Hilbert space representations of a q-deformed Minkowski algebra

Dissertation der Fakultät für Physik
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Bianca Letizia Cerchiai
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Zusammenfassung

In dieser Arbeit wird ein q-deformierter Minkowski-Raum untersucht und eine Hilbertraumdarstellung der zugehörigen Vertauschungsrelationen konstruiert.

Die q-deformierte Poincaré-Hopf-Algebra (ko-)wirkt als Symmetriealgebra auf diesem Raum, und es ist nicht möglich, den q-deformierten Phasenraum von der q-deformierten Lorentz-Algebra zu trennen, weil die Vertauschungsrelationen der Orts- und Impulsvariablen eine Realisierung der q-Lorentz-Algebra enthalten.

Die Unteralgebra, die von den q-deformierten Rotationen gebildet wird, wird in dieser Darstellung der q-deformierten Lorentz-Algebra identifiziert. Wie im undeformierten Fall kommutieren die q-deformierte Rotationen mit der Zeitkoordinate und bilden die kleine Gruppe des Ruhesystems.

Dies ermöglicht die Konstruktion einer Spin-0-Darstellung der Minkowski-Algebra und der Poincaré-Algebra. Es handelt sich dabei um eine unendlich-dimensionale unitäre Darstellung. Die Zeit X^0, die invariante Länge $X \cdot X$, der Casimir-Operator der Rotationen und die dritte Komponente des Drehimpulses bilden einen vollständigen Satz von Operatoren, mit denen man die Hilbertraumzustände beschreiben kann.

Es stellt sich heraus, daß diese Operatoren ein rein punktuelles (diskretes) Spektrum mit abzählbar vielen Eigenwerten aufweisen. Deshalb wird die Theorie automatisch auf einem Gitter dargestellt und es besteht die Hoffnung, daß die Feldtheorie, die auf diesem Quantenraum entwickelt werden kann, bei kleinen Abständen nicht die Divergenzen der üblichen Feldtheorie zeigt.

Man hat eine Gitterstruktur mit exponentiell wachsenden Gitterabständen. In diesem Phasenraum treten Lichtkegelkoordinaten in natürlicher Weise auf.

Die Eigenwerte von X^3 werden auch ausgerechnet. Man sieht, daß sie unsymmetrisch bezüglich eines Vorzeichenwechsels sind. Um dieselbe Anzahl von positiven und negativen Eigenwerten zu haben, muß man zwei verschiedene Darstellungen addieren.

Wenn man die Ortskoordinaten und die Impulsvariablen gleichzeitig betrachtet, ist es nicht möglich, eine Darstellung auf dem Lichtkegel zu finden. Er enthält aber Akkumulationspunkte der Spektren von X^0 und $X \cdot X$.

Abstract

In this thesis a q-deformed Minkowski space is studied and a Hilbert space representation of it is constructed.

The symmetry Hopf algebra (co-)acting on this phase-space is the q-deformed Poincaré Hopf algebra. The q-deformed Minkowski space cannot be separated from the q-deformed Lorentz algebra, because a realization of the latter appears in the commutation relations of the position coordinates with the momentum observables.

The q-deformed rotations are identified as a subalgebra of this realization of the q-deformed Lorentz algebra. As in the undeformed case the q-deformed rotations commute with the time observable and make up the stability group of the rest frame.

Using this fact, a spin-0-representation of the Minkowski phase-space together with the Poincaré algebra is constructed. It is a unitary infinite-dimensional representation. As a complete set of commuting observables the time component X^0, the invariant length $X^2 = X \cdot X$, the Casimir operator of the rotations and the third component of the angular momentum are chosen.

It turns out that the spectrum of the observables is discrete, so that the theory is automatically confined on a lattice. This gives rise to the hope that the field theory constructed on this quantum space would be more regular at short distances than ordinary field theory and that this method provides a good regularization procedure.

The distance between the points of the lattice grows exponentially. The lattice naturally admits light-cone coordinates.

The eigenvalues of X^3 are also determined. It can be seen that they are asymmetric with respect to a change of sign. In order to have the same number of positive and negative eigenvalues two different representations have to be added.

If the momentum is taken into account, there is no representation on the light-cone, but the spectrum of X^0 and of the spatial radius have accumulation points on the latter.

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Chapter 1

Introduction

Symmetries, and in particular symmetries described by groups, are one of the most important concepts in physics today. One very striking application of this is the description due to Wigner [47] of the free particles in field theory as unitary irreducible representations of the Poincaré group, which is regarded as the symmetry group of the Minkowski space-time.

Quantum groups [7, 17, 37] are a mathematically well-defined way of deforming and generalizing the idea of a symmetry (Lie) group in the framework of non-commutative geometry. Deformation is a procedure which is very common in physics, the most notable example being the transition from classical to quantum mechanics by turning the Poisson brackets into commutators. In fact, the name quantum groups suggests that they can be obtained through such a quantization process from Lie groups.

There are different ways of introducing quantum groups. The main idea is that it is possible to recover all the informations about a group manifold just by studying the C^*-algebra of continuous complex-valued functions on it, which turns out to be a commutative Hopf algebra [1].Via an application of the Gelfand-Naimark theorem [14] the manifold is reconstructed as the spectrum of the algebra. Quantum groups can be introduced as deformations of this structure [48, 49], in such a way that it becomes non-commutative, but still remains a Hopf algebra. In a different approach, which turns out to be dual to the previous one and is equivalent to it, quantum groups are constructed as deformations of the enveloping algebra of the Lie algebra corresponding to the group [7, 17]. Notice that quantum groups are not really groups, because the Gelfand-Naimark theorem fails in the non-commutative case.

Quantum groups have interesting applications in physics. Originally they were developed as symmetries underlying the integrability of certain lattices in statistical mechanics [44].

But the most important observation is that quantum groups are Hopf algebras and this allows the construction of objects on which they act as symmetries, such as e.g. the quantum spaces. First, the quantum plane has been introduced

[23, 24] and in the sequel the quantum Euclidean space [37] and the q-deformed Minkowski space [2, 4, 39, 26, 27]. On these spaces it is possible to define a differential calculus [46, 3, 10, 45].

There are still some problems in the study of quantum spaces. Notably, the inhomogeneous quantum groups cannot be equipped with a $*$-Hopf algebra structure, because the $*$-operation does not commute with the co-product, that part of the Hopf algebra structure which allows the construction of tensor products of representations and hence the definition of many-particle states and of the center of mass frame.

An extra scalar generator which acts as a dilaton has to be introduced [30], making it necessary to extend the algebra to the conformal one.

Also, Connes [6] has developed a standard set of tools in his approach to non-commutative geometry to explicitly reconstruct all the geometric characteristics of non-commutative manifolds starting from the algebraic properties of the latter. It has not yet been possible to apply completely this scheme to quantum spaces, because the operators defining the quantum groups do not belong to any of the ideals for which it is valid.

Nevertheless quantum spaces are very promising from most other points of view, especially as a tool to solve the inconsistencies which appear in field theory at short distances and which lead to the necessity of renormalization. The most interesting characteristic of quantum spaces is that in the representation theory of the corresponding $*$-algebras the spectrum of the observables is discrete. This is true in one dimension [41, 15, 40], as well as in higher dimensions [16, 45, 11, 12, 32]. This means that the theory is automatically forced on a lattice and that a cut-off is introduced in a natural way at short distances. It should be expected that the corresponding field theory needs no further regularization and no ultraviolet divergences should appear. Space-time itself is quantized in such a way that the propagators are always well-defined.

Quantum spaces give rise to lattices in which the distance between the points grows exponentially. Such lattices, unlike usual quadratic lattices, preserve the invariance under the action of a symmetry Hopf algebra and therefore the corresponding theories are solvable. Quantum groups can be interpreted as dynamical or hidden symmetries of the models defined on them [22] as well. At least in the 1-dimensional case it is possible to express the generators of the deformed phase-space in terms of the generators of the undeformed one [9] and therefore quantum groups can be used to construct quantum mechanical models which are easily solved when expressed in the deformed variables, but are very complicated when written in the undeformed ones [9, 19]. Embeddings of the deformed algebras in the undeformed ones are also studied in [43, 13].

Unlike in ordinary quantum mechanics quantum spaces admit a whole family of inequivalent representations depending on a continuous parameter. This parameter sets the scale of the lattice. It is a phenomenon very similar to spontaneous symmetry breaking. Space-time would exist in a phase in which it is continuous when the deformation parameter $q = 1$ and in a phase in which it is on a lattice when $q \neq 1$, and a phase-transition should occur e.g. at the Planck scale. The particular symmetry breaking mechanism should determine the dependence of the parameter fixing the scale on the deformation parameter q of the quantum group. However, the symmetry breaking mechanism cannot be described in the framework of this theory, but should be determined by a more general theory incorporating this one.

The harmonic oscillator, the simplest but the most important model, has already been solved [42, 20]. This gives rise to the hope that it will be possible to construct a consistent field theory on quantum spaces.

The four-dimensional Minkowski space is particularly interesting, because it is the basis to construct field theory. At the moment various types of deformed Lorentz and Poincaré algebras are actively studied by groups in Munich [27, 21], in Warsaw [36, 35, 34], in Wroclaw [18], in Italy [5]. Each group has developed its own version and uses a different approach. Quantum mechanics on this space has been studied in [25]. Some results on the representation theory of the Munich q-Poincaré algebra have been developed in [31, 32, 29, 28, 33].

This thesis deals with the representation theory of the Minkowski phase-space and of the Poincaré algebra acting on it. It is structured in the following way.

Chap. 2 contains a definition of the algebra describing the q-deformed Minkowski phase-space and the q-deformed Poincaré algebra (sec. 2.1,2.2), as it is given in [27, 21]. It is interesting to notice that, unlike ordinary spaces, the quantum phase-space cannot be separated from the symmetry acting on it, because a realization of the q-deformed Lorentz algebra appears in the commutation relations of the position coordinates with the momentum observables. In order to study the representation theory, the first step is to identify the q-deformed rotations as a subalgebra of this realization of the q-deformed Lorentz algebra. A good Ansatz is to look for quantities which commute with the time coordinate (sec. 2.3). Then it is checked that these quantities effectively realize the commutations relations of $SO_q(3)$ and their action on the coordinates (sec. 2.4) and on the generators R^A,S^A and U of the q-deformed Lorentz algebra (sec. 2.5) is found.

In chap. 3 a representation of the position coordinates is constructed. As a complete set of commuting observables the time component X^0, the invariant length $X^2 = X \cdot X$, the Casimir operator of the rotations and the third component of the angular momentum are chosen. The eigenvalues of X^0 and $X^2 = X \cdot X$ are determined and the eigenvalues of time are plotted versus those of the spatial radius (sec. 3.3.2). The results are compared with those which were previously known [32]. In complete analogy with the lower-dimensional cases [41, 15, 9, 45, 11, 12] the spectrum of the operators is discrete and there is a continuous family of inequivalent representations. Then the matrix elements of the coordinates are computed (sec 3.4). The eigenvalues of X^3 are also found and plotted (sec. 3.5). It can be seen that they are asymmetric with respect to a change of sign. In order to have the same number of positive and negative eigenvalues two different representations have to be added, in other words the forward and backward cone are both needed.

In chap. 4 the matrix elements of the generators of the q-deformed Lorentz algebra are calculated.

In chap. 5 a representation of the momentum coordinates P^a is constructed. It is particularly interesting that when the momentum is introduced, it is no longer possible to give a representation of the algebra on the light-cone. Nevertheless, X^0 and the spatial radius have accumulation points on it

It would be interesting to continue this study. The Fourier transform of the energy and of P^2 have to be studied. As in the 1-dimensional case [15, 40] it has to be expected that they are not self-adjoint operators and that it is necessary to find a self-adjoint extension of them. The Klein-Gordon and Dirac equations should be investigated and the propagator should be calculated, e.g for the free relativistic particle. An embedding of the deformed phase-space in the undeformed should be found in order to compare the results with the undeformed ones.

Chapter 2

The q-deformed Minkowski space

2.1 Definition of the algebra

The construction of the q-deformed Minkowski phase-space is based on its symmetry, namely the q deformed Poincaré algebra and has been performed in a long series of papers starting at the beginning of the '90 [27, 21, 2, 4, 39, 26].

Starting point are the $\hat{R}$-matrices defining the q-deformed Lorentz algebra: $\hat{R}_I$ and $\hat{R}_{II}$. In the appendix A their explicit expressions and their projector decomposition are given, together with the expressions for η and ε, which play the roles of a q-deformed metric and ε-tensor respectively. Notice that throughout this work indices with capital letters $A, B, \ldots$ can take the values $+, -, 3$, whereas indices with small letters $a, b, \ldots$ can take the values $0, +, -, 3$.

The q-deformed Minkowski phase-space is defined in ref. [21] as the algebra generated by the elements X^a, P^a, V^{ab}, U, $\Lambda^{\frac{1}{2}}$, $\Lambda^{-\frac{1}{2}}$ divided by the ideal $\mathcal{I}$ given by the following relations:

$$\begin{aligned} X^0 X^A &= X^A X^0 \\ \varepsilon_{CB}{}^A X^B X^C &= (1-q^2) X^0 X^A \end{aligned} \tag{2.1}$$

$$\begin{aligned} P^0 P^A &= P^A P^0 \\ \varepsilon_{CB}{}^A P^B P^C &= (1-q^2) P^0 P^A \end{aligned} \tag{2.2}$$

$$P_A{}^{rs}{}_{ad} \eta_{bc} V^{ab} V^{cd} = \frac{1}{1+q^2} U V^{rs} \tag{2.3}$$

$$V^{ab} U = U V^{ab} \tag{2.4}$$

$$V^{ab} X^f = P_A{}^{ab}{}_{cd} X^c \left\{-(q+q^{-1}) V^{df} + q^{-1} \eta^{df} U\right\} \tag{2.5}$$

$$V^{ab} P^f = P_A{}^{ab}{}_{cd} P^c \left\{-(q+q^{-1}) V^{df} + q^{-1} \eta^{df} U\right\} \tag{2.6}$$

$$
\begin{aligned}
UX^a &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^aU - \frac{1}{2q}(q^2-1)^2\eta_{bc}X^bV^{ca} \\
UP^a &= \frac{1}{q}\frac{q^4+1}{q^2+1}P^aU - \frac{1}{2q}(q^2-1)^2\eta_{bc}P^bV^{ca}
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
P^aX^b - q^{-2}\hat{R}_{II}^{-1\,ab}{}_{cd}X^cP^d = \\
-\tfrac{i}{2}\Lambda^{-1/2}\left\{(1+q^4)\eta^{ab}U + q^2(1-q^4)V^{ab}\right\}
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
\Lambda^{-1/2}X^a &= qX^a\Lambda^{-1/2} \\
\Lambda^{-1/2}P^a &= q^{-1}P^a\Lambda^{-1/2} \\
\Lambda^{-1/2}V^{ab} &= V^{ab}\Lambda^{-1/2} \\
\Lambda^{-1/2}U &= U\Lambda^{-1/2}
\end{aligned}
\tag{2.9}
$$

Observe that it is not possible to close the algebra generated by X,P alone, e.g. because in the XP-commutation relation (2.8) the tensor V^{ab} and the scalar U appear. It will be shown that V^{ab} provides a representation of the q-deformed Lorentz algebra in the Minkowski space. This means that the symmetry acting on the space naturally arises from the space itself and the phase-space is no longer decoupled from the symmetry acting on it.

In [21] it is shown that $U \to 1$ in the limit $q \to 1$, so that in the undeformed case it reduces to the identity.

Further, it can be noticed that a generator Λ appears. From eqns. (2.9) it can be deduced that this operator acts as a scalar on X and P. This suggests that in reality the underlying symmetry algebra is a conformal algebra. In the limit for $q \to 1$ it is easily seen [30] that $\Lambda \to 1$ so that it disappears.

