Hilbert Lattice Equations

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OL, OML, modular OL

An *ortholattice* (OL) is an algebra $\langle A, \vee, \wedge, ' >$ in which the following conditions hold:

$$\begin{aligned} a \vee b &= b \vee a && (1) \\ (a \vee b) \vee c &= a \vee (b \vee c) && (2) \\ a'' &= a && (3) \\ a \vee (a \wedge b) &= a && (4) \\ a \wedge b &= (a' \vee b')' && (5) \end{aligned}$$

An *orthomodular lattice* (OML) is an OL in which the orthomodular law holds:

$$a \vee b = ((a \vee b) \wedge b') \vee b \qquad (6)$$

A *modular* OL is an OL in which the modular law holds:

$$a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c) \qquad (7)$$

$\mathcal{C}(\mathcal{H})$

A *Hilbert space* is a (for us, complex) vector space with an inner product, which is complete in the induced metric.

The set of closed subspaces of a (possibly infinite-dimensional) Hilbert space $\mathcal{H}$ is denoted $\mathcal{C}(\mathcal{H})$. It is a lattice; in particular, it is an orthomodular lattice (OML). This fact provides a primary motivation for studying the properties of OMLs.

$\mathcal{C}(\mathcal{H})$ is also called a *Hilbert lattice*.

$\mathcal{C}(\mathcal{H})$ lattice operations

The orthocomplement a' of a closed subspace a (or actually any $a \subseteq \mathcal{H}$) is the set of vectors orthogonal to all vectors in a:

$$a' \stackrel{\text{def}}{=} \{x \in \mathcal{H} : (\forall y \in a)\ (x, y) = 0\} \tag{8}$$

where $(\cdot, \cdot)$ is the Hilbert vector space inner product. Note that a' is a closed subspace for any $a \subseteq \mathcal{H}$, and a'' (the closure of a) is the smallest closed subspace containing a.

The meet operation is just set-theoretical intersection:

$$a \wedge b \stackrel{\text{def}}{=} a \cap b \tag{9}$$

$\mathcal{C}(\mathcal{H})$ lattice operations (cont.)

Ordering, join, unit, and zero can be defined in terms of these. (We also define *commutes* and *Sasaki implication* for later use.)

$$a \le b \quad \overset{\text{def}}{\Leftrightarrow} \quad a = a \wedge b \quad \Leftrightarrow \quad a \subseteq b \tag{10}$$

$$a \vee b \quad \overset{\text{def}}{=} \quad (a' \wedge b')' = (a{+}b)'' = (a \cup b)'' \tag{11}$$

$$0 \quad \overset{\text{def}}{=} \quad a \wedge a' = \{\mathbf{0}\} = \mathcal{H}' \tag{12}$$

$$1 \quad \overset{\text{def}}{=} \quad 0' = \mathcal{H} \tag{13}$$

$$aCb \quad \overset{\text{def}}{\Leftrightarrow} \quad a = (a \wedge b) \vee (a \wedge b') \quad \text{(commutes)} \tag{14}$$

$$a \rightarrow_1 b \quad \overset{\text{def}}{=} \quad a' \vee (a \wedge b) \quad \text{(Sasaki implication)} \tag{15}$$

where $+$ is subspace sum, $\cup$ is set-theoretical union, and $\mathbf{0}$ is the zero vector. Note that $\mathcal{C}(\mathcal{H})$ itself can be defined as

$$\mathcal{C}(\mathcal{H}) \quad \overset{\text{def}}{=} \quad \{x \subseteq \mathcal{H} : x = x''\} \tag{16}$$

An open problem in $\mathcal{C}(\mathcal{H})$

"An open problem is to determine all equations satisfied by hilbertian lattices [i.e. $\mathcal{C}(\mathcal{H})$], which would make possible the separation of the 'purely logic' part from the above axiomatics. It is not known if this problem is solvable, for it is not certain that these equations form a recursively enumerable set."

