, etc.) on a discrete level through the use of discrete integral identities (e.g., Green's theorem). Many of the standard finite difference methods are special cases of the SO method. Unlike elementary FD and FE methods, the SO method can be used to construct finite difference schemes on meshes of arbitrary structure. In this manner, the SO method is connected to finite volume (FV) methods. An exhaustive analysis of the SO method can be found in [67].

The accuracy of a discretization as well as complexity of computations are very important for engineering applications. Similarities between MFE, FD, SO and FV methods, when they are applied to elliptic equations, have been investigated by several researchers [8, 24, 27, 59]. The MFE method produces a velocity field everywhere over the entire domain. Furthermore, imposing boundary conditions with FV, FD or SO methods often poses nontrivial problems, while it is straightforward with MFE methods. The MFE method is actually a family of methods from which one can pick a scheme of arbitrary high accuracy [18, 64]. This has still to be done with FV methods. It was shown in a number of publications that MFE methods were more robust than FE methods [59] and control FV methods [27]. However, under some specific restrictions, FV methods are superior than MFE methods (more accurate results with less unknowns) [24]. It is pertinent to recall here that SO methods are much less investigated than FE and FD methods. The boundary element method has also been adapted to poroelastic problems [25] but its discussion is beyond the scope of this thesis.

The theory of MFE methods states that a discrete solution is estimated from above by the best interpolation error if the stability conditions for both continuous and discrete problems hold (see, for example, [18, 75, 64]). Unfortunately, there is no such a proposition for SO discretizations. In Chapter 5 we established the convergence of mimetic discretizations for a 2D elliptic problem by developing connections between

SO and MFE methods. One can extrapolate this idea to the coupled problem. However, even for a simple 2D elliptic problem, convergence of mimetic discretizations on general meshes is still an open question.

1.2 Review of solution methods

The classical approach to solving coupled problems is based on alternating-direction (or operator splitting) methods introduced in the mid-1950's by D.Peaceman and H.Rachford, and fractional step methods investigated some years later by G.I.Marchuk, N.Yanenko, R.Temam, J.L.Lions, A.Bensoussan and others. The augmented Lagrangian and operator-splitting methods have been applied by R.Glowinski to the solution of problems from non-Newtonian fluid mechanics, non-linear elasticity and petroleum engineering seismic explorations (see [26, 32, 33] and reference therein). For the Biot poroelastic model, operator splitting methods separate the elastic operator terms from the diffusion operator terms. Then, each of the problems may be efficiently solved by available iterative methods. However, preconditioners for the Lamé and diffusion operators are the only methods we need to solve simultaneously for the porous media displacement, fluid mass flux, and pore pressure. This is one of the main results of this thesis.

The mathematical formulation of the Biot model (2.9) results in a saddle point problem. A general framework for constructing preconditioners for saddle point problems is formulated in Lemma 3.1 which was independently proved by Yu.Kuznetsov [43] and T.Rusten *et al.* [66]. It states that a saddle point problem can be efficiently preconditioned by a block diagonal matrix with as many blocks as there are physical variables. We shall show in Chapter 2 that the block diagonal preconditioner includes preconditioners for the Lamé and diffusion equations. The problem is further simplified by the Korn inequality which says that a bilinear form associated with the elastic energy generates a norm in a space of displacements equivalent the to H^1-norm provided that the displacement is fixed on a measurable subset of the domain boundary. Therefore, spectral properties of the block diagonal preconditioner will depend on spectral properties of a preconditioner for an elliptic operator.

A theory of preconditioners for elliptic problems is a very well elaborated field in numerical analysis. In this thesis we concentrate on two classes of preconditioners: separable and multilevel preconditioners.

The separable preconditioners (matrices) are usually related to regular geometric structures such as cubic meshes. The classical example here is a finite difference approximation of the Dirichlet boundary value problem for the Poisson equation on a cubic mesh [20]. More general separable matrices have been studied in [41, 50, 74]. In Chapter 4 we extend the concept of separable matrices to a class of matrices appearing from MFE discretizations on a cubic mesh. The main difficulty here is that the degrees of freedom are associated with faces of the cubic mesh; namely, a matrix sparsity structure is given by an 11-point stencil describing connections between faces belonging to the same mesh cell.

The multigrid method is one of the fastest methods for solving systems of equations involving a large number of unknowns. The method was proposed by R.P.Fedorenko [29], who formulated it as a two-grid method, and three years later as a multigrid method [30]. He showed that the method has optimal arithmetical complexity. Later, N.S.Bachvalov extended the method to a general elliptic problem in the unit square [6]. The basic multigrid idea is as follows. First, we carry out several relaxation steps in order to strongly damp high-frequency components in the error. Then, we go to a coarser grid and approximate the remaining smooth part. The algorithm may be symmetrized by repeating the same number of relaxation steps after a coarse grid correction.