Eqn.(2.3) shows that, apart from the presence of the scalar U, the V-tensor is an eigenvector of the antisymmetric projector, therefore it has only 6 independent generators. (This should be expected, because the Lorentz algebra has six generators.) In some cases it can be more handy to decompose V^{ab} into its selfdual and antiselfdual components:

$$
R^A = P_+{}^{A0}{}_{cd}V^{cd} = -\frac{1}{q^4-1}(V^{A0} + q^2V^{0A}) \tag{2.10}
$$

$$
S^A = q^{-2}P_-{}^{A0}{}_{cd}V^{cd} = \frac{1}{q^4-1}(q^2V^{A0} + V^{0A})
$$

The inverse relation is:

$$
\begin{aligned}
V^{A0} &= R^A + q^2S^A \\
V^{0A} &= -q^2R^A - S^A \\
V^{AB} &= \varepsilon_C{}^{AB}(R^C - S^C) \\
V^{00} &= 0
\end{aligned}
\tag{2.11}
$$

The following star operation is compatible with the algebra:

$$\begin{aligned} \overline{X^0} &= X^0, \qquad \overline{X^A} = g_{AB} X^B \\ \overline{P^0} &= P^0, \qquad \overline{P^A} = g_{AB} P^B \\ \overline{R^A} &= -g_{AB} S^B, \qquad \overline{U} = U \\ \overline{\Lambda^{1/2}} &= q^4 \Lambda^{-1/2} \end{aligned} \tag{2.12}$$

An extra relation which is postulated and can be shown to be consistent with the previous relations is:

$$U^2 - 1 = \frac{1}{2}(q^4 - 1)^2 (R \circ R + S \circ S) \tag{2.13}$$

where the symbol $\circ$ is the three-dimensional scalar product (as defined in eqn. (A.3) of the appendix A). This shows that U is an extra generator which is a root of the Casimir $1 + \frac{1}{2}(q^4 - 1)^2 (R \circ R + S \circ S)$ of the algebra generated by R, S. Further from the equations (2.3-2.8) it follows that:

$$X^a \varepsilon_{abcd} V^{cd} = 0 \tag{2.14}$$

$$P^a \varepsilon_{abcd} V^{cd} = 0$$

This again has as a consequence that:

$$\varepsilon_{abcd} V^{ba} V^{cd} = 0 \tag{2.15}$$

Such a relation characterizes the particular representation of the Lorentz algebra acting on the Minkowski space.

2.2 The algebra in terms of R and S

In terms of R,S the VV relations (2.3) become

$$\begin{aligned} \varepsilon_{DA}{}^{K} R^A R^D &= \frac{1}{1+q^2} U R^K \\ \varepsilon_{DA}{}^{K} S^A S^D &= -\frac{1}{1+q^2} U S^K \\ R^A S^B &= q^2 \hat{R}^{AB}{}_{CD} S^C R^D \end{aligned} \tag{2.16}$$

The $\hat{R}$-matrix appearing in this equation is the $\hat{R}$-matrix of $SO_q(3)$ and is explicitly given in eqn. (A.1) of Appendix A. Eqns.(2.16) show that R,S are constructed in such a way as to transform like vectors under the rotations. [1] Through application of the expression (A.16) the last identity in (2.16) can be written as:

$$R^A S^B = q^2 S^A R^B - \frac{1}{q^2}\varepsilon_L{}^{AB}\varepsilon_{CD}{}^L S^D R^C - \frac{1}{q^2}(q^2-1)g^{AB} S\circ R \tag{2.17}$$

and its inverse as:

$$S^A R^B = \frac{1}{q^2} R^A S^B - \frac{1}{q^2}\varepsilon_L{}^{AB}\varepsilon_{CD}{}^L R^D S^C + (q^2-1)g^{AB} R\circ S \tag{2.18}$$

The VU-relation (2.4) is:

$$UR^A = R^A U, \qquad US^A = S^A U \tag{2.19}$$

The extra relation (2.14) becomes:

$$\begin{aligned} g_{AB}X^A(R^B - q^2 S^B) &= 0 \\ X^0(S^A - q^2 R^A) - \varepsilon_{CB}{}^A X^B(R^C + S^C) &= 0 \end{aligned} \tag{2.20}$$

which imply:

$$R\circ R = S\circ S \tag{2.21}$$

This relation is very important, because it means that the two Casimir elements $R\circ R$ and $S\circ S$ are no longer distinct, so that they can be identified with only one Casimir element U^2 by (2.13), which becomes:

$$U^2 - 1 = (q^4-1)^2 R\circ R = (q^4-1)^2 S\circ S \tag{2.22}$$

That an extra relation of the kind (2.21) for the Casimir operators holds in the realization in terms of R, S of the q-deformed Lorentz algebra, had to be expected. In fact, eqn. (2.8) allows to interpret V^{ab} as orbital angular momentum and to express it in terms of P,X, and not all the representations of the q-deformed Lorentz algebra admits such an interpretation. Eqn. (2.21) guarantees that this is consistent. It means that the representations of the two subalgebras generated by S, U and R, U in the Minkowski space have always the same dimension.

Notice that (2.21) could alternatively be obtained by means of the conjugation properties (2.12) of R^A and S^A and is necessary for them to be consistent.

From (2.5) the RX-relations follow:

$$\begin{aligned} R^A X^0 &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^0R^A + \frac{1}{q}\frac{q^2-1}{q^2+1}\varepsilon_{LM}{}^A X^M R^L - \frac{q}{(1+q^2)^2}X^A U \\ R^A X^B &= \frac{1}{1+q^2}\Big[q(1+q^2)X^A R^B - \frac{1}{q}(q^2-1)\varepsilon_C{}^{AB}X^0R^C \\ &- \frac{1}{q}(q^2-1)g^{AB}g_{MC}X^M R^C - \frac{2}{q}\varepsilon^{ABG}\varepsilon_{STG}X^T R^S \\ &- \frac{1}{q}\frac{1}{1+q^2}g^{AB}X^0U + \frac{1}{q}\frac{1}{1+q^2}\varepsilon_M{}^{AB}X^M U\Big] \end{aligned} \tag{2.23}$$

[1] This will appear more clearly in sect. 2.6, where the action of the rotations on R and S is explicitly derived.

Analogously for the SX-relations it holds:

$$\begin{aligned} S^A X^0 &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^0 S^A + \frac{1}{q}\frac{q^2-1}{q^2+1}\varepsilon_{LM}{}^A X^M S^L - \frac{1}{q(1+q^2)^2}X^A U \\ S^A X^B &= \frac{1}{1+q^2}\Big[\frac{1}{q}(1+q^2)X^A S^B - \frac{1}{q}(q^2-1)\varepsilon_C{}^{AB}X^0 S^C \\ &+ q(q^2-1)g^{AB}g_{MC}X^M S^C - \frac{2}{q}\varepsilon^{ABG}\varepsilon_{STG}X^T S^S \\ &- \frac{q}{1+q^2}g^{AB}X^0 U - \frac{1}{q}\frac{1}{1+q^2}\varepsilon_M{}^{AB}X^M U\Big] \end{aligned} \tag{2.24}$$

Notice that only an explicit use of (2.20) allows to obtain (2.23) and (2.24), otherwise R and S would be mixed.

The RP and thr SP-relations are the same as the RX and the SX-relations respectively, when X is substituted by P.

In terms of R,S the UX-relations are:

$$\begin{aligned} UX^0 &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^0 U - \frac{1}{q}(q^2-1)^2 X \circ R \\ UX^A &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^A U - q(q^2-1)^2 X^0 R^A \\ &-\frac{1}{q}(q^2-1)^2\varepsilon_{CB}{}^A X^B R^C \end{aligned} \tag{2.25}$$

By using (2.20) these relations can be written in the equivalent form:

$$\begin{aligned} UX^0 &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^0 U - q(q^2-1)^2 X \circ S \\ UX^A &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^A U - \frac{1}{q}(q^2-1)^2 X^0 S^A \\ &+\frac{1}{q}(q^2-1)^2\varepsilon_{CB}{}^A X^B S^C \end{aligned} \tag{2.26}$$

The UP-relations are the same as the UX-relations, where X is substituted by P.

As in the sequel the relations appearing in this section are often used, in appendix B many of them are written out explicitly.

2.3 Identification of the rotations

In the Euclidean case the tensor appearing on the right-hand side of the PX-commutation relations contains a term of the form $\varepsilon_C{}^{AB}L^C$ [16, 45, 19, 21], where L^C is related to the angular momentum. Therefore it provides a realization of $SO_q(3)$. Analogously it can be expected that the tensor V^{ab} realizes the q-deformed Lorentz-algebra in the Minkowski space. Finding the explicit expressions of the generators of the q-deformed Lorentz algebra in terms of R and S is interesting in view of the representation theory. In particular, it is necessary to identify the $SO_q(3)$-subalgebra, because as in the undeformed case it is the stability algebra of any four-momentum with non-vanishing square.

Classically the rotations commute with X^0, so that a good Ansatz is to find expressions in R,S commuting with X^0 and which tend to the undeformed angular momentum when $q \to 1$.

The first idea would be to try a linear Ansatz, e.g. $L^A \sim \varepsilon_{CB}{}^A V^{BC} \sim R^A - S^A$, because by construction it transforms like a vector and it has the right classical limit. Unfortunately, eqns. (2.23) immediately show that this linear Ansatz does not work: it is not possible to take into account the extra generator U.

A better Ansatz, which is quadratic, could be of the form $L^A \sim U(R^A - S^A)$. This also transforms like a vector and has the right classical limit, because $U \to 1$ for $q \to 1$. But the RR and SS-commutation relations (2.16) show that this is equivalent to considering terms of the form $\varepsilon_{CB}{}^A R^B R^C$ and $\varepsilon_{CB}{}^A S^B S^C$ and this suggests that it is necessary to consider also terms of the form $\varepsilon_{CB}{}^A R^B S^C$. At last an Ansatz which works is of the form:

$$L^A = \alpha U(R^A - S^A) + \gamma Z^A \tag{2.27}$$

with

$$Z^A = \varepsilon_{BC}{}^A R^C S^B \tag{2.28}$$

Now, it is possible to determine the constants α and γ in such a way that:

$$L^A X^0 = X^0 L^A \tag{2.29}$$

It has to be expected that $\gamma \to 0$ in the limit $q \to 1$, so that the extra term γZ^A disappears.

As the calculations are very long, the first step is to determine the constants by requiring (2.29) only for one component and then by checking that (2.29) holds in general.

Starting point is the expression: [2]

$$L^+ = \alpha U R^+ + \beta U S^+ + \gamma R^3 S^+ + \delta R^+ S^3 \tag{2.30}$$

The constants α, β, γ and δ are to be determined so that:

$$L^+ X^0 = X^0 L^+ \tag{2.31}$$

[2]The constants β and δ are introduced to distinguish the terms resulting from the commutation of each one of the four expressions UR^+, US^+, R^3S^+ and R^+S^3 with X^0, but it is expected and it will actually turn out that $\beta = -\alpha$ and $\delta = -q^2\gamma$.

The next table shows which terms appear, when commuting L^+ through X^0 by using eqns. (B.5, B.6, B.7, B.8):

L^+X^0	α	β	γ	δ
X^0US^+	$\frac{(q^2-1)^2}{(1+q^2)^2}$	$\frac{(q^4+1)^2}{q^2(1+q^2)^2}$	$-\frac{q^2-1}{q^2(q^2+1)^3}$	$\frac{q^2-1}{(q^2+1)^3}$
X^0R^+U	$\frac{(q^4+1)^2}{q^2(1+q^2)^2}$	$\frac{(q^2-1)^2}{(1+q^2)^2}$	$\frac{q^2-1}{q^2(q^2+1)^3}$	$-\frac{q^2-1}{(q^2+1)^3}$
$X^0R^3S^+$	$\frac{(q^2-1)^3}{q^2+1}$	$-\frac{(q^2-1)^3}{q^2+1}$	$\frac{q^6+q^4-q^2+3}{(q^2+1)^2}$	$\frac{(q^2-1)^2}{(q^2+1)^2}$
$X^0R^+S^3$	$-\frac{q^2(q^2-1)^3}{q^2+1}$	$\frac{q^2(q^2-1)^3}{q^2+1}$	$\frac{(q^2-1)^2}{(q^2+1)^2}$	$\frac{(3q^6-q^4+q^2+1)}{q^2(q^2+1)^2}$
X^+U^2	$-\frac{q^4+1}{(1+q^2)^3}$	$-\frac{1}{q^2}\frac{q^4+1}{(1+q^2)^3}$	$-\frac{1}{q^2}\frac{1}{(1+q^2)^4}$	$\frac{1}{(1+q^2)^4}$
X^3US^+	$-\frac{(q^2-1)^2}{(q^2+1)^2}$	$\frac{(q^4+1)(q^2-1)}{q^2(q^2+1)^2}$	$\frac{-q^6-q^4+q^2-1}{q^2(q^2+1)^3}$	$-\frac{q^2-1}{(q^2+1)^3}$
X^+US^3	$\frac{q^2(q^2-1)^2}{(q^2+1)^2}$	$-\frac{(q^4+1)(q^2-1)}{(q^2+1)^2}$	$-\frac{q^2-1}{(q^2+1)^3}$	$-\frac{2q^2}{(q^2+1)^3}$
X^3R^+U	$\frac{(q^4+1)(q^2-1)}{q^2(q^2+1)^2}$	$\frac{(q^2-1)^2}{q^2(q^2+1)^2}$	$-\frac{q^4+q^2-2}{q^2(q^2+1)^3}$	$-\frac{2}{(q^2+1)^3}$
X^+R^3U	$\frac{-(q^4+1)(q^2-1)}{(q^2+1)^2}$	$-\frac{(q^2-1)^2}{(q^2+1)^2}$	$-\frac{2}{(q^2+1)^3}$	$\frac{q^2-1}{(q^2+1)^3}$
$X^3R^+S^3$	$-\frac{2q^2(q^2-1)^2}{q^2+1}$	$\frac{(q^2-1)^3}{q^2+1}$	$-\frac{(q^2-1)(q^4+q^2-2)}{(q^2+1)^2}$	$\frac{(q^2-1)(-q^4+2q^2+1)}{q^2(q^2+1)^2}$
$X^+R^3S^3$	$\frac{q^2(q^2-1)^3}{q^2+1}$	$-\frac{q^2(q^2-1)^3}{q^2+1}$	$-\frac{2q^2(q^2-1)}{(q^2+1)^2}$	$-\frac{2q^2(q^2-1)}{(q^2+1)^2}$
$X^3R^3S^+$	$-\frac{(q^2-1)^3}{q^2+1}$	$-\frac{2(q^2-1)^2}{q^2+1}$	$\frac{(q^2-1)(3-q^4)}{(q^2+1)^2}$	$-\frac{(q^2-1)^2}{(q^2+1)^2}$
$X^-R^+S^+$	$\frac{1}{q}(q^2-1)^2$	$\frac{1}{q^3}(q^2-1)^2$	$\frac{(q^2-1)(q^6+2q^4-1)}{q^3(q^2+1)^2}$	$\frac{q^2-1}{q(q^2+1)}$
$X^+R^-S^+$	0	$q(q^2-1)^2$	$-\frac{q(q^2-1)}{q^2+1}$	0
$X^+R^+S^-$	$q^3(q^2-1)^2$	0	0	$-\frac{q(q^2-1)}{q^2+1}$

In this table each column shows the coefficients of the different terms multiplying the same constant, in such a way that the first column gives the commutation relation of UR^+ with X^0, the second of US^+ with X^0, the third of R^3S^+ with X^0 and the last of R^+S^3 with X^0. The constants have to be chosen in such a way that for each row the sums of the coefficients multiplied by the corresponding constant vanishes, e.g. the first row yields the equation:

$$\frac{(q^2-1)^2}{(1+q^2)^2}\alpha + \frac{(q^4+1)^2}{q^2(1+q^2)^2}\beta - \frac{q^2-1}{q^2(q^2+1)^3}\gamma + \frac{q^2-1}{(q^2+1)^3}\delta = 0 \tag{2.32}$$

the second row implies:

$$\frac{(q^4+1)^2}{q^2(1+q^2)^2}\alpha + \frac{(q^2-1)^2}{(1+q^2)^2}\beta + \frac{q^2-1}{q^2(q^2+1)^3}\gamma - \frac{q^2-1}{(q^2+1)^3}\delta = 0 \tag{2.33}$$

and so on.

Notice that the relations (2.20) have been used here, because they are necessary to obtain (B.5,B.6,B.7,B.8) and to express the UX-relations (2.7) once in terms of R and once in terms of S. But in this way the expressions appearing in the table are ordered in such a way that they are always of the form XRS, so that they can be added.

It can be verified that (2.31) is satisfied if the following relations hold for the constants:

$$\alpha = -\beta, \quad \gamma = (q^4-1)\beta, \quad \delta = -q^2(q^4-1)\beta \tag{2.34}$$

so that it turns out:

$$L^A = \beta\left(U(S^A - R^A) + (q^4-1)Z^A\right). \tag{2.35}$$

The overall normalization constant β can be determined by studying the algebra generated by the L and by requiring that it is the same as in the three-dimensional Euclidean case [21, 19]. This is done in the next section.