—René Mayet, "Varieties of orthomodular lattices related to states," *Algebra Universalis 20* (1985), 368-396

Equations known to hold in $\mathcal{C}(\mathcal{H})$

(See also last year's slide show)

- Orthomodular (OML) law (Husumi, 1937)
- Orthoarguesian (OA) law (Alan Day, 1975)
- Godowski's state-related equations (Godowski, 1981)
- Mayet's state-related equations (Mayet, 1985)
- n-orthoarguesian (n-OA) laws (Megill/Pavičić, 2000)
- The modular law **does not hold** in (∞-dimensional) $\mathcal{C}(\mathcal{H})$

Equations known to hold in $\mathcal{C}(\mathcal{H})$ (cont.)

OML law (Husumi, 1937):

$$((a \vee b) \wedge b') \vee b \;=\; a \vee b \tag{17}$$

OA law (Alan Day, 1975):

$$\begin{aligned} a \wedge (((a \wedge b) \vee ((a \rightarrow_1 d) \wedge (b \rightarrow_1 d))) \vee \\ (((a \wedge c) \vee ((a \rightarrow_1 d) \wedge (c \rightarrow_1 d))) \wedge \\ ((b \wedge c) \vee ((b \rightarrow_1 d) \wedge (c \rightarrow_1 d))))) \;\leq\; b' \rightarrow_1 d \end{aligned} \tag{18}$$

Equations known to hold in $\mathcal{C}(\mathcal{H})$ (cont.)

n-OA example, $n = 5$ (Megill/Pavičić, 2000):

$$
\begin{aligned}
& a \wedge ((((a \wedge b) \vee ((a \to_1 d) \wedge (b \to_1 d))) \vee \\
& (((a \wedge e) \vee ((a \to_1 d) \wedge (e \to_1 d))) \wedge \\
& ((b \wedge e) \vee ((b \to_1 d) \wedge (e \to_1 d))))) \vee \\
& ((((a \wedge c) \vee ((a \to_1 d) \wedge (c \to_1 d))) \vee \\
& (((a \wedge e) \vee ((a \to_1 d) \wedge (e \to_1 d))) \wedge \\
& ((c \wedge e) \vee ((c \to_1 d) \wedge (e \to_1 d))))) \wedge \\
& (((b \wedge c) \vee ((b \to_1 d) \wedge (c \to_1 d))) \vee \\
& (((b \wedge e) \vee ((b \to_1 d) \wedge (e \to_1 d))) \wedge \\
& ((c \wedge e) \vee ((c \to_1 d) \wedge (e \to_1 d))))))) \ \leq \ b' \to_1 d
\end{aligned}
\tag{19}
$$