For the last four decades, the theory of multigrid methods was extensively elaborated (see e.g., [7, 14, 15, 35, 42, 56, 76]). We mention a few methods related to this thesis. In a number of approaches [1, 4, 7, 42], the underlying sequence of grids is still coarsened in the classical (geometric) way but the interpolation and restriction operators (matrices) depend on a fine grid matrix. For these methods, simple pointwise relaxation steps may fail to provide a robust convergence rate and an ILU-type smoother must be used instead [34]. In other approaches [7, 70, 73], both the interpolation and restriction operators and a coarse grid are matrix-dependent. In contrast to previously mentioned geometric multigrid methods, simple pointwise relaxation steps are sufficient to provide a robust purely algebraic algorithm.

Grauschopf *et al.* [34] presented a detailed numerical analysis of BPX [14], Dendy [1] and AMG [70] multigrid methods. It seems that for the second order elliptic problems, the AMG method provides the most robust preconditioner. However, an efficient parallel implementation of algebraic multigrid methods is still an open question, in contrast to geometric multigrid methods whose parallel properties are usually more transparent. In Chapter 4 we describe and study numerically a multilevel method which has been proposed recently in [44] and may be classified as a geometric multigrid method. The method is characterized by a new coarse grid correction space which is an image space of a low rank matrix. The method is applicable for arbitrary meshes provided that a hierarchical structure of mesh superelements exists. Since the hierarchical structure may be matrix-dependent, the method has some similarities with the AMGe method by Brezina *et al.* [15].

1.3 Thesis outline

The thesis outline is as follows. In Section 2.1 of **Chapter 2** we describe the Biot poroelastic model coupling a solid deformation with fluid processes. The model is mathematically formulated as a coupled system of the Lamé and diffusion equations.

Using the theory of saddle point problems, we prove existence and uniqueness of a weak solution. The problem is discretized by two popular methods. In Section 2.2 we consider mixed finite elements and show solvability of a discrete problem. We use piecewise linear FEs for porous media displacement, lowest order Raviart-Thomas FEs for fluid mass flux and discontinuous piecewise constant FEs for pore pressure. Application of these FEs is restricted to tetrahedral, prismatic or hexahedral meshes. In Section 2.3 we consider a mimetic finite difference discretization based on the SO methodology. The SO methodology enables us to derive discrete equations for arbitrary meshes.

Chapter 3 is devoted to an iterative solution of the discrete problem. In Section 3.1 we formulate a general framework (Lemma 3.1) for deriving block diagonal spectrally equivalent preconditioners for a class of saddle point matrices. Lemma 3.1 states that a saddle point problem can be efficiently preconditioned by a block diagonal matrix with as many blocks as there are physical variables. In Section 3.2 we describe a preconditioner for a discrete Lamé operator. Using the Korn inequality we show that a bilinear form associated with the elastic energy generates a norm in a space of displacements equivalent to the H^1-norm. Therefore, a preconditioner for a discrete elliptic operator will be a preconditioner for the discrete Lamé operator. Section 3.3 is the core of this Chapter. In Lemmas 3.2–3.4 we describe an abstract way of constructing preconditioners for a discrete diffusion operator given in a mixed form. For the case of topologically cubic meshes, Lemmas 3.5 and 3.7 specify abstract blocks in the preconditioner given by Lemma 3.4. The main result of Chapter 3 is summarized in Theorem 3.1. In Section 3.1 we present results of numerical experiment with few preconditioners. We shall show that the Dendy multigrid with a Jacobi smoother is not robust enough for problems with severe jumps in coefficients.

In **Chapter 4** we hybridize the original discrete problem. In Section 4.1 we apply an algebraic hybridization procedure to the diffusion block and reformulate the coupled problem in terms of displacements and Lagrange multipliers. The resulting problem is solved by an iterative method with a 2x2 block diagonal preconditioner. The first block in the preconditioner is the previously elaborated preconditioner for the discrete Lamé operator. The second block is a preconditioner for a Lagrange multiplier block. In Section 4.2 we formulate a new separable preconditioner for this block. The preconditioner can be applied for the case of anisotropic meshes and anisotropic diffusion tensors. However, its construction is restricted to topologically cubic meshes. In Section 4.3 we describe a multilevel preconditioner which works for general meshes but is sensitive to anisotropy of both the mesh and diffusion tensor. We shall prove its optimal arithmetical complexity and study numerically its preconditioning properties.

In **Chapter 5** we analyze convergence of mimetic discretizations for a 2D diffusion equation. There is a lack of theoretical analysis of this type of the discretization. One can extrapolate our analysis to the coupled problem from Section 2.3. Unfortunately, the complete analysis of the coupled problem is a good topic for a separate dissertation and will not be discussed here. In Section 5.1 we recall the basic error estimates for MFE methods on triangular and quadrilateral grids. Using an appropriate quadrature rule to compute the scalar product of two fluxes, we obtain a new proof of the error estimates for the lowest order Raviart-Thomas (RT) finite elements on quadrilateral grids [75, 69]. Then, our error analysis proceeds by formalizing the relationship between the SO discretization and the lowest order RT MFE methods. The main results are formulated in Section 5.2 in Lemmas 5.2 and 5.3, and Theorem 5.4 where we prove optimal error estimates for the SO discretization on both triangular and quadrilateral grids. The validity of the theoretical results is demonstrated with numerical experiments in Section 5.3.

The main results of Chapters 3–5 have been published in [8, 47] and submitted for a journal publication [48].