Then it can be checked that (2.29) holds for L^A in general. It is a lengthy calculation. The commutation relations (2.23, 2.24, 2.25, 2.26) and the identity (2.20) have to be used. Repeated application of the eqns. (A.7)-(A.14) for the ε-tensor in appendix A is necessary. The following table shows which terms appear when this reduction has been performed:

$L^A X^0$	US^A	$-UR^A$	$(q^4-1)Z^A$	Sum
$X^0 U S^A$	$\frac{(q^4+1)^2}{q^2(q^2+1)^2}$	$-\frac{(q^2-1)^2}{(q^2+1)^2}$	$-\frac{(q^4+1)(q^2-1)^2}{q^2(q^2+1)^2}$	1
$X^0 U R^A$	$\frac{(q^2-1)^2}{(q^2+1)^2}$	$-\frac{(q^4+1)^2}{q^2(q^2+1)^2}$	$\frac{(q^4+1)(q^2-1)^2}{q^2(q^2+1)^2}$	-1
$X^0 Z^A$	$-\frac{(q^2-1)^3}{q^2+1}$	$-\frac{(q^2-1)^3}{q^2+1}$	$\frac{(q^2-1)(3q^4-2q^2+3)}{q^2+1}$	(q^4-1)
$X^A U^2$	$-\frac{q^4+1}{q^2(q^2+1)^3}$	$\frac{q^4+1}{(q^2+1)^3}$	$-\frac{(q^4+1)(q^2-1)}{q^2(q^2+1)^3}$	0
${\varepsilon_{BC}}^A X^C U S^B$	$\frac{(q^4+1)(q^2-1)}{q^2(q^2+1)^2}$	$\frac{(q^2-1)^2}{(q^2+1)^2}$	$-\frac{(q^2-1)(2q^4-q^2+1)}{q^2(q^2+1)^2}$	0
${\varepsilon_{BC}}^A X^C R^B U$	$\frac{(q^2-1)^2}{q^2(q^2+1)^2}$	$-\frac{(q^4+1)(q^2-1)}{q^2(q^2+1)^2}$	$\frac{(q^2-1)(q^4-q^2+2)}{q^2(q^2+1)^2}$	0
$X \circ R\, S^A$	$-\frac{(q^2-1)^2(q^4+1)}{q^2(q^2+1)}$	$-\frac{(q^2-1)^3}{q^2+1}$	$\frac{(q^2-1)^2(2q^4-q^2+1)}{q^2(q^2+1)}$	0
$X^A R \circ S$	0	$q^2(q^2-1)^2$	$-q^2(q^2-1)^2$	0
${\varepsilon_{BC}}^D {\varepsilon_{ED}}^A X^C R^B S^E$	$-\frac{(q^2-1)^3}{q^2(q^2+1)}$	$-2\frac{(q^2-1)^2}{q^2+1}$	$\frac{(q^2-1)^2(3q^2-1)}{q^2(q^2+1)^2}$	0

2.4 The algebra generated by the L

Now it must be verified that the L^A, which have been found in the previous chapter, effectively provide a realization of the q-deformed rotations (C.7). This also determines the expression for the operator W which has not yet been found and the overall normalization constant β. An extensive use of the relations listed in the appendix B2 is necessary to establish these results.

The first step is to calculate $\varepsilon_{BC}{}^A L^C L^B$. It turns out that:

$$\varepsilon_{BC}{}^A L^C L^B = \beta \left(-\frac{U^2}{q^2+1} + q^2(q^4-1)(q^2-1) R{\circ}S \right) L^A \tag{2.36}$$

The table shows in more detail the terms appearing when performing this calculation. The eqns. in (B.27) are applied to compute $\varepsilon_{BC}{}^A L^C S^B$, $\varepsilon_{BC}{}^A L^C R^B$,$\varepsilon_{BC}{}^A L^C Z^B$ and then (B.29) to commute $R\circ S$ with S^C and R^C in (B.27):

$\varepsilon_{BC}{}^A L^C L^B$	US^A	$-UR^A$	$(q^4-1)Z^A$	Sum
$U^3 S^A$	$-\frac{1}{q^2+1}$	0	0	$-\frac{1}{q^2+1}$
$U^3 R^A$	0	$\frac{1}{q^2+1}$	0	$\frac{1}{q^2+1}$
$U^2 Z^A$	$-q^2$	1	0	$1-q^2$
$UR^A\ S{\circ}S$	$q^2(q^4-1)$	0	$-q^2(q^4-1)$	0
$U\ R{\circ}S\ S^A$	$-q^2(q^4-1)$	0	$q^4(q^4-1)$	$q^2(q^2-1)(q^4-1)$
$US^A\ R{\circ}R$	0	q^4-1	$-(q^4-1)$	0
$UR{\circ}S\ R^A$	0	$-q^4(q^4-1)$	$q^2(q^4-1)$	$-q^2(q^2-1)(q^4-1)$
$R{\circ}SZ^A$	0	0	$q^2(q^2-1)(q^4-1)^2$	$q^2(q^2-1)(q^4-1)^2$

(2.37)

From (B.29) it immediately follows that:

$$R{\circ}S\ L^A = L^A\ R{\circ}S \tag{2.38}$$

As U commutes with R^A and S^A and hence with L^A, too, the whole expression $-\frac{U^2}{q^2+1} + q^2(q^4-1)(q^2-1)R{\circ}S$ commutes with L^A and is consistent to identify:

$$-\frac{W}{q^2} = \beta \left(-\frac{U^2}{q^2+1} + q^2(q^4-1)(q^2-1) R{\circ}S \right) \tag{2.39}$$

so that the first of eqns. (C.7) is satisfied:

$$\varepsilon_{BC}{}^A L^C L^B = -\frac{W}{q^2} L^A \tag{2.40}$$

To calculate the overall normalization constant β it is necessary to establish the relation between W and $L\circ L$, which can be computed from (B.26) by expressing $R\circ R + S\circ S$ in terms of U^2 through (2.13):

$$\begin{aligned} L\circ L &= \beta^2\left(\frac{U^4-1}{(q^4-1)^2} - 2q^2U^2\ R\circ S + q^4(q^4-1)^2(R\circ S)^2\right) \\ &= \left(\frac{1}{q^4}\frac{1}{(q^2-1)^2}W^2 - \frac{\beta^2}{(q^4-1)^2}\right) \end{aligned} \tag{2.41}$$

Requiring the last of eqns. (C.7) to hold:

$$q^4(q^2-1)^2 L\circ L = W^2 - 1$$

fixes β^2:

$$\beta^2 = \frac{(q^2+1)^2}{q^4} \tag{2.42}$$

By choosing the phase to be 1 the following realization of the commutation relations (C.7) for the L-algebra in the Minkowski space is obtained:

$$\begin{aligned} L^A &= \frac{q^2+1}{q^2}(US^A - UR^A + (q^4-1)Z^A) \\ W &= U^2 - q^2(q^4-1)^2 R\circ S \end{aligned} \tag{2.43}$$

The conjugation properties of L^A can be derived from the conjugation properties of R^A,S^A,U (2.12) and Z^A (B.25):

$$\overline{L^A} = g_{AB}L^B \tag{2.44}$$

By remembering that $\overline{R\circ S} = R\circ S$ (B.23) it immediately follows that:

$$\overline{W} = W \tag{2.45}$$

(2.44) and (2.45) coincide with (C.11) and hence are the right conjugation properties for the rotations. Therefore the choice of the phase for the constant β done in (2.43) is consistent.

By means the map (C.8) defined in the appendix C the L-algebra is related to the T-algebra and hence to $SU_q(2)$.

2.5 Action of the L-algebra on X

In this section the action of the q-deformed rotations (2.43) on X^A is computed.

To obtain the commutation relations of L^A with X^A eqns.(2.23, 2.24,2.25,2.26) and the identity (2.20) have to be applied. In order to reduce the terms with 3, 4 or more ε-tensors to terms with 2 or 3 ε-tensors repeated use of the identities

(A.7)-(A.14) for the ε-tensor in appendix A is necessary. The next table shows which terms come out. All the terms which appear can be brought to one of the types listed in the table by use of the formulas in appendix A, so that it is possible to sum them up.

L^AX^B	US^AX^B	$-UR^AX^B$	$(q^4-1)Z^AX^B$	Sum
$\varepsilon_T{}^{AB}X^TU^2$	$-\frac{(q^4+1)}{q^2(q^2+1)^3}$	$-\frac{(q^4+1)}{q^2(q^2+1)^3}$	$\frac{(q^2-1)^2}{q^2(q^2+1)^3}$	$-\frac{1}{q^2(q^2+1)}$
$g^{AB}X^0U^2$	$-\frac{(q^4+1)}{(q^2+1)^3}$	$\frac{(q^4+1)}{q^2(q^2+1)^3}$	$\frac{(q^2-1)(q^4+1)}{q^2(q^2+1)^3}$	0
X^AUS^B	$-\frac{(q^4+1)(q^2-1)}{q^2(q^2+1)^2}$	$-\frac{(q^2-1)^2}{(q^2+1)^2}$	$\frac{(q^2-1)(2q^4-q^2+1)}{q^2(q^2+1)^2}$	0
$g^{AB}X\circ S\,U$	$\frac{q^4+1}{q^2+1}$	0	$\frac{-q^2(q^2-1)}{q^2+1}$	1
$\varepsilon_{MN}{}^A\varepsilon_T{}^{MB}X^NUS^T$	$-\frac{2(q^4+1)}{q^2(q^2+1)^2}$	$-\frac{(q^2-1)^2}{q^2(q^2+1)^2}$	$\frac{2(q^2-1)^2}{q^2(q^2+1)^2}$	$-\frac{1}{q^2}$
X^AUR^B	$\frac{(q^2-1)^2}{(q^2+1)^2}$	$-\frac{(q^4+1)(q^2-1)}{(q^2+1)^2}$	$\frac{(q^4-q^2+2)(q^2-1)}{(q^2+1)^2}$	0
$g^{AB}X\circ R\,U$	0	$-\frac{q^4+1}{q^2(q^2+1)}$	$-\frac{q^2-1}{q^2(q^2+1)}$	-1
$\varepsilon_{MN}{}^A\varepsilon_T{}^{MB}X^NUR^T$	$\frac{(q^2-1)^2}{q^2(q^2+1)^2}$	$\frac{2(q^4+1)}{q^2(q^2+1)^2}$	$-\frac{2(q^2-1)^2}{q^2(q^2+1)^2}$	$\frac{1}{q^2}$
$g^{AB}X\circ Z$	$-(q^2-1)^2$	0	$2q^2(q^2-1)$	q^4-1
X^AZ^B	$\frac{2(q^2-1)^2}{q^2+1}$	$-\frac{(q^2-1)^3}{q^2+1}$	$\frac{(q^2-1)(q^4-4q^2+3)}{q^2+1}$	0
$\varepsilon_{MN}{}^A\varepsilon_T{}^{MB}X^NZ^T$	$\frac{2(q^2-1)^2}{q^2(q^2+1)}$	$\frac{2(q^2-1)^2}{q^2(q^2+1)}$	$\frac{(q^2-1)(-q^4-6q^2+3)}{q^2(q^2+1)}$	$-\frac{1}{q^2}(q^4-1)$
$\varepsilon_{MN}{}^AX^NR^MS^B$	$-\frac{1}{q^2}(q^2-1)^2$	$-\frac{1}{q^2}(q^2-1)^2$	$\frac{1}{q^2}2(q^2-1)^2$	0
$\varepsilon_T{}^{AB}X\circ R\,S^T$	$-\frac{1}{q^2}(q^2-1)^2$	$-\frac{1}{q^2}(q^2-1)^2$	$\frac{1}{q^2}2(q^2-1)^2$	0
$\varepsilon_T{}^{AB}X^TR\circ S$	$\frac{2(q^2-1)^2}{q^2+1}$	$\frac{2(q^2-1)^2}{q^2+1}$	$\frac{(q^2-1)^2(q^4+2q^2-3)}{q^2+1}$	$(q^4-1)(q^2-1)$
$X^0R^AS^B$	$-(q^2-1)^2$	$\frac{1}{q^2}(q^2-1)^2$	$\frac{1}{q^2}(q^2-1)^3$	0
$\varepsilon_T{}^{AB}X^0Z^T$	$\frac{2(q^2-1)^2}{q^2+1}$	$\frac{(q^2-1)^3}{q^2(q^2+1)}$	$-\frac{(q^2-1)^2(3q^2-1)}{q^2q^2+1}$	0
$g^{AB}X^0\ R\circ S$	$-\frac{q^2(q^2-1)^3}{q^2+1}$	$\frac{(q^2-1)^3}{q^2+1}$	$\frac{(q^2-1)^4}{q^2+1}$	0
$\varepsilon_T{}^{AB}X^0US^T$	$-\frac{(q^2-1)(q^4+1)}{q^2(q^2+1)^2}$	$\frac{(q^2-1)^2}{q^2(q^2+1)^2}$	$\frac{(q^2-1)(q^4-q^2+2)}{q^2(q^2+1)^2}$	0
$\varepsilon_T{}^{AB}X^0UR^T$	$\frac{(q^2-1)^2}{(q^2+1)^2}$	$\frac{(q^2-1)(q^4+1)}{q^2(q^2+1)^2}$	$-\frac{(q^2-1)(2q^4-q^2+1)}{q^2(q^2+1)^2}$	0

The result is:

$$\begin{aligned} L^A X^B &= -\frac{1}{q^4}\varepsilon_C{}^{AB} X^C W - \frac{1}{q^2}\varepsilon_{KC}{}^{A}\varepsilon_D{}^{KB} X^C L^D + g^{AB} X \circ L \\ &= X^A L^B - \frac{1}{q^2}\varepsilon_C{}^{AB}\varepsilon_{KD}{}^{C} X^D L^K - \frac{1}{q^4}\varepsilon_C{}^{AB} X^C W \end{aligned} \quad (2.46)$$

The second form of the action follows from the first by applying (A.9). The first thing that one notices when studying the action (2.46) is that there is an extra term containing $X \circ L$ which is not present in the three-dimensional Euclidean case [21, 19]. However, this is not astonishing, because in the Euclidean case the extra relation:

$$X \circ L = 0 \quad (2.47)$$

holds and the term automatically vanishes. In the Minkowski space the analogous to the relation (2.47) are the eqns. (2.14, 2.15).

The next step is to calculate the commutation relations between W and X^A. An alternative method can be used to achieve this result which is shorter and more general than having to commute W with X^A by means of the RX,SX and UX-relations.

First one contracts the LX-relation (2.46) with $\varepsilon_{BA}{}^S$:

$$\varepsilon_{BA}{}^S L^A X^B = -\frac{1}{q^4} X^S W + (1 - q^2 - \frac{1}{q^2})\varepsilon_{BA}{}^S X^A L^B \quad (2.48)$$

Then conjugation of the LX-relation (2.46) gives $X^B L^A$.

$$X^B L^A = -\frac{1}{q^4}\varepsilon_C{}^{BA} W X^C - \frac{1}{q^2}\varepsilon_D{}^{BK}\varepsilon_{EK}{}^{A} L^D X^E + g^{BA} X \circ L \quad (2.49)$$

Notice that:

$$\overline{X \circ L} = L \circ X = X \circ L \quad (2.50)$$

Now by contracting (2.49) with $\varepsilon_{AB}{}^S$ it can be verified that: [3]

$$\varepsilon_{AB}{}^S X^B L^A = -\frac{1}{q^4}(1 + q^4) W X^S + (1 - q^2 - \frac{1}{q^2})\varepsilon_{AB}{}^S L^B X^A \quad (2.51)$$

At last $\varepsilon_{AB}{}^S L^B X^A$ has to be expressed in terms of $\varepsilon_{AB}{}^S X^B X^A$ by means of (2.48) and the result is:

$$W X^A = (q^2 + \frac{1}{q^2} - 1) X^A W + (q^2 - 1)^2 \varepsilon_{DC}{}^{A} X^C L^D \quad (2.52)$$

This is exactly the same expression which holds in the three-dimensional case.

[3] This equation may have been obtained more easily by directly conjugating (2.48), but it is useful to have (2.49) written out explicitly.

It is worth mentioning explicitly that:

$$WX^0 = X^0 W \tag{2.53}$$

as one can see from the LL and $L^A X^0$-commutation relations.

Moreover, using the action (2.46) of L^A on X^A, the relations (2.13,2.20) and the SS-relations (2.16) the following important commutation relations follow:

$$L^A\, X\circ L = X\circ L\, L^A \tag{2.54}$$

$$X\circ L = \frac{1}{q^2(q^2-1)} X^0 (W-1)$$

The last equation substitutes the relation (2.47) which holds in the three-dimensional case, and reduces to it when $X^0 = 0$.

2.6 Action of the L-algebra on R, S

The next commutation relations which are necessary are the commutation relations between L^A and R^A,S^A; they can be obtained by using the eqns. (2.16, 2.17, 2.18, 2.19, B.28).

The result is:

$$\begin{aligned} L^A R^B &= -\frac{1}{q^4}\varepsilon_C{}^{AB} R^C W - \frac{1}{q^2}\varepsilon_{KC}{}^{A}\varepsilon_D{}^{KB} R^C L^D + g^{AB} R\circ L \\ &= R^A L^B - \frac{1}{q^2}\varepsilon_C{}^{AB}\varepsilon_{KD}{}^{C} R^D L^K - \frac{1}{q^4}\varepsilon_C{}^{AB} R^C W \end{aligned} \tag{2.55}$$

and analogously:

$$\begin{aligned} L^A S^B &= -\frac{1}{q^4}\varepsilon_C{}^{AB} S^C W - \frac{1}{q^2}\varepsilon_{KC}{}^{A}\varepsilon_D{}^{KB} S^C L^D + g^{AB} S\circ L \\ &= S^A L^B - \frac{1}{q^2}\varepsilon_C{}^{AB}\varepsilon_{KD}{}^{C} S^D L^K - \frac{1}{q^4}\varepsilon_C{}^{AB} S^C W \end{aligned} \tag{2.56}$$

This means that the L^B act on R^A and S^A exactly as they act on X^A. In fact, this had to be expected, because R and S are constructed in such a way as to behave like vectors under the rotations. This also explains why $R\circ S$ commutes with the L^A: by construction it is a scalar.