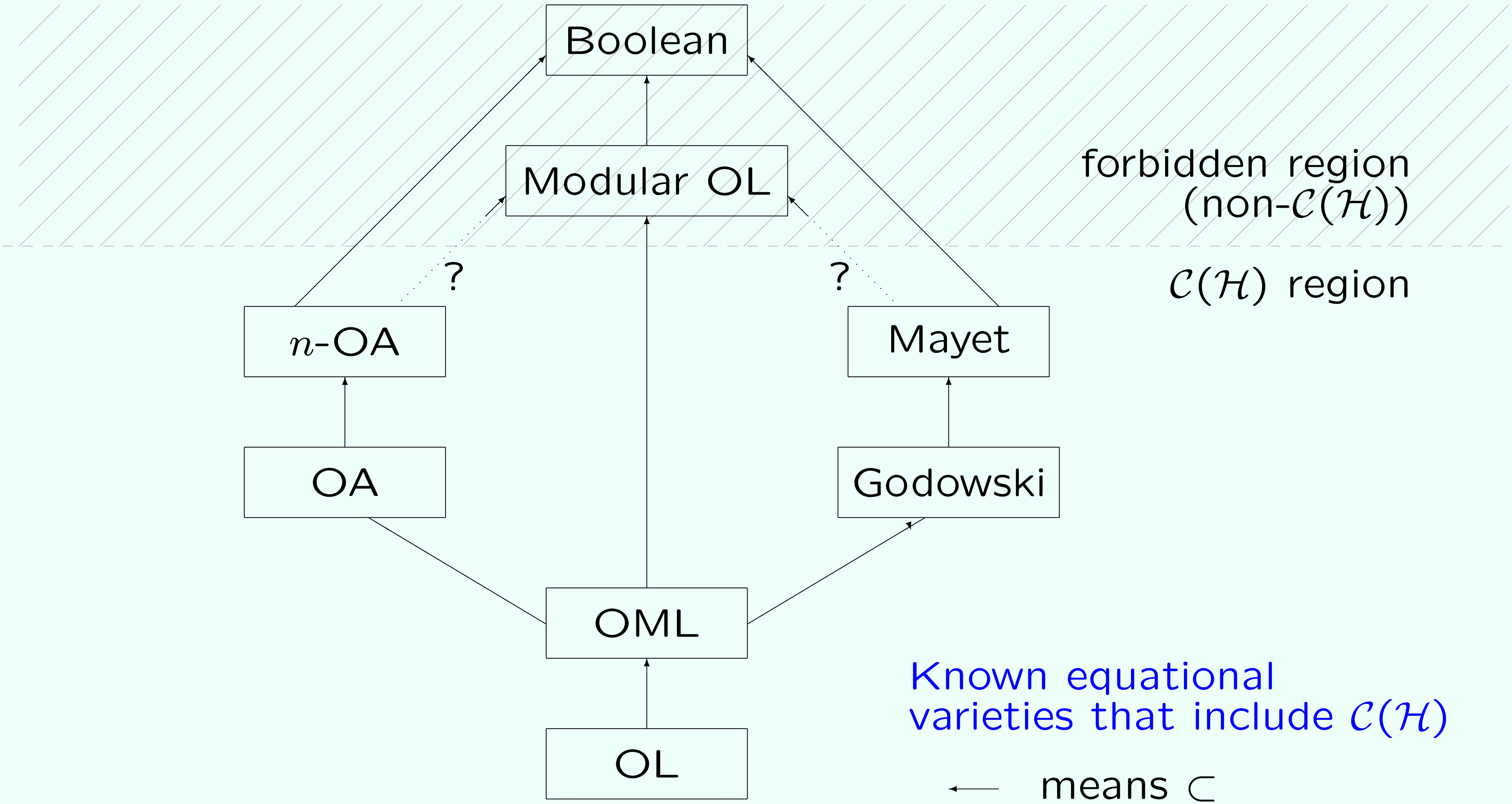
Equations known to hold in $\mathcal{C}(\mathcal{H})$ (cont.)

Example of Godowski equation (Godowski, 1981):

$$(a \to_1 b) \wedge (b \to_1 c) \wedge (c \to_1 a) \ \le \ a \to_1 c \tag{20}$$

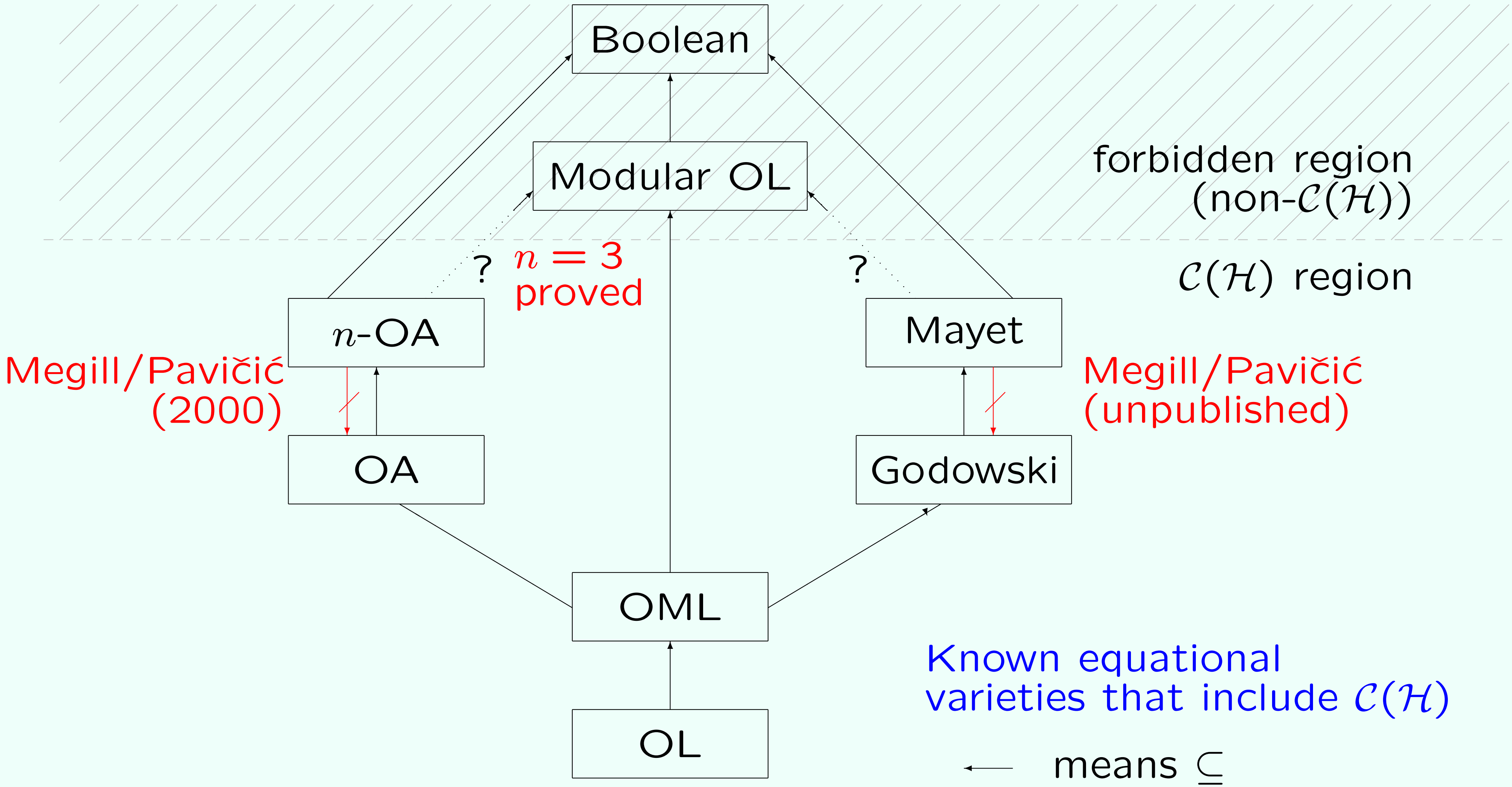
Example of Mayet equation (Megill/Pavičić, unpublished):

$$((a \to_1 b) \to_1 (c \to_1 b)) \wedge (a \to_1 c) \wedge (b \to_1 a) \ \le \ c \to_1 a \tag{21}$$



Known equational varieties that include $\mathcal{C}(\mathcal{H})$

$\longleftarrow$ means $\subset$



Known equational
varieties that include $\mathcal{C}(\mathcal{H})$

$\longleftarrow$ means $\subseteq$

Modular pairs

The *modular pair* relation between two lattice elements a, b, denoted $(a, b)M$, is defined as

$$(a, b)M \overset{\text{def}}{\Leftrightarrow} (\forall x)\ [x \le b \ \Rightarrow \ x \vee (a \wedge b) = (x \vee a) \wedge b] \quad (22)$$

The *dual modular pair* relation between two lattice elements a, b, denoted $(a, b)M^*$, is defined as

$$(a, b)M^* \overset{\text{def}}{\Leftrightarrow} (\forall x)\ [x \ge b \ \Rightarrow \ x \wedge (a \vee b) = (x \wedge a) \vee b] \quad (23)$$

M-symmetry

The set of closed subspaces of infinite-dimensional Hilbert space, $\mathcal{C}(\mathcal{H})$, has a remarkable property: it is *M-symmetric*.