Notice that the the LS-relation can alternatively be computed by conjugating the LR-relation.

At last, only the action of W on R^A and S^A is missing. A method similar to the one used to calculate the action of W on X^A can be applied, but now R^A is no longer symmetric. Nevertheless conjugating R^A gives S^A, so that it is possible to obtain the $R^B L^A$-relation and hence the WR^A-relation by conjugating the $L^A S^B$-relation and vice versa. Paying attention to this, exactly the same reasoning goes through as for WX^A and therefore the result is:

$$\begin{aligned} WR^A &= (q^2 + \frac{1}{q^2} - 1)R^A W + (q^2-1)^2 \varepsilon_{DC}{}^A R^C L^D \\ WS^A &= (q^2 + \frac{1}{q^2} - 1)S^A W + (q^2-1)^2 \varepsilon_{DC}{}^A S^C L^D \end{aligned} \tag{2.57}$$

2.7 Identification of the boosts

As the main interest in this thesis is to study the representation theory of the Minkowski phase-space and of the Poincaré algebra co-acting on it, the identification of the q-deformed rotations was necessary.

For the sake of completeness in this section the results of ref. [38] are reviewed, in which the boosts identified in the R,S-algebra.

The q-deformed Lorentz-algebra has been defined in [27, 2, 4, 39, 26] as the algebra generated by $\tau^1, \tau^2, \tau, T^+, T^-, S^1, \sigma^2$ with the following relations:

$$\begin{aligned} \tau^1 T^+ &= T^+\tau^1 + \lambda T^2 \\ \tau^1 T^- &= \frac{1}{q^2} T^-\tau^1 - \lambda S^1 \\ \tau^1 T^2 &= q^2 T^2 \tau^1 \\ \tau^1 S^1 &= S^1 \tau^1 \\ \tau^1 \sigma^2 &= \sigma^2 \tau^1 + q\lambda^3 T^2 S^1 \\ \sigma^2 T^+ &= T^+ \sigma^2 - q^2 \lambda \tau T^2 \\ \sigma^2 T^- &= q^2 T^- \sigma^2 - q^2 \lambda S^1 \\ \sigma^2 T^2 &= \frac{1}{q^2} T^2 \sigma^2 \\ \sigma^2 S^1 &= S^1 \sigma^2 \\ \tau T^+ &= \frac{1}{q^4} T^+ \tau \\ \tau T^- &= q^4 T^- \tau \\ \tau T^2 &= \frac{1}{q^4} T^2 \tau \\ \tau S^1 &= q^4 S^1 \tau \\ \tau\tau^1 &= \tau^1 \tau \\ \tau\sigma^2 &= \sigma^2 \tau \\ T^+ T^2 &= \frac{1}{q^2} T^2 T^+ \\ T^+ S^1 &= q^2 S^1 T^+ + \lambda^{-1}(\tau\tau^1 - \sigma^2) \\ T^+ T^- &= q^2 T^- T^+ + q\lambda^{-1}(1-\tau) \end{aligned} \tag{2.58}$$

$$\begin{aligned} T^-T^2 &= T^2T^- + \lambda^{-1}(\sigma^2 - \tau^1) \\ T^-S^1 &= S^1T^- \\ T^2S^1 &= S^1T^2 \end{aligned}$$

and the extra relation:

$$\tau^1\sigma^2 - q^2\lambda^2T^2S^1 = 1 \tag{2.59}$$

In these formulas the following abbreviation is used:

$$\lambda = q - \frac{1}{q} \tag{2.60}$$

Obviously T^+, T^-, τ form a closed subalgebra corresponding to $SO_q(3)$.

The relation (2.59) is necessary because there is an extra generator in comparison with the undeformed case $q = 1$. Therefore a further relation must be used to obtain the correct classical limit.

The operators $\tau^1, \tau^2, S^1, \sigma^2$ admit the following realization in the R,S-algebra:

$$\begin{aligned} T^2 &= \frac{\sqrt{(q^2+1)^3}}{q}R^+ \\ \tau^1 &= -\big((q^4-1)R^3 + U\big) \\ S^1 &= -\sqrt{(q^2+1)^3}\tau^{\frac{1}{2}}S^- \\ \sigma^2 &= \tau^{\frac{1}{2}}\big((q^4-1)S^3 - U\big) \end{aligned} \tag{2.61}$$

This shows that the algebra generated by R, S, U really provides a realization of the q-deformed Lorentz algebra.

An important remark which is done also in the Appendix C is that it is necessary to introduce the extra generator $\tau^{\frac{1}{2}}$ in order to define the mapping (2.61) properly.

Chapter 3

The matrix elements of the coordinates

Now that the q-deformed rotations have been identified it is possible to begin the study of the representation theory of the Minkowski phase-space and of the Poincaré algebra.

As a complete set of commuting observables the operators $\vec{T}^2$ defined in (C.9), $\tau^{\frac{1}{2}}$, X^0, $X \cdot X$ are chosen.

3.1 The matrix elements of the q-deformed rotations

The representation theory of the $SO_q(3)$-algebra (C.3) for q real is well-known. Its finite-dimensional representations are labeled by the eigenvalues of the Casimir $[j][j+1]$, where j is a positive semi-integer number $j = 0, \frac{1}{2}, 1, \frac{3}{2}$ and

$$[j] = \frac{q^j - q^{-j}}{q - q^{-1}} \tag{3.1}$$

The operators $\vec{T}^2, \tau^{\frac{1}{2}}$ provide a complete set of commuting observables. The eigenvalues of $\tau^{\frac{1}{2}}$ are q^{-2m} with $m = -j, -j+1, \cdots, j-1, j$.

In complete analogy with the undeformed case, denoting the common eigenfunctions of $\vec{T}^2, \tau^{\frac{1}{2}}$ with $|j,m\rangle$, the T-algebra admits the following representation:

$$\begin{aligned} \vec{T}^2 \ |j,m\rangle &= [j][j+1] \ |j,m\rangle \\ T^+ \ |j,m\rangle &= q^{-m-\frac{3}{2}} \sqrt{[j+m+1][j-m]} \ |j,m+1\rangle \\ T^- \ |j,m\rangle &= q^{-m+\frac{3}{2}} \sqrt{[j+m][j-m+1]} \ |j,m-1\rangle \\ \tau \ |j,m\rangle &= q^{-4m} \ |j,m\rangle \end{aligned} \tag{3.2}$$

3.2 The Wigner-Eckart theorem

In this section the dependence of the X^A-matrix elements on m is determined. This is a simple application of the Wigner-Eckart theorem. It means that it is possible to calculate the m-dependence by using only the TX-commutation relations which follow from the LX-commutation relations (2.46), in other words the $SO_q(3)$-symmetry of the spatial coordinates completely fixes the m-dependence of their matrix elements. As P^A,R^A,S^A also behave like vectors under the action of the q-deformed rotations, the same calculation which is carried out here for X^A also holds for the P^A,R^A,S^A-matrix elements.

The TX-commutation relations are easily obtained by expressing the T-algebra in terms of the L-algebra and then by using the LX-relations (2.46) found in the previous chapter.

The action on the coordinates is:

$$\begin{aligned}
T^3X^3 &= X^3T^3 \\
T^3X^+ &= q^{-4}X^+T^3 + q^{-1}(1+q^{-2})X^+ \\
T^3X^- &= q^4X^-T^3 - q(1+q^2)X^- \\
T^+X^3 &= X^3T^+ + q^{-2}\sqrt{1+q^2}X^+ \\
T^+X^+ &= q^{-2}X^+T^+ \\
T^+X^- &= q^2X^-T^+ + q^{-1}\sqrt{1+q^2}X^3 \\
T^-X^3 &= X^3T^- + q\sqrt{1+q^2}X^- \\
T^-X^+ &= q^{-2}X^+T^- + \sqrt{1+q^2}X^3 \\
T^-X^- &= q^2X^-T^-
\end{aligned} \tag{3.3}$$

Remembering that in the undeformed case the $SO(3)$-algebra is the algebra generated by the the elements $J^{\pm}$, J^3 modulo the ideal given by the relations:

$$[J^3, J^{\pm}] = \pm J^{\pm}, \quad [J^+, J^-] = 2J^3$$

the Wigner-Eckart theorem [47, 8] states the following:

Wigner-Eckart Theorem If a quantity X_J^M carries the J-th representation of the rotations, e.g.

$$\begin{aligned}
[J^{\pm}, X_J^M] &= \sqrt{(J \mp M)(J \pm M + 1)}X_J^{M\pm 1} \\
[J^3, X_J^M] &= MX_J^M
\end{aligned} \tag{3.4}$$

then the matrix elements of X_J^M depend on m only via the Clebsch-Gordan coefficients $C(j, J, j'; m, M, m')$:

$$\langle j', m', \mu'|X_J^M|j, m, \mu\rangle = C(j, J, j'; m, M, m')\langle j', \mu'\|X_J^M\|j, \mu\rangle \tag{3.5}$$

$\langle j', \mu'\|X_J^M\|j, \mu\rangle$ does not depend on m. It is called the reduced matrix element. μ denotes the set of eigenvalues which correspond to the other observables different from $\vec{J}^2, J^3$ labeling the representation. The proof of the theorem just consists in taking the matrix elements of the commutation relations (3.4). The only important fact that has to be taken into account are the conjugation properties of the rotations.

Exactly the same can be done in the case $q \neq 1$, too.

This leads to the following result:

$$\begin{aligned}
\langle j, m+1, \mu | X^+ | j, m, \nu\rangle &= -q^{m+2}\sqrt{[j+m+1][j-m]}\langle j,\mu\|X^-\|j,\nu\rangle \\
\langle j+1, m+1, \mu | X^+ | j, m, \nu\rangle &= q^{m-2j}\sqrt{[j+m+1][j+m+2]} \\
&\quad \cdot\langle j+1,\mu\|X^-\|j,\nu\rangle \\
\langle j-1, m+1, \mu | X^+ | j, m, \nu\rangle &= q^{m+2j+2}\sqrt{[j-m][j-m-1]} \\
&\quad \cdot\langle j-1,\mu\|X^-\|j,\nu\rangle
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\langle j, m-1, \mu | X^- | j, m, \nu\rangle &= q^{m}\sqrt{[j+m][j-m+1]}\langle j,\mu\|X^-\|j,\nu\rangle \\
\langle j+1, m-1, \mu | X^- | j, m, \nu\rangle &= q^{m}\sqrt{[j-m+1][j-m+2]} \\
&\quad \cdot\langle j+1,\mu\|X^-\|j,\nu\rangle \\
\langle j-1, m-1, \mu | X^- | j, m, \nu\rangle &= q^{m}\sqrt{[j+m][j+m-1]}\langle j-1,\mu\|X^-\|j,\nu\rangle
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\langle j, m, \mu | X^3 | j, m, \nu\rangle &= q^{\frac{3}{2}}\frac{\sqrt{1+q^2}}{q^2-1}\{q^{2m} - \frac{q^{2j+1}+q^{-2j-1}}{q+q^{-1}}\}\langle j,\mu\|X^-\|j,\nu\rangle \\
\langle j+1, m, \mu | X^3 | j, m, \nu\rangle &= q^{m-j-\frac{1}{2}}\sqrt{1+q^2}\sqrt{[j-m+1][j+m+1]} \\
&\quad \cdot\langle j+1,\mu\|X^-\|j,\nu\rangle \\
\langle j-1, m, \mu | X^3 | j, m, \nu\rangle &= -q^{m+j+\frac{1}{2}}\sqrt{1+q^2}\sqrt{[j-m][j+m]} \\
&\quad \cdot\langle j-1,\mu\|X^-\|j,\nu\rangle
\end{aligned} \tag{3.8}$$

Proof:

As the first step, taking the matrix elements of the last of the eqns. (3.3) leads to:

$$\langle j', m', \mu | T^- X^- | j, m, \nu\rangle = q^2 \langle j', m', \mu | X^- T^- | j, m, \nu\rangle$$

But then $\overline{T^+} = \frac{1}{q^2}T^-$ and (3.2) imply:

$$\begin{aligned}
&\langle j', m'+1, \mu | X^- | j, m, \nu\rangle q^{-m'+\frac{1}{2}}\sqrt{[j'+m'+1][j'-m']} = \\
&\qquad \langle j', m', \mu | X^- | j, m-1, \nu\rangle q^{-m+\frac{7}{2}}\sqrt{[j+m][j-m+1]}
\end{aligned}$$

For $m' = m-2$ the following recursion relation follows:

$$\frac{\langle j', m-1, \mu | X^- | j, m, \nu\rangle q^{-1}}{\sqrt{[j+m][j-m+1]}} = \frac{\langle j', m-2, \mu | X^- | j, m-1, \nu\rangle}{\sqrt{[j'+m-1][j'-m+2]}} \tag{3.9}$$

For $j' = j$, $j' = j+1$, $j' = j-1$ this recursion relation gives the first, second and third of the eqns. (3.7) respectively. These are the only matrix elements of X^- which are different from 0, as can be seen by taking the matrix elements of the third of the TX-relations (3.3).

Now, the matrix elements of X^3 can be computed in terms of the reduced matrix elements of X^- by taking the matrix elements of the equation:

$$T^+X^- = q^2 X^- T^+ + q^{-1}\sqrt{1+q^2}X^3$$

and inserting the expressions for $\langle j', m-1, \mu | X^- | j, m, \nu \rangle$.

At last, the matrix elements of X^+ are obtained by taking the matrix elements of:

$$T^-X^+ = q^{-2}X^+T^- + \sqrt{1+q^2}X^3$$

and inserting the expression for the X^3-matrix elements.

In this way all the matrix elements are expressed in terms of the reduced matrix elements of X^-. The TX-relations (3.3) imply relations between the reduced matrix elements of X^+, X^- and X^3. As an example it would have been possible to calculate the reduced X^+-matrix elements by taking the matrix elements of $T^+X^+ = q^{-2}X^+T^+$. For $j' = j$ this would have led to the result:

$$\langle j, m+1, \mu | X^+ | j, m, \nu \rangle = q^m \sqrt{[j+m+1][j-m]} \langle j, \mu \| X^+ \| j, \nu \rangle$$

and hence by comparison with the first of (3.6) to the relation:

$$\langle j, \mu \| X^+ \| j, \nu \rangle = -q^2 \langle j, \mu \| X^- \| j, \nu \rangle \tag{3.10}$$

Analogously, the following relations are proven:

$$\begin{aligned} \langle j+1, \mu \| X^+ \| j, \nu \rangle &= q^{-2j} \langle j+1, \mu \| X^- \| j, \nu \rangle \\ \langle j-1, \mu \| X^+ \| j, \nu \rangle &= q^{2j+2} \langle j-1, \mu \| X^- \| j, \nu \rangle \end{aligned} \tag{3.11}$$

$$\begin{aligned} \langle j, \mu \| X^3 \| j, \nu \rangle &= \frac{q^2}{(q^2-1)\sqrt{[2]}} \langle j, \mu \| X^- \| j, \nu \rangle \\ \langle j+1, \mu \| X^3 \| j, \nu \rangle &= q^{-j}\sqrt{[2]} \langle j+1, \mu \| X^- \| j, \nu \rangle \\ \langle j-1, \mu \| X^3 \| j, \nu \rangle &= -q^{j+1}\sqrt{[2]} \langle j-1, \mu \| X^- \| j, \nu \rangle \end{aligned} \tag{3.12}$$

□

The conjugation properties of the reduced matrix elements can also be established by conjugating the eqns. (3.6,3.7,3.8) and remembering the conjugation properties of the X^A (2.12). [1]

$$\begin{aligned} \langle j, \mu \| X^- \| j, \nu \rangle &= \overline{\langle j, \nu \| X^- \| j, \mu \rangle} \\ \langle j+1, \mu \| X^- \| j, \nu \rangle &= -q^{2j+2}\overline{\langle j, \nu \| X^- \| j+1, \mu \rangle} \\ \langle j-1, \mu \| X^- \| j, \nu \rangle &= -q^{-2j}\overline{\langle j, \nu \| X^- \| j-1, \mu \rangle} \end{aligned} \tag{3.13}$$

[1] These are the only results in this section which do not hold for the reduced matrix elements of R^A,S^A, as they have different conjugation properties.

The same conjugation properties also hold for the P^A-matrix elements.

3.3 Eigenvalues of X^0 and $X \cdot X$

3.3.1 Computation of the eigenvalues

In this section the eigenvalues of X^0 and $X \cdot X$ are determined. To achieve this result the commutation relations of $R \circ X$ (B.10) and U (2.25) with X^0 are used.

Previously, some remarks have to be done. The representation which is studied in this thesis is a spin-0 representation, the eigenvalue j gives only the orbital angular momentum. In section 3.4 this will be shown explicitly. Therefore, j can assume only integer values, and not semi-integer.

The following remark about the reduced matrix elements $\langle j', \mu \| X^- \| j, \nu \rangle$ has to be done. The complete set of commuting observables contains also $X \cdot X$ and X^0 besides $\vec{T}^2, \tau^{\frac{1}{2}}$. Therefore $\mu = (s^2, t)$, where s^2 is the eigenvalue of $X \cdot X$ and t the eigenvalue of X^0 respectively. A reduced matrix element may be denoted as $\langle j', s'^2, t' \| X^- \| j, s^2, t \rangle$, and a Hilbert space state as $|j, m, s^2, t\rangle$.

The states $|j, m, s^2, t\rangle$ are a complete orthonormal system for the Hilbert space:

$$\langle j', m', s'^2, t | j, m, s^2, t \rangle = \delta_{j',j} \delta_{m',m} \delta_{s'^2, s^2} \delta_{t',t} \tag{3.14}$$

By definition X^- commutes with $X \cdot X$ and X^0. As a consequence it does not change the eigenvalues s^2, t:

$$\langle j', s'^2, t' \| X^- \| j, s^2, t \rangle = 0 \qquad \text{for} \qquad s'^2 \neq s^2, t' \neq t \tag{3.15}$$

By means of the eqns. (3.6,3.7,3.8) it is possible to calculate the expression for the scalar product $A \circ B$ of two vectors A,B and thus get, in particular, an equation for the matrix elements of $R \circ X = g_{AB} X^A R^B$. The action of the q-deformed rotations on vectors imply that the scalar product $A \circ B$ commutes with the T-algebra and therefore:

$$\langle j', m', s'^2, t' | A \circ B | j, m, s^2, t \rangle = 0 \quad \text{for} \quad j' \neq j, m' \neq m \tag{3.16}$$

For $j' = j$, $m' = m$:

$$\begin{aligned}
&\langle j, m, s''^2, t'' | A \circ B | j, m, s^2, t \rangle = \\
&\langle j, s''^2, t'' \| A \circ B \| j, s^2, t \rangle = \\
&\sum_{s'^2, t'} \langle j, s''^2, t'' \| A^- \| j, s'^2, t' \rangle \langle j, s'^2, t' \| B^- \| j, s^2, t \rangle \frac{1}{q + q^{-1}} q^2 [2j+2][2j] \\
&- \langle j, s''^2, t'' \| A^- \| j+1, s'^2, t' \rangle \langle j+1, s'^2, t' \| B^- \| j, s^2, t \rangle q^2 [2j+2][2j+3] \\
&- \langle j, s''^2, t'' \| A^- \| j-1, s'^2, t' \rangle \langle j-1, s'^2, t' \| B^- \| j, s^2, t \rangle q^2 [2j][2j-1]
\end{aligned} \tag{3.17}$$

with A, B vectors.

The result does not depend on m. This had to be expected, because the scalar product is by construction invariant under the action of q-rotations.

Now, (3.17) can be applied to the matrix elements of $X \circ R$ in (B.10) and (2.25):

$$\begin{aligned} X \circ R\, X^0 &= \frac{2q}{1+q^2} X^0\, X \circ R - \frac{q}{(1+q^2)^2} X \circ X\, U \\ U X^0 &= \frac{1}{q}\frac{q^4+1}{q^2+1} X^0 U - \frac{1}{q}(q^2-1)^2 X \circ R \end{aligned}$$

This yields the two equations:

$$\langle j,m,s^2,t'|U|j,m,s^2,t\rangle = \frac{(2t' - (q+q^{-1})t)\,(q^2+1)}{r'^2} \langle j,m,s^2,t'|X \circ R|j,m,s^2,t\rangle \tag{3.18}$$

and

$$\begin{aligned} &\langle j,m,s^2,t'|U|j,m,s^2,t\rangle = \\ &-\left(t - \frac{1}{q}\frac{q^4+1}{q^2+1}t'\right)^{-1} \frac{1}{q}(q^2-1)^2 \langle j,m,s^2,t'|X \circ R|j,m,s^2,t\rangle \end{aligned} \tag{3.19}$$

respectively. Here

$$r^2 = s^2 + t^2 \tag{3.20}$$

is an eigenvalue of the spatial radius $X \circ X$. Remark that $s^2 = s'^2$, because the q-deformed Lorentz generators U,R and S commute with the four-dimensional invariant length $X \cdot X$.

By comparison of (3.18) and (3.19) it turns out that:

$$\langle j,m,s^2,t'\|X \circ R\|j,m,s^2,t\rangle \left\{ \left(t - \frac{2q}{1+q^2}t'\right) - \frac{(q^2-1)^2 r'^2}{(1+q^2)^2} \frac{1}{t - \frac{1}{q}\frac{q^4+1}{q^2+1}t'} \right\} = 0 \tag{3.21}$$

In order to have non-vanishing elements of $\langle j,m,s^2,t'\|X \circ R\|j,m,s^2,t\rangle$ the expression in the curly bracket of (3.21) have to vanish. The possible eigenvalues of X^0 are fixed by the equation:

$$t^2 + 2\frac{q^4+1}{(q^2+1)^2}t'^2 - \frac{1}{q}(q^2+1)tt' - \frac{(q^2-1)^2}{(q^2+1)^2}r'^2 = 0 \tag{3.22}$$

Notice that this equation is not valid for $r' = 0$, because it is necessary to divide by r'^2 in (3.18). By substituting $r'^2 = s^2 + t'^2$ it is equivalent to:

$$t'^2 + t^2 - \frac{1}{q}(q^2+1)t't - \frac{(q^2-1)^2}{(q^2+1)^2}s^2 = 0 \tag{3.23}$$

The equation is completely symmetric in t, t'. Given t it is possible to determine the value of t':

$$t' = \frac{1}{2}(q + \frac{1}{q})t \pm \frac{1}{2}(q - \frac{1}{q})\sqrt{t^2 + \frac{4q^2}{(q^2+1)^2}s^2} \tag{3.24}$$

For $s^2 = 0$ on the light-cone t, t' must satisfy:

$$t'^2 + t^2 - \frac{1}{q}(q^2+1)t't = 0 \tag{3.25}$$

By starting with the value $t = t_0 = \tau_0$ the result for t' is:

$$t_1 = \tau_0 q \quad \text{or} \quad t_{-1} = \tau_0 q^{-1} \tag{3.26}$$

And by induction t' could assume the values t_n:

$$t_n = \tau_0 q^n \quad \text{for} \quad s^2 = 0 \tag{3.27}$$

In fact, inserting $t = t_{n-1}$ in (3.25).

$$(t'^2 - \tau_0(q^2+1)q^{n-2}t' + \tau_0^2 q^{2(n-1)} = 0 \tag{3.28}$$

yields:

$$t_n = \tau_0 q^n \quad \text{or} \quad t_{n-2} = \tau q^{-n-2} \tag{3.29}$$

For $s^2 < 0$ time-like, assume that there is an eigenstate with $t = \tilde{t}_0$, $r = 0$, s^2 fixed to be $s^2 = -\tilde{t}_0^2$. Then (3.24) would yield:

$$t_1 = \tilde{t}_0 \frac{q^2 + q^{-2}}{q + q^{-1}} \qquad t_{-1} = \tilde{t}_0 \frac{2}{q + q^{-1}} \tag{3.30}$$

But t_{-1} cannot be an eigenvalue because it would correspond to a negative value of r'^2 and this is not possible.

Then it can shown by induction that the eigenvalues for X^0 are:

$$t_n = \tilde{t}_0 \frac{q}{1+q^2}(q^{n+1} + q^{-n-1}) \quad \text{for } s^2 \text{ time-like}, \quad n = 0, 1, \ldots \tag{3.31}$$

In fact, for $n > 0$, $s^2 = -\tilde{t}_0^2$, $t = t_{n-1} = \frac{q}{1+q^2}(q^n + q^{-n})$ eqn. (3.22) can be written:

$$t'^2 - \frac{1}{q}(q^2+1)\tilde{t}_0 \frac{q}{1+q^2}(q^n + q^{-n})t' + \tilde{t}_0^2 \frac{(q^2-1)^2 + (q^n + q^{-n})^2}{(q^2+1)^2} = 0 \tag{3.32}$$

which has the solutions:

$$t_n = \tilde{t}_0 \frac{q}{1+q^2}(q^{n+1} + q^{-n-1}) \quad \text{or} \quad t_{n-2} = \tilde{t}_0 \frac{q}{1+q^2}(q^{n-1} + q^{-n+1}) \tag{3.33}$$

Notice that the condition $n > 0$ holds.

The next case to be analyzed is the case $s^2 > 0$ space-like. Assume that there is an eigenstate with $t' = 0$, $r' = \tilde{l}_0$, therefore $s^2 = \tilde{l}_0^2$.

Then eqn. (3.22) becomes:

$$t'^2 - \frac{(q^2-1)^2}{(q^2+1)^2} l_0^2 = 0 \tag{3.34}$$

which admits the solutions:

$$t_1 = \tilde{l}_0 \frac{q^2-1}{q^2+1} \quad \text{or} \quad t_{-1} = -\tilde{l}_0 \frac{q^2-1}{q^2+1} \tag{3.35}$$

More in general, the spectrum of X^0 can contain the points:

$$t_n = \tilde{l}_0 \frac{q}{1+q^2}(q^n - q^{-n}) \quad \text{for } s^2 \text{ space-like}, \quad n \in \mathbb{Z} \ldots \tag{3.36}$$

In fact, by induction, for $s^2 = \tilde{l}_0^2$, $t = t_{n-1} = \frac{q}{1+q^2}(q^{n-1} + q^{-(n-1)})$ eqn. (3.22) can be written:

$$t'^2 - \frac{1}{q}(q^2+1)\tilde{l}_0 \frac{q}{1+q^2}(q^{n-1} + q^{-(n-1)})t' - \tilde{l}_0^2 \frac{(q^2-1)^2 - (q^{n-1} - q^{-(n-1)})^2}{(q^2+1)^2} = 0 \tag{3.37}$$

which has the solutions:

$$t_n = \tilde{l}_0 \frac{q}{1+q^2}(q^n - q^{-n}) \quad \text{or} \quad t_{n-2} = \tilde{l}_0 \frac{q}{1+q^2}(q^{n-2} + q^{-n+2}) \tag{3.38}$$

Notice that for s^2 space-like there is no restriction $n \geq 0$.

Also, observe that not all the points listed in eqns. (3.27,3.31,3.36) are necessarily contained in the spectrum of X^0. The condition (3.22) is necessary but not sufficient. It is a purely algebraic condition, which does not state anything neither about the actual existence of an eigenvalue nor on the multiplicity which it could have. In fact it will turn out that, as soon as the momentum operators P^a are introduced in the algebra, there are no points on the light-cone, as it is proven in Chapter 5. [2]

But this is not the end of the story. Consider the operator Λ and its commutation relations (2.9) with the coordinates X^a. It is immediate that if $|j, m, s^2, t\rangle$ is an eigenfunction of X^0 and $X \cdot X$ with eigenvalues t and s^2 respectively, then

$$X^0(\Lambda^{-\frac{1}{2}}|j, m, s^2, t\rangle) = \frac{1}{q}\Lambda^{-\frac{1}{2}} X^0 |j, m, s^2, t\rangle = \frac{1}{q} t(\Lambda^{-\frac{1}{2}}|j, m, s^2, t\rangle) \tag{3.39}$$

$$X \cdot X(\Lambda^{-\frac{1}{2}}|j, m, s^2, t\rangle) = \frac{1}{q^2}\Lambda^{-\frac{1}{2}} X \cdot X |j, m, s^2, t\rangle = \frac{1}{q^2} s^2(\Lambda^{-\frac{1}{2}}|j, m, s^2, t\rangle) \tag{3.40}$$

[2] Such points exist, however, as long as the coordinates alone, or equivalently the momentum coordinates alone, are represented and one is restricted to representations of the Poincaré algebra.

and therefore also $\Lambda^{-\frac{1}{2}}|j, m, s^2, t\rangle$ is an eigenfunction of X^0 and $X \cdot X$ with eigenvalues $\frac{1}{q}t$ and $\frac{1}{q^2}s^2$ respectively. It holds:

$$\Lambda^{\frac{1}{2}}|j, m, s^2, t\rangle \sim |j, m, s^2q^2, tq\rangle \tag{3.41}$$

For this reason the expression for the eigenvalues of X^0 is obtained by rescaling the parameter $\tilde{t}_0$ or $\tilde{l}_0$ with q^M, which is free:

$$\tilde{t}_0 = t_0 q^M, \quad \tilde{l}_0 = l_0 q^M$$

with $M \in \mathbb{Z}$.

In other words the eigenvalues of $X \cdot X$ are fixed by M, namely:

$$X \cdot X|j, m, s^2, t\rangle = l_0 q^{2M}|j, m, s^2, t\rangle \quad \text{for } s^2 \text{ space-like} \tag{3.42}$$

and

$$X \cdot X|j, m, s^2, t\rangle = -t_0 q^{2M}|j, m, s^2, t\rangle \quad \text{for } s^2 \text{ time-like} \tag{3.43}$$

For the light-cone the eigenvalues of X^0 take the values $t = \tau_0 q^n$ with $n \in \mathbb{Z}$ and rescaling τ^0 just induces a rescaling of n and therefore it is not necessary. In fact, on the light-cone $s^2 = 0$ and therefore it is obtained by taking the limit $M \to -\infty$, while keeping $M + n$ finite. This means that $n \to \infty$ and that the number n appearing in (3.27) can be interpreted as the limit of $M+n$ for $s^2 \to 0$.

The final result is:

$$t_{M,n} = t_0 q^M \frac{q}{1+q^2}(q^{n+1} + q^{-n-1}) \quad \text{for } s^2 < 0 \text{ time-like}, \quad M \in \mathbb{Z},\ n = 0, 1, \ldots$$

$$t_{M,n} = l_0 q^M \frac{q}{1+q^2}(q^n - q^{-n}) \quad \text{for } s^2 > 0 \text{ space-like}, \quad M, n \in \mathbb{Z}, \tag{3.44}$$

$$t_n = \tau_0 q^n \quad \text{for} \quad s^2 = 0,\ n \in \mathbb{Z}$$

The parameters τ, t_0 and l_0 are free. It can be shown that the representations with different values of the parameters τ, t_0 and l_0 are not equivalent. There is one representation for each choice. As the aim is to construct a theory on the whole configuration space the matching conditions are imposed:

$$t_0 = l_0 = [2]\tau_0 \tag{3.45}$$

Notice that for the light-cone and in the case s^2 time-like, the forward region is described by a positive value of the parameter τ_0 and t_0 respectively, while the backward region is found by taking $-\tau_0$ and $-t_0$ respectively. Therefore it is necessary to consider the direct sum of two representations with parameters which have different signs. In complete analogy with the 1-dimensional case [15] this is also necessary in order to represent the momentum coordinates P^a as self-adjoint operators. This problem is currently under investigation and will be studied elsewhere.

The parameter t_0 carries the dimension and thus fixes the scale of the theory. This is an interesting remark, because q is dimensionless, and hence cannot fix the scale of the theory. It is tempting to interpret the fact that the theory is on a lattice when $q \neq 1$ and tends to the classical theory, hence to a continuous theory, for $q \to 1$ as a process of spontaneous symmetry breaking. Space-time would exist in a phase where it is continuous and in a phase in which it is a lattice and chooses one particular value of the parameter. With this supposition the parameter t_0 would depend on q, but the way it does depend on it is determined by the particular process which causes the symmetry breaking. The symmetry breaking mechanism cannot be described in the framework of this theory, and up to the moment it has not been studied yet in any particular quantum group model, not even for the lower-dimensional cases.

Knowing the eigenvalues of X^0 and of $X \cdot X$ the eigenvalues r^2 of the spatial radius $X \cdot X$ are easily computed through the definition: $r^2 = s^2 + t^2$. The result is:

$$
\begin{aligned}
r^2_{M,n} &= t_0^2 q^{2M} \frac{q^2}{(1+q^2)^2}(q^{n+2} - q^{-n-2})(q^n - q^{-n}) \\
&\quad \text{for } s^2 < \text{time-like}, \quad M \in \mathbb{Z},\ n = 0, 1, \ldots \\
r^2_{M,n} &= l_0^2 q^{2M} \frac{q^2}{(1+q^2)^2}(q^{n+1} + q^{-n-1})(q^{n-1} + q^{-n+1}) \\
&\quad \text{for } s^2 > 0 \text{ space-like}, \quad M, n \in \mathbb{Z} \\
r^2_n &= \tau_0^2 q^{2n} \qquad \text{for } s^2 = 0, \quad n \in \mathbb{Z}
\end{aligned}
\tag{3.46}
$$

In [31],[32] the eigenvalues of X^0 and $X \circ X$ are also studied. However, only the case $s^2 < 0$ is considered there. This has the following explanation. As the algebra defining the Minkowski space is completely symmetric in X and P it is possible to interpret $X \cdot X$ as mass, and then only positive values of the mass are interesting and the positive-mass-case in the momentum space corresponds to the time-like case in the coordinate space.