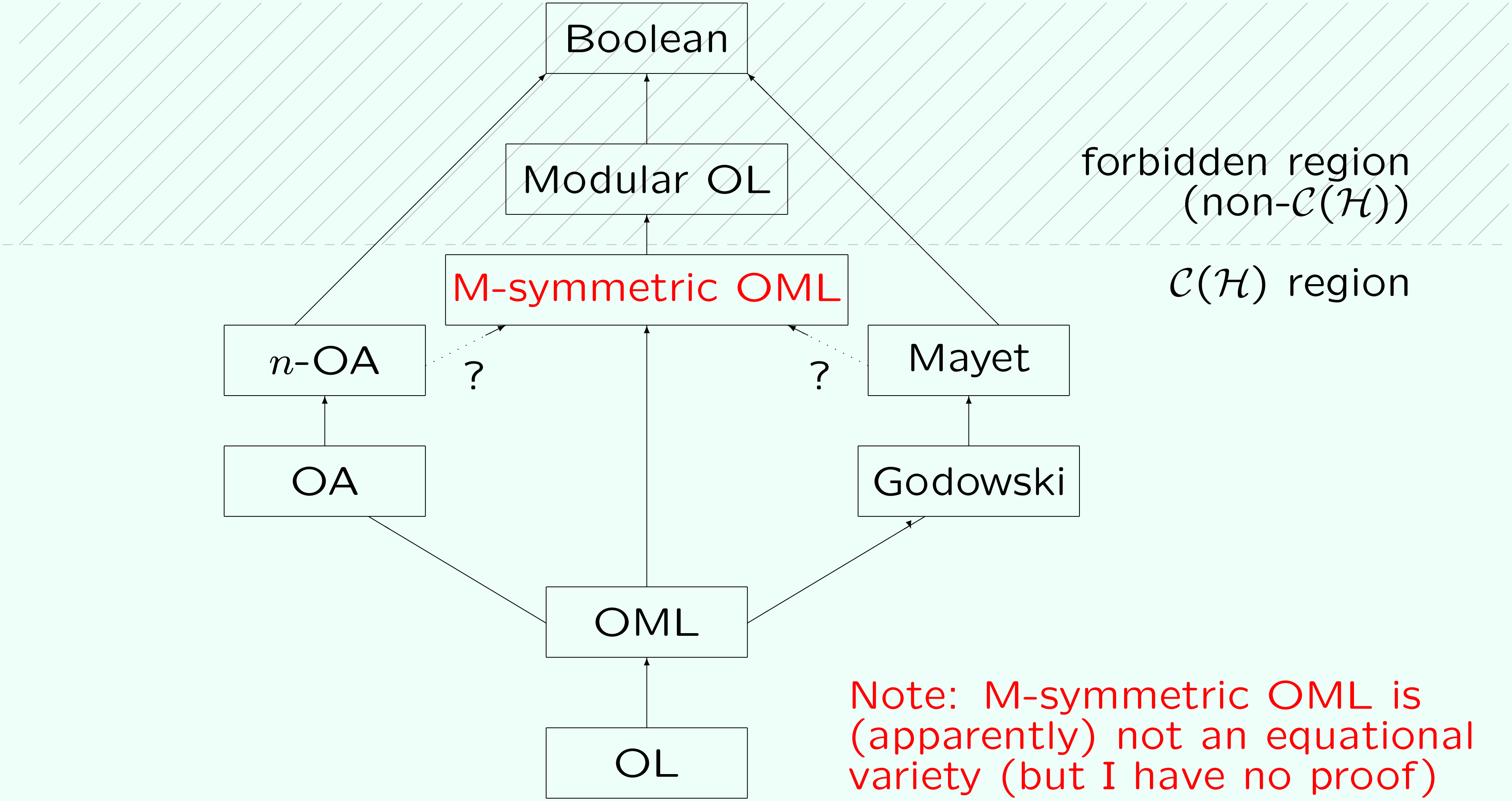
$$(a,b)M \quad \Leftrightarrow \quad (b,a)M \tag{24}$$