In [31, 32] the eigenvalues depend on the integer numbers N,F and r and the continuous parameter d_0. By identifying:

$$
2N - r = n, \quad F = M, \quad d_0 = -\frac{t_0}{\sqrt{q^2+1}} \tag{3.47}
$$

it can be checked that the eigenvalues given here for s^2 time-like coincide with those found in [31, 32].

3.3.2 Plots of the space-time lattice

As already observed, the most striking fact about the spectrum of X^0,r^2 and s^2 is that it is discrete and hence the theory is automatically forced on a lattice. This gives rise to the hope that the field theory which could be developed starting from this model does not need to be regularized at short distances, because it has natural cut-off provided by the lattice. That is the most important reason for studying this kind of models.

To understand how the q-deformed Minkowski phase-space looks like, it is a good idea to plot the admissible eigenvalues t of X^0 versus the square root r of those of $X \circ X$. The Maple-program doing this is reported in Appendix E. The result are the pictures appearing in the next pages. The plot is done only for a positive value of $t_0 = l_0$ ($t_0 = l_0 = 1$). The plot of the spectrum for the negative value $-t_0$ would be the same as the one for t_0 reflected along the axis $t = 0$.

The distance between the lattice points grow exponentially. This is a typical phenomenon for the lattices resulting from quantum group symmetries which is true already in smaller dimensions [15, 16, 45].

There are three kinds of very interesting structures emerging from the plots.

The first can best be seen in the following way. The points lying on the light-cone are accumulation points. More precisely, a point with coordinates $(t_0\frac{q^v}{[2]}, \frac{q^v}{[2]})$ is accumulation point for the points lying on the curve defined by the condition $M + n = v$, when the limit $M \to -\infty, n \to \infty$ is taken, in such a way that v is kept finite.

Another way to describe this curve is:

$$r_v(t) = \frac{1}{q}(q^2+1)\sqrt{\tau_0^2 q^{2v} - t\tau_0 q^v + t^2\frac{q^2}{(q^2+1)^2}} \tag{3.48}$$

where v is fixed.

By deriving this equation in $r = t$, one sees that all these curves have the same slope $-2(q^2 + q^{-2})^{-1}$ which tends to -1 in the limit $q \to 1$. Hence in this limit the curves become parallel to each other and are orthogonal to the light-cone. In some sense they define a sort of light-cone coordinates. By calculating the second derivative, it can be verified that they are nearly straight lines in a neighbourhood of the light-cone.

The more the value of v increases, the more these curves are separated from one another. This means that if the plot is interpreted in the momentum space, more and more energy is required to move from one line to the next.

There is also another kind of interesting structure.

The scalar generator $\Lambda^{\frac{1}{2}}$ maps the point with coordinates (t, r) to (qt, qr) and the points $(q^M t, q^M r)$ describe straight lines starting at the origin. It is possible to move along one of these lines, by fixing n and varying M.

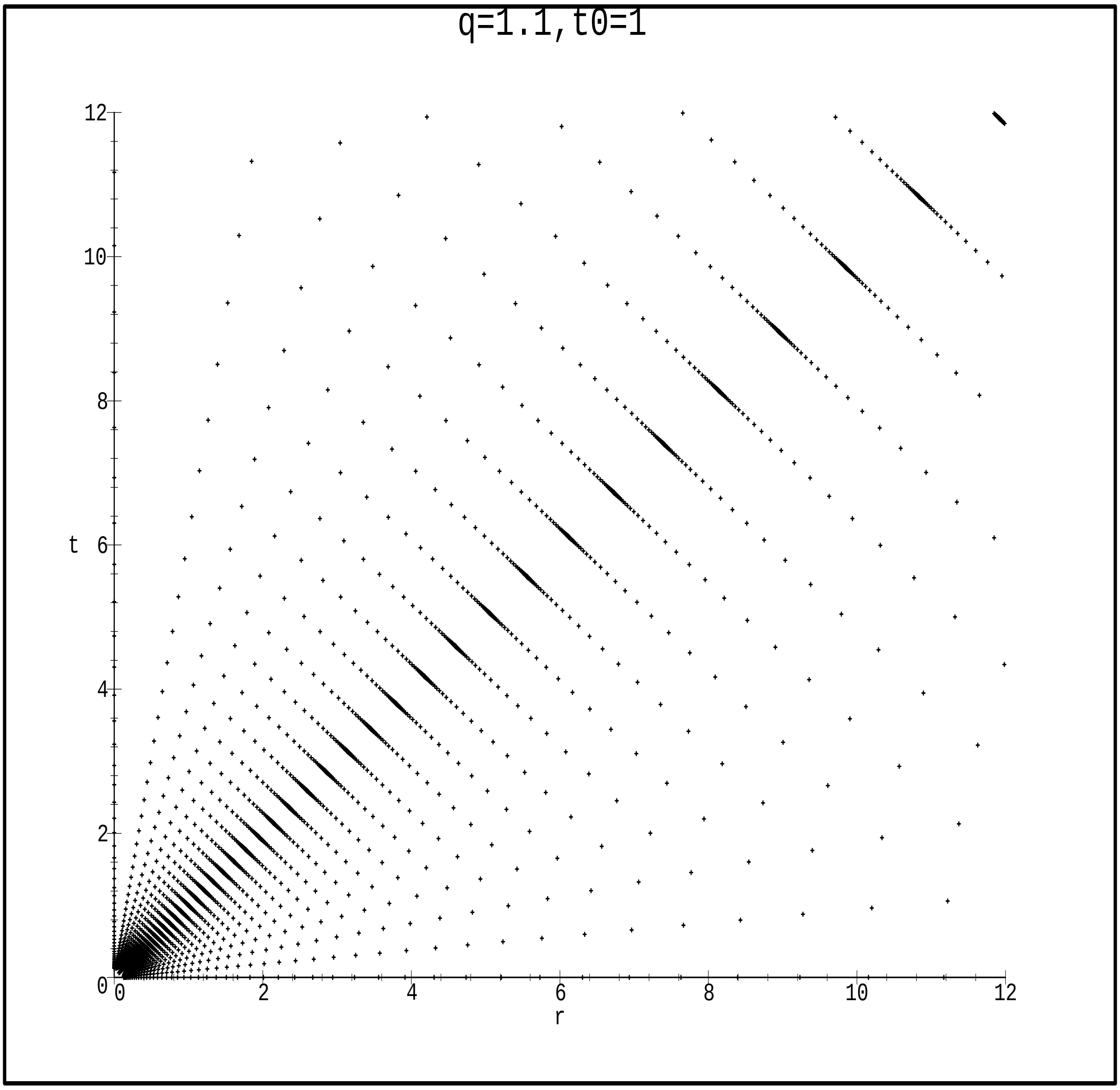

Figure 3.1: Admissible values of t versus those of r for $q = 1.1$ and $t_0 = 1$.

Then there are the hyperbolas determined by the points with the same invariant length s^2, or in other words by the points with the same value of M. In the figure in the next page they emerge better. Changing n means to move along one of such hyperbolas. In the picture the points lying on the hyperbolas with $|s|$ greater than a certain value are cut off, so that the remaining hyperbolas can be clearly observed.

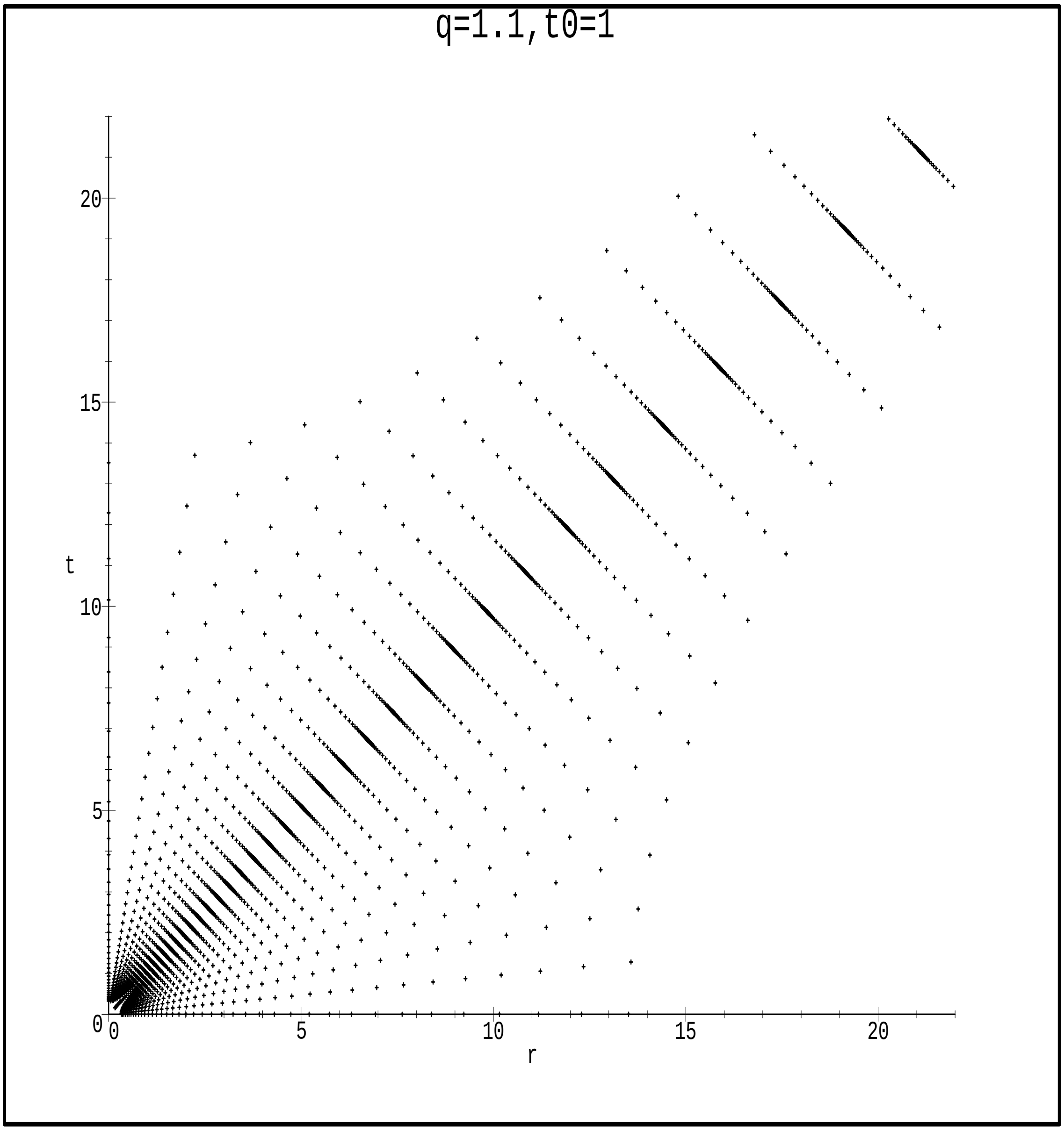

Figure 3.2: Admissible values of t versus those of r for $q = 1.1$, $t_0 = 1$ and $|s| < 14.5$.

The points become the more and more dense as $q \to 1$, as can be seen by comparing the next figure with the first one. In the limit $q \to 1$ the whole time-like part of the lattice collapses to the point $(0, t_0)$, the light-cone to the point (τ_0, τ_0) and the space-like part to $(l, 0)$. To recover the ordinary undeformed results in that limit it is necessary to consider densities of the quantities, hence to divide by the singular factor $q - 1$. Notice that points around the origin are more and more dense. [3]

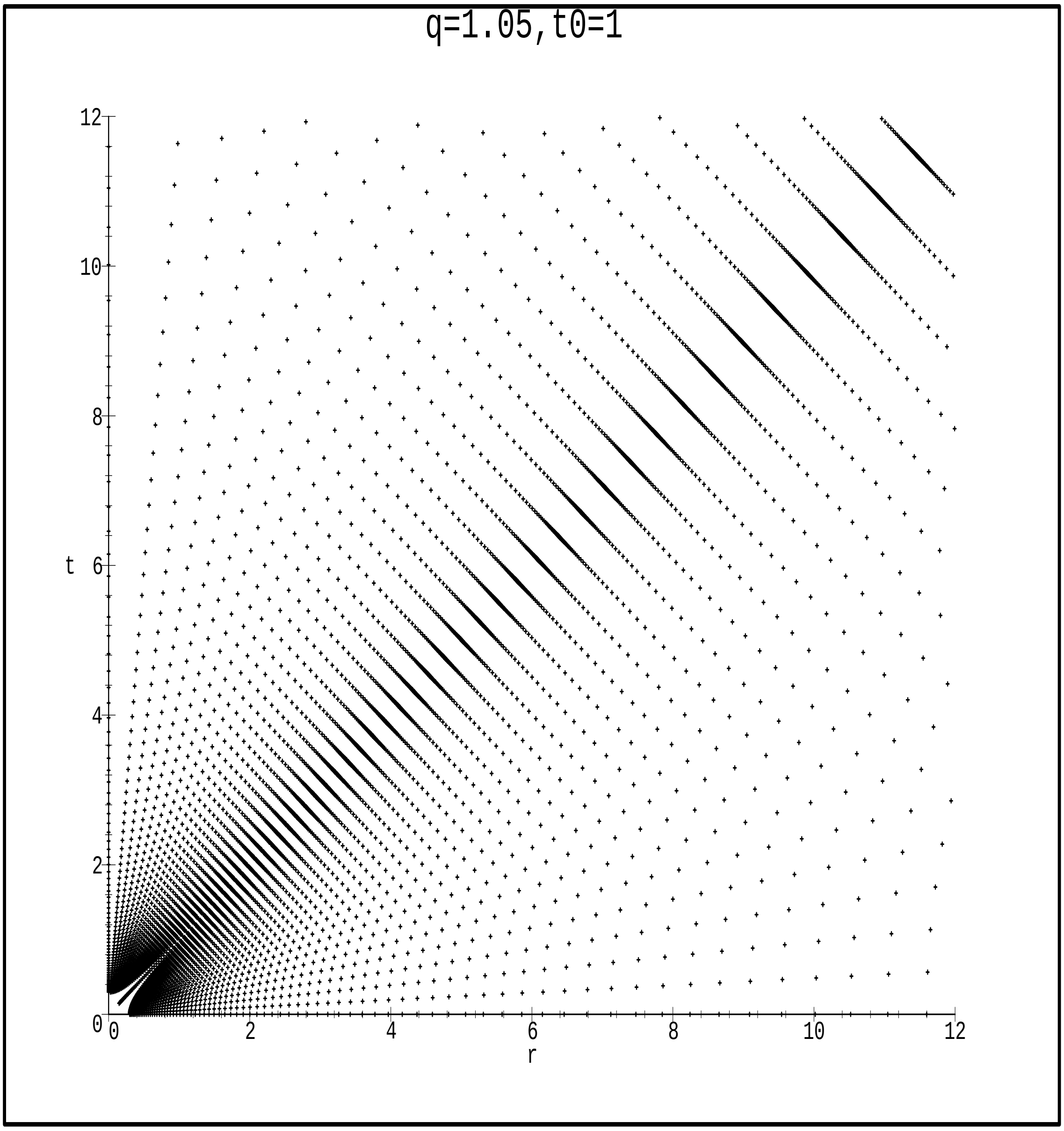

Figure 3.3: Admissible values of t versus those of r for $q = 1.05$ and $t_0 = 1$.

[3] It is only a feature of the Maple program drawing the plot that it does not appear so.

3.4 Study of the XX-relations

In this section the reduced matrix elements $\langle j',\mu\|X^-\|j,\nu\rangle$ are calculated. A reduced matrix element may be denoted as $\langle j',s'^2,t'\|X^-\|j,s^2,t\rangle$ and a state of the Hilbert space as $|j,m,s^2,t\rangle$. Alternatively, as the eigenvalues of $X\cdot X$ are labeled by M and those of X^0 are labeled by n, it is possible to denote the eigenstates as $|j,m,n\rangle$ for $s^2=0$ and as $|j,m,M,n\rangle$ for $s^2\neq 0$, where $M\in\mathbb{Z}$, $n\in\mathbb{Z}$ for s^2 space-like, $n=0,1\ldots$ for s^2 time-like. In the next sections both notations are used, depending on which one is more convenient to write a given formula.

The first important observation is that by definition X^- commutes with $X\cdot X$ and X^0. As a consequence it does not change the eigenvalues s^2,t:

$$\langle j',s'^2,t'\|X^-\|j,s^2,t\rangle = 0 \qquad \text{for} \quad (s'^2,t')\neq(s^2,t) \tag{3.49}$$

The XX-relations (2.1) are the starting point for the calculation of the reduced matrix elements $\langle j',s^2,t\|X^-\|j,s^2,t\rangle$.

Computation of the matrix elements of the first of the XX-commutation relations (B.1):

$$X^3X^+ - q^2X^+X^3 = (1-q^2)X^0X^+ \tag{3.50}$$

for $j'=j+1$ and insertion of the expressions (3.6, 3.8) for the X^+,X^3-matrix elements respectively yields the following equations for the reduced matrix elements:

$$\begin{aligned}
&\langle j+1,s^2,t\|X^-\|j+1,s^2,t\rangle\langle j+1,s^2,t\|X^-\|j,s^2,t\rangle\\
&\cdot\sqrt{[j+m+1][j+m+2]}\tfrac{q^{m-2j+2}}{\sqrt{[2]}}\left(\tfrac{q^{2j+5}-q^{-2j-3}}{q^2-1}\right)\\
&-\langle j+1,s^2,t\|X^-\|j,s^2,t\rangle\langle j,s^2,t\|X^-\|j,s^2,t\rangle\\
&\cdot\sqrt{[j+m+1][j+m+2]}\tfrac{q^{m-2j+2}}{\sqrt{[2]}}\left(\tfrac{q^{2j+1}-q^{-2j+1}}{q^2-1}\right)\\
&=(1-q^2)q^{m-2j}\sqrt{[j+m+1][j+m+2]}\,t\langle j+1,s^2,t\|X^-\|j,s^2,t\rangle
\end{aligned}$$

X^0 commutes with X^- and therefore it is replaced with its eigenvalue t.

By dividing both sides of this relation through the common coefficient $\frac{1}{\sqrt{[2]}}q^{m-2j+2}\sqrt{[j+m+1][j+m+2]}$ (this is possible, because $j+m\geq 0$ and this factor never vanishes) one obtains:
For $j\geq 0$:

$$\begin{aligned}
&[2j+4]\langle j+1,s^2,t\|X^-\|j+1,s^2,t\rangle\langle j+1,s^2,t\|X^-\|j,s^2,t\rangle\\
&-[2j]\langle j+1,s^2,t\|X^-\|j,s^2,t\rangle\langle j,s^2,t\|X^-\|j,s^2,t\rangle\\
&=\sqrt{[2]}(1-q^2)q^{-2}t\langle j+1,s^2,t\|X^-\|j,s^2,t\rangle
\end{aligned} \tag{3.51}$$

In a similar way, computation of the matrix elements X^3X^+-commutation relation (3.50) for $j'=j$ gives:

For $j \geq 1$:

$$\begin{aligned} &-[2j+3]\langle j, s^2, t\|X^-\|j+1, s^2, t\rangle\langle j+1, s^2, t\|X^-\|j, s^2, t\rangle \\ &+[2j-1]\langle j, s^2, t\|X^-\|j-1, s^2, t\rangle\langle j-1, s^2, t\|X^-\|j, s^2, t\rangle \\ &+\tfrac{[2j]-[2j+2]}{[2]}\langle j, s^2, t\|X^-\|j, s^2, t\rangle\langle j, s^2, t\|X^-\|j, s^2, t\rangle \\ &= \tfrac{q^2-1}{\sqrt{[2]}}q^2 t\langle j, s^2, t\|X^-\|j, s^2, t\rangle \end{aligned} \tag{3.52}$$

Notice that to obtain (3.52) it is necessary to divide the resulting equation by $\sqrt{[2]}q^m[j-m][j+m+1]$. This factor vanishes for $j=m$ and therefore the equation is valid only for $j>0$, but not for $j=0$.

By dividing both sides of eq. (3.51) by $\langle j+1, s^2, t\|X^-\|j, s^2, t\rangle$ a recursion formula in j for $\langle j, s, t\|X^-\|j, s, t\rangle$ follows:

$$\langle j+1, s^2, t\|X^-\|j+1, s^2, t\rangle = \frac{[2j]}{[2j+4]}\langle j, s^2, t\|X^-\|j, s^2, t\rangle + \frac{\sqrt{[2]}(1-q^2)q^{-2}}{[2j+4]}t \tag{3.53}$$

This recursion relation can be solved:

$$\begin{aligned} \langle j, s^2, t\|X^-\|j, s^2, t\rangle &= (1-q^2)q^{-2}\sqrt{[2]}t\frac{\sum_{k=0}^{j-1}(q^{2k+2}-q^{-2k-2})}{[2j][2j+2](q-q^{-1})} \\ &= (1-q^2)q^{-2}\sqrt{[2]}t\frac{[j][j+1]}{[2j][2j+2]} \\ &= (1-q^2)q^{-2}\sqrt{[2]}t\frac{1}{\{j\}\{j+1\}} \end{aligned} \tag{3.54}$$

which is valid for $j \geq 1$.

Here the notation is used:

$$\{a\} = q^a + q^{-a} \tag{3.55}$$

For $j=0$ it is:

$$\langle 0, s^2, t\|X^-\|0, s^2, t\rangle = 0 \quad \text{for} \quad j=0 \tag{3.56}$$

because X^- would change $m=0$ to $m=-1$ and this is not possible for $j=0$.

The recursion relation which follows from (3.52) is not enough to determine $\langle j+1, s^2, t\|X^-\|j, s^2, t\rangle$, because it does not fix $\langle 1, s^2, t\|X^-\|0, s^2, t\rangle$.

For this reason the next step is to use the eqn. (3.17) for the scalar product $A \circ B$ of two vectors A,B to determine an equation for the matrix elements of $X \circ X = g_{AB}X^A X^B$.

With the notation:

$$\begin{aligned}\rho(j+1) &= [2j+1][2j+3]\langle j,s^2,t\|X^-\|j+1,s^2,t\rangle\langle j+1,s^2,t\|X^-\|j,s^2,t\rangle \\ &= -q^{-2j-2}\mid \langle j+1,s^2,t\|X^-\|j,s^2,t\rangle \mid^2\ [2j+1][2j+3] \qquad (3.57)\end{aligned}$$

eqn. (3.17) yields for $X\circ X$:

$$\begin{aligned}r^2 &= \frac{(1-q^2)^2[2]q^{-4}t^2}{\{j\}^2\{j+1\}^2}\frac{1}{q+q^{-1}}q^2[2j+2][2j] \qquad (3.58)\\ &\quad -\rho(j+1)q^2\frac{[2j+2]}{[2j+1]}-\rho(j)q^2\frac{[2j]}{[2j+1]}\end{aligned}$$

where $r^2=s^2+t^2$ is the eigenvalue of $X\circ X$. Notice that for $j=0$ this means:

$$r^2=-q^2[2]\rho(1) \qquad (3.59)$$

Eqn. (3.58) is a recursion relation which fixes $\rho(j)$. It could be solved directly, but it is easier to apply a different method, namely use (3.52).

By substituting (3.57) and (3.54) in (3.52) for $j>0$, $\rho(j)$ has to solve also the recursion relation:

$$\rho(j+1)-\rho(j)=(1-q^2)^2[2]q^{-4}t^2\frac{[2j+1]}{\{j+1\}^2\{j\}^2} \qquad (3.60)$$

Now (3.60) can be used to express $\rho(j+1)$ in terms of $\rho(j)$ in (3.58) and thus to get $\rho(j)$.

By shifting $j\to j+1$, the result is:

$$\begin{aligned}\rho(j+1) &= -\frac{r^2}{q^2[2]}+\frac{[2j]q^{-4}(1-q^2)^2}{\{j\}^2\{j+1\}^2}(1+\frac{[2j+2]}{[2]})t^2 \qquad (3.61)\\ &= \frac{1}{q^2[2]}\left\{-r^2+\frac{[j][j+2](q-q^{-1})^2}{\{j+1\}^2}t^2\right\}\end{aligned}$$

As a byproduct the interesting summation formula is found:

$$\sum_{l=1}^{j}\frac{[2l+1]}{\{l\}^2\{l+1\}^2}=\frac{[2j]}{[2]\{j\}^2\{j+1\}^2}(1+\frac{[2j+2]}{[2]}) \qquad (3.62)$$

It can be interpreted as a consistency condition for the two recursion relations (3.58) and (3.60) to be compatible.

Namely, (3.60) is formally solved by:

$$\rho(j+1)=\rho(1)+(1-q^2)^2[2]q^{-4}t^2\sum_{l=1}^{j}\frac{[2l+1]}{\{l\}^2\{l+1\}^2} \qquad (3.63)$$

and (3.62) follows by comparison between (3.61) and (3.63) by means of (3.59). Eqn. (3.62) can be shown directly by induction and thus everything is consistent. In the limit $q \to 1$ it reduces to the well-known formula:

$$\sum_{l=1}^{j} 2l + 1 = j(j+1) + j = j(j+2) \tag{3.64}$$

The following important consequence can be drawn from formula (3.61). The definition (3.57) of ρ implies that in any case $\rho(j) \leq 0$, because it must be $| \langle j+1 \| X^- \| j \rangle |^2 \geq 0$. If one takes the limit $j \to \infty$ of the summation formula and substitutes $s^2 = -t^2 + r^2$ one gets:

$$\rho(j) \to -\frac{s^2}{q^2[2]} \quad \text{for} \quad j \to \infty$$

which is positive for $s^2 < 0$. This means that for s^2 time-like j has to be bounded, and cannot increase arbitrarily if the time is fixed. The recursion relations (3.58) and (3.60) have to break down for a certain value of j.

Now, as the eigenvalues of X^0 and $X \circ X$ are known, it is possible to insert these values in the equations (3.54) and (3.61) and to get the explicit expressions for $\langle j, M, n \| X^- \| j, M, n \rangle$ and ρ.
For $s^2 = 0$, $j \geq 1$, $n \in \mathbb{Z}$:

$$\langle j, n \| X^- \| j, n \rangle = (1 - q^2) q^{-2} \tau_0 q^n \sqrt{[2]} \frac{1}{\{j\}\{j+1\}}$$

For s^2 time-like, $j \geq 1$, $M \in \mathbb{Z}$, $n \geq 0$:

$$\langle j, M, n \| X^- \| j, M, n \rangle = (1 - q^2) q^{-2} \frac{t_0 q^M \{n+1\}}{\sqrt{[2]}\{j\}\{j+1\}} \tag{3.65}$$

For s^2 space-like, $j \geq 1$, $M, n \in \mathbb{Z}$:

$$\langle j, M, n \| X^- \| j, M, n \rangle = (1 - q^2)(q - q^{-1}) q^{-2} \frac{l_0 q^M [n]}{\sqrt{[2]}\{j\}\{j+1\}}$$

For $s^2 = 0$, $n \in \mathbb{Z}$, $j \geq 0$:

$$\rho(j+1) = -\tau_0^2 q^{2n-2} \frac{[2]}{\{j+1\}^2}$$

For s^2 time-like, $M \in \mathbb{Z}$, $n \geq 0$, $0 \leq j \leq n$:

$$\rho(j+1) = -q^{2M-2} t_0^2 (q - q^{-1})^2 \frac{[n-j][n+j+2]}{\{j+1\}^2[2]} \tag{3.66}$$

For s^2 space-like, $M, n \in \mathbb{Z}$, $j \geq 0$:

$$\rho(j+1) = -q^{2M-2} l_0^2 \frac{\{n-j-1\}\{n+j+1\}}{\{j+1\}^2[2]}$$

In the case s^2 time-like negativity of $\rho(j)$ imposes the condition:

$$(q^{n-j} - q^{-n+j})(q^{n+j+2} - q^{-n-j-2}) \geq 0 \tag{3.67}$$

and hence:

$$j \leq n \quad \text{for } s^2 \text{ time-like.} \tag{3.68}$$

In fact, it can be verified that if $j = n$, then $\rho(j+1) = 0$ and hence

$$\langle j{+}1 = n{+}1, M, n \| X^- \| j = n, M, n \rangle = \langle j = n, M, n \| X^- \| j{+}1 = n{+}1, M, n \rangle = 0$$

so that the recursion relations (3.58) and (3.60) effectively break down as anticipated.

Notice that for s^2 time-like $\rho(j+1) \to 1$ for $q \to 1$. This is consistent with the previous observation that $r^2 \to 0$ in this case.

Notice also that for $r^2 = 0$ the consistency condition (3.68) $j \leq n$ implies that $j = 0$. This means that the spin in this representation is 0. The same condition implies that j is integer, otherwise it could never be $j = n$ and the recursion relations (3.58) and (3.60) would not break down and it would be necessary to allow for ρ to become negative, which is not possible. This is consistent with the fact that j describes only the orbital angular momentum.

The question remains open, whether it would be possible to suppose, e.g. in the case $s^2 < 0$, that there is no state for which $r = 0$ in the spectrum. In this case (3.24) would imply that the spectrum of X^0 is $s\frac{q^{x+n+1}+q^{-x-n-1}}{[2]}$ with $0 < x < 1$. But this is not compatible with the condition $j = n+x$ for j integer, which would have to be satisfied for (3.58) and (3.60) to break down. Therefore $x = 0$.

Through (3.57) it is possible to compute $\langle j{+}1, M, n \| X^- \| j, M, n \rangle$ from $\rho(j{+}1)$ up to a phase. This phase cannot be fixed by the algebraic relations alone, the algebra allows a freedom in fixing the phase between eigenfunctions which belong different eigenvalues. So it is possible to simply put the phase to 1 and the final result is:

For $s^2 = 0$, $n \in \mathbb{Z}$, $j \geq 0$:

$$\langle j+1, n \| X^- \| j, n \rangle = \tau_0 \frac{q^{n+j}\sqrt{[2]}}{\{j+1\}\sqrt{[2j+1][2j+3]}}$$

For s^2 time-like, $M \in \mathbb{Z}$, $n \geq 0$, $0 \leq j \leq n$:

$$\langle j+1, M, n \| X^- \| j, M, n \rangle = t_0 q^{M+j}(q - q^{-1}) \frac{\sqrt{[n-j][n+j+2]}}{\{j+1\}\sqrt{[2][2j+1][2j+3]}} \tag{3.69}$$

For s^2 space-like, $M, n \in \mathbb{Z}$, $j \geq 0$:

$$\langle j+1, M, n \| X^- \| j, M, n\rangle = l_0 q^{M+j} \frac{\sqrt{\{n+j+1\}\{n-j-1\}}}{\{j+1\}\sqrt{[2][2j+1][2j+3]}}$$

Applying (3.13) it follows:
For $s^2 = 0$, $n \in \mathbb{Z}$, $j \geq 1$:

$$\langle j-1, n \| X^- \| j, n\rangle = -\tau_0 \frac{q^{n-j-1}\sqrt{[2]}}{\{j\}\sqrt{[2j-1][2j+1]}}$$

For s^2 time-like, $M \in \mathbb{Z}$, $n \geq 0$, $1 \leq j-1 \leq n$:

$$\langle j-1, M, n \| X^- \| j, M, n\rangle = -t_0 q^{M-j-1}(q-q^{-1}) \frac{\sqrt{[n-j+1][n+j+1]}}{\{j\}\sqrt{[2][2j-1][2j+1]}} \quad (3.70)$$

For s^2 space-like, $M, n \in \mathbb{Z}$, $j \geq 1$:

$$\langle j-1, M, n \| X^- \| j, M, n\rangle = -l_0 q^{M-j-1} \frac{\sqrt{\{n+j\}\{n-j\}}}{\{j\}\sqrt{[2][2j-1][2j+1]}}$$

and

$$\langle j', M', n' \| X^- \| j, M, n\rangle = 0 \quad \text{for} \quad j' \neq j \pm 1, j; \quad M' \neq M, n' \neq n \quad (3.71)$$

3.5 Eigenvalues of X^3

As a next step the eigenvalues of X^3 are studied. The operators X^0, $X \cdot X$, X^3, $X_T := -qX^+X^- - \frac{1}{q}X^-X^+$ and τ commute. It would have been possible to choose (X^0, X^3, τ, X_T) as a complete set of commuting observables.

First, the eigenvalues can be determined by a purely algebraic method, exactly as it has been done in sect. 3.3.1 for the eigenvalues of X^0. Observe that this does not provide any information about the actual existence and on the multiplicity of the eigenvalues, nor on the self-adjointness of X^3. These problems are only partially treated here.

The idea is to commute $X^0 - X^3$ with X^+. By means of (B.1) it follows:

$$X^+(X^0 - X^3) = \frac{1}{q^2}(X^0 - X^3)X^+ \quad (3.72)$$

and therefore the eigenvalues of $X^0 - X^3$ are fixed by $d_0' q^{2\kappa}$. The eigenvalues of X^3 are given by $-(d_0' q^{2\kappa} - t)$. By choosing $d_0' = t_0 q^{M-n}$ for s^2 time-like and $d_0' = l_0 q^{M-n+1}$ for s^2 space-time, the eigenvalues of X^3 are given by:

For s^2 time-like

$$z_{\kappa,M,n} = -\frac{t_0 q^M}{[2]}(q^{2\kappa-n+1} + q^{2\kappa-n-1} - q^{n+1} - q^{-n-1}) \tag{3.73}$$

with $\kappa \in \mathbb{Z}$, $0 \le \kappa \le n$
For s^2 space-like

$$z_{\kappa,M,n} = -\frac{l_0 q^M}{[2]}(q^{2\kappa-n+2} + q^{2\kappa-n} - q^n + q^{-n}) \tag{3.74}$$

with $\kappa \le n-1$.