Other related symmetry properties also hold:

$$(a,b)M^* \quad \Leftrightarrow \quad (b,a)M^* \quad (\textit{M}^*\textit{-symmetric}) \tag{25}$$

$$(a,b)M \quad \Leftrightarrow \quad (b',a')M \quad (\textit{O-symmetric}) \tag{26}$$

$$(a,b)M \quad \Leftrightarrow \quad (b,a)M^* \quad (\textit{cross-symmetric}) \tag{27}$$



Note: M-symmetric OML is (apparently) not an equational variety (but I have no proof)

Can M-symmetry help us find a new $\mathcal{C}(\mathcal{H})$ equation?

The M- (or M*-)symmetry property is not an equation, because in prenex normal form it has an existential quantifier. To obtain an equation, one possible approach (that we are investigating) is to find a quantifier-free expression (polynomial equations connected with 'and') $E(a, b, \ldots)$ s.t.

$$E(a, b, \ldots) \;\Rightarrow\; (b, a)M^* \tag{28}$$

holds in OML (or in some other known $\mathcal{C}(\mathcal{H})$ condition). Then

$$E(a, b, \ldots) \;\Rightarrow\; (a, b)M^* \tag{29}$$

will also hold in $\mathcal{C}(\mathcal{H})$ and (after removal of $(a, b)M^*$ quantifier) will be an equational inference that holds in $\mathcal{C}(\mathcal{H})$, hopefully stronger than the first condition.

Extending M*-symmetry

A detailed analysis of Maeda's M*-symmetry proof reveals that it actually proves something subtly "stronger."

The M*-symmetry proof makes use of the fact that $\mathcal{C}(\mathcal{H})$ is a *relatively atomic* lattice satisfying the *exchange axiom*.

So we need more definitions...

More definitions

A lattice element b *covers* a, written $a \lessdot b$, if a is less than b and there is nothing in between. An *atom* is a lattice element that covers 0.

$$a \lessdot b \overset{\text{def}}{\Leftrightarrow} a < b \;\&\; \neg(\exists x)\; a < x < b \tag{30}$$

$$p \text{ is an atom} \overset{\text{def}}{\Leftrightarrow} 0 \lessdot p \tag{31}$$

$\mathcal{C}(\mathcal{H})$ is *relatively atomic*:

$$a < b \;\Rightarrow\; (\exists \text{ an atom } p)\; a < a \vee p \leq b \tag{32}$$

and satisfies the *exchange axiom*:

$$a \wedge b \lessdot b \;\Leftrightarrow\; a \lessdot a \vee b \tag{33}$$

Extending M*-symmetry (cont.)

The hypothesis of the M*-symmetry proof does not require the full strength of $(b,a)M^*$ (Eq. 23) to get $(a,b)M^*$, but only the following "weaker" condition:

$$(\forall \text{ atoms } p)\ [p \geq a \ \Rightarrow\ p \wedge (b \vee a) = (p \wedge b) \vee a] \tag{34}$$

Thus we can exploit special properties of atoms in an attempt to prove the condition of Eq. 28. For example, using

$$p \not\leq a \vee b \ \Rightarrow\ (a \vee p) \wedge (a \vee b) = a \tag{35}$$

we can weaken Eq. 28 (that we want to prove) into the following form, where p is (optionally) an atom and cannot occur in $E(a,b,\ldots)$:

$$E(a,b,\ldots)\ \&\ a \leq p \leq a \vee b \ \Rightarrow\ p \leq (p \wedge b) \vee a \tag{36}$$

Extending M*-symmetry (cont.)

Among other properties available to exploit, there is an extensionality-like law for eliminating atoms:

$$(\forall \text{ atoms } p)\ (p \le a \ \Rightarrow\ p \le b) \quad \Leftrightarrow \quad a \le b \tag{37}$$

and a closure law for modular pairs that allows working directly in Hilbert space with vectors and subspace sums:

$$(a,b)M \quad \Leftrightarrow \quad a{+}b = a \vee b \tag{38}$$

Also, an atom p forms a modular pair with any other lattice element:

$$(a,p)M,\ (p,a)M,\ (a,p)M^*,\ \text{etc.} \tag{39}$$

Our current working conjecture

We (myself and Mladen Pavičić) are trying to prove or disprove that this condition holds in all OMLs:

$$\begin{aligned} aCc \ \& \ c\wedge d\le a \ \& \ b\le d \ \& \ b\le (a\wedge d)\vee c \ \& \\ p\le a\vee b \ \& \ a\le p \ \Rightarrow \ p\le (p\wedge b)\vee a \end{aligned} \tag{40}$$

So far we haven't found a counterexample.

On the other hand, the conclusion using M*-symmetry,

$$\begin{aligned} aCc \ \& \ c\wedge d\le a \ \& \ b\le d \ \& \ b\le (a\wedge d)\vee c \ \& \\ \& \ b\le e \ \Rightarrow \ e\wedge (a\vee b)\le (e\wedge a)\vee b, \end{aligned} \tag{41}$$

fails in almost all non-modular OMLs and appears much stronger than, and certainly independent from, any known Hilbert lattice equation.

Our current working conjecture (cont.)

Motivation. The following informal heuristic motivates our conjecture. Trivially, we have

$$a = b \ \Rightarrow \ (b,a)M^* \tag{42}$$

The general idea is to disconnect $a = b$, then reconnect a and b via a modular or modular-like law.

$$\begin{aligned} a = (\text{l.h.s. of modular law}) \ \& \ (\text{r.h.s. of modular law}) = b \\ \Rightarrow \quad (b,a)M^* \end{aligned} \tag{43}$$

If we select the direction and form of the modular law "just right," the hope is that the strength of the hypotheses will effectively "cancel" the strength of $(b,a)M^*$, but won't "cancel" the strength of $(a,b)M^*$. Very roughly, we observe this kind of bias with simple lattice experiments. By trial-and-error we added the additional hypothesis aCc in Eq. 40 to get Eq. 43 to pass in known OMLs.

Our current working conjecture (cont.)

Eq. 40 is equivalent to the following closed form:

$$p \wedge (a \vee ((((a \wedge d) \vee (c \wedge (a \vee (c \wedge a')))) \wedge d) \wedge b)) \\ \leq ((((c \wedge d) \vee a) \vee p) \wedge b) \vee ((c \wedge d) \vee a) \tag{44}$$

Expressed as a Mace4 problem:

```
% Use as the denial in the Mace4 example "nonmodular-oml.in"
(E ^ (A v ((((A ^ D) v (C ^ (A v (C ^ c(A))))) ^ D) ^ B)))
v (((((C ^ D) v A) v E) ^ B) v ((C ^ D) v A))
!= ((((C ^ D) v A) v E) ^ B) v ((C ^ D) v A).
```

References

Most of the references for this material can be found at:

`http://us.metamath.org/qlegif/mmql.html#ref`

More miscellaneous stuff can be found at:

`http://us.metamath.org/award2003.html`