The range of κ is determined by observing that it must be

$$r^2 \ge z^2$$

If the eigenvalues of $X_T = -qX^+X^- - \frac{1}{q}X^-X^+$ are denoted by

$$r_T := r^2 - z^2 \tag{3.75}$$

this is just the condition:

$$r_T \ge 0 \tag{3.76}$$

that the eigenvalues of the operator X_T are not negative.

Hence, for s^2 time-like

$$\frac{q^2}{(1+q^2)^2}(q^n - q^{-n})(q^{n+2} - q^{-n-2}) \ge \left(\frac{1}{[2]}(q^{2\kappa-n+1} + q^{2\kappa-n-1} - q^{n+1} - q^{-n-1})\right)^2 \tag{3.77}$$

This is a second order equation in $q^{2\kappa}$ and provides the bounds:

$$0 \le \kappa \le n \qquad \text{for} \quad s^2 \text{ time-like} \tag{3.78}$$

For s^2 space-like:

$$\frac{q^2}{(1+q^2)^2}(q^{n+1} + q^{-n-1})(q^{n-1} + q^{-n+1}) \ge \left(\frac{1}{[2]}(q^{2\kappa-n+2} + q^{2\kappa-n} - q^n + q^{-n})\right)^2 \tag{3.79}$$

It is a second order equation in $q^{2\kappa}$. Nevertheless it gives only an upper bound, because the lowest solution is negative and $q^{2\kappa}$ cannot become negative. The result is:

$$\kappa \le n-1 \qquad \text{for} \quad s^2 \text{ space-like} \tag{3.80}$$

For s^2 time-like it gives the condition $0 \le \kappa \le n$. Once that the eigenvalues of X^3 are known, the eigenvalues r_T^2 of X_T are computed by means of the definition $r_T^2 = r^2 - z^2$. The result is the following:

For s^2 time-like

$$r_T^2 = \frac{t_0^2 q^{2M}}{[2]} \left(2q^{2k+1} + 2q^{2(k-n)-1} - [2](q^{2(2k-n)} + 1)\right) \tag{3.81}$$

For $n \to \infty$ for $\kappa = 0$ it holds $r_T \to \frac{t_0^2 q^{2M}(q-\frac{1}{q})}{[2]}$. Therefore it will never vanish and there is a vertical asymptote in the plot of the eigenvalues of X^3 versus those of X_T.
For s^2 space-like

$$r_T^2 = \frac{l_0^2 q^{2M}}{[2]} \left(2q^{2k+1} - 2q^{2(k-n)+1} - [2](q^{2(2k-n+1)} - 1)\right) \tag{3.82}$$

Notice that in the limit $\kappa \to -\infty$, r_T^2 tends to the particular value $\frac{l_0 q^M}{[2]}$ and there is a vertical asymptote.

For $s^2 = 0$ it is possible to choose $d_0' = [2]\tau_0 q^{-n+1}$. Then the eigenvalues of X^3 are determined by the condition:
For $s^2 = 0$

$$z_n = -\tau_0(q^{2\kappa-n+2} + q^{2\kappa-n} - q^{-n}) \tag{3.83}$$

The eigenvalues of X_T are:

$$r_T^2 = \tau_0^2[2] \left(2q^{2\kappa+1} - [2]q^{2(2\kappa-n+1)}\right) \tag{3.84}$$

and the condition $r_T^2 \geq 0$ imposes:

$$2q - [2]q^{2(\kappa-n+1)} \geq 0 \tag{3.85}$$

so that there is the bound:

$$k \leq n-1 \qquad \text{for} \quad s^2 = 0 \tag{3.86}$$

Again there is a vertical asymptote: $r_T \to 0$ for $\kappa \to \infty$.

For $q \to 1$ the whole spectrum of X^3 collapses to 0 for s^2 time-like, to $-l_0 q^M$ for s^2 space-like, and to $-\tau_0$ for $s^2 = 0$. Analogously $r_T \to 0$ (both for $s^2 = 0$ and for $s^2 \neq 0$.) Only densities of these quantities on the lattice give the classical limit.

The picture in the next page shows a plot of the admissible eigenvalues z of X^3 versus those r_T of X_T. The Maple program drawing this figure is reported in Appendix E.

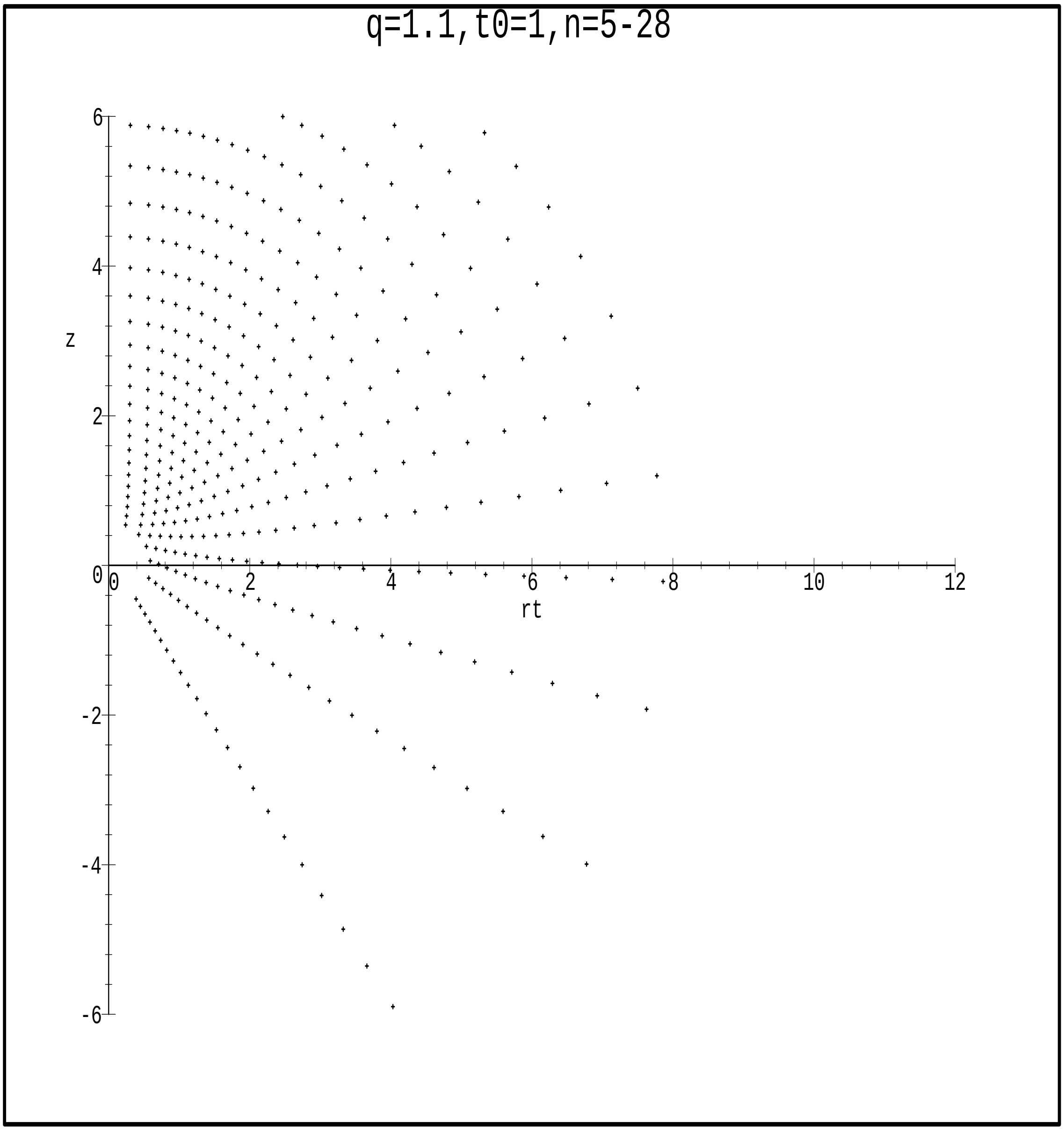

Figure 3.4: Eigenvalues of X^3 versus those of X_T for $q = 1.1$ in the case s^2 time-like.

A very striking fact is that they are completely asymmetric: if t_0 is positive there are much more positive eigenvalues than negative eigenvalues. This had to be expected. The eigenvalues of X^0 with which the calculation has been made were all positive. In order to obtain the same number of positive and negative eigenvalues it is necessary to take the direct sum of two representations, one with positive parameter t_0 and the other with negative parameter $-t_0$.

τ has eigenvalues q^{-4m} and this is not symmetric with respect to the transformation $m \to -m$. Once again, this shows that there is no symmetry between positive and negative values, if not more than one representation is considered. In fact, the momentum operators are are not even self-adjoint otherwise.

As q decreases the points become more and more dense.

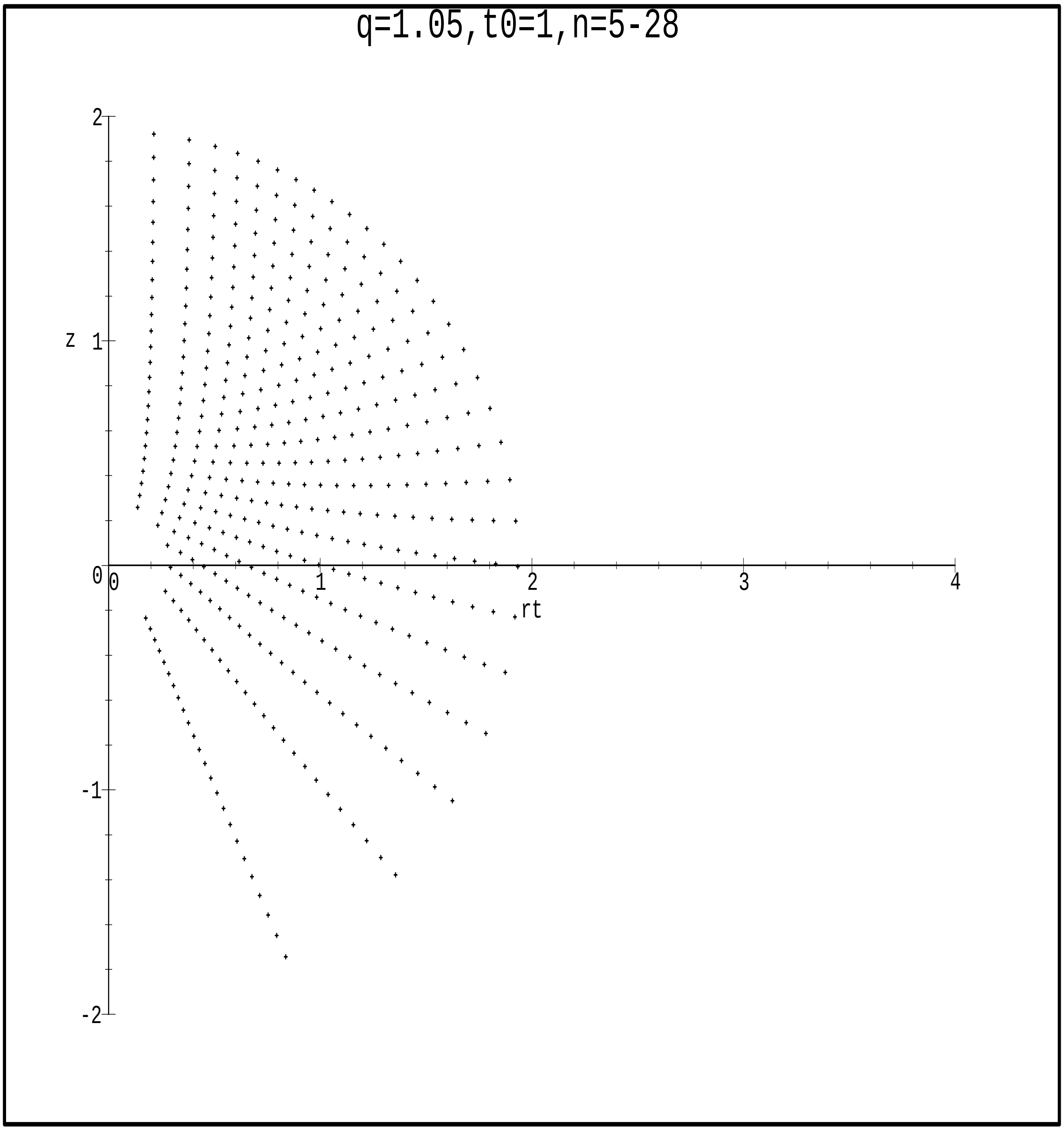

Figure 3.5: Eigenvalues of X^3 versus those of X_T for $q = 1.05$ in the case s^2 time-like.

The following figure shows the eigenvalues of X^3 plotted versus those of r_T in the case that s^2 space-like. The program drawing this plot can be found in

Appendix E.

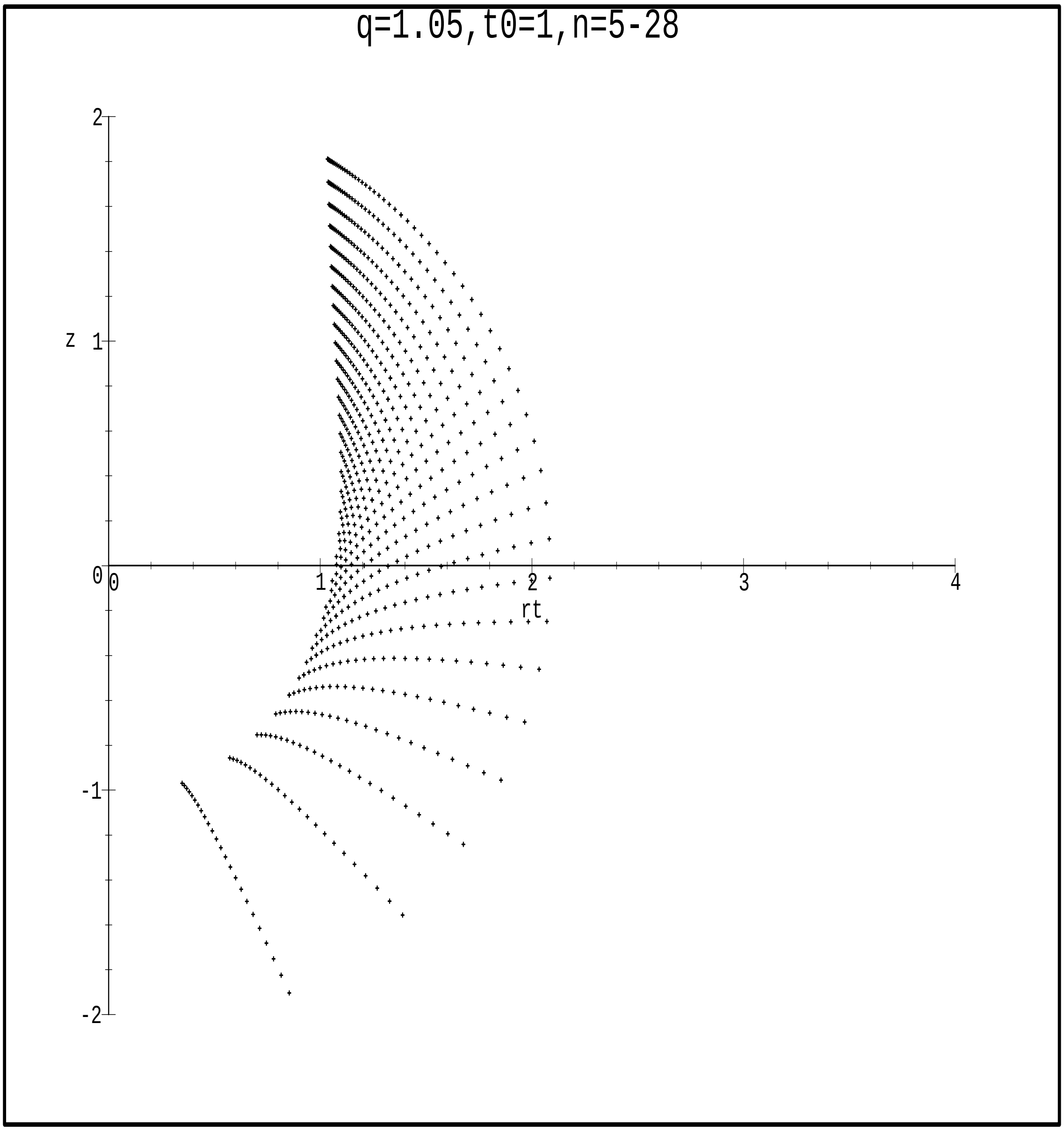

Figure 3.6: Eigenvalues of X^3 versus those of X_T for $q = 1.05$ in the case s^2 space-like.

The last figure in the next page is a plot of the eigenvalues of X^3 versus the values of r_T in the case $s^2 = 0$. Like in the case s^2 space-like there is no lower bound for κ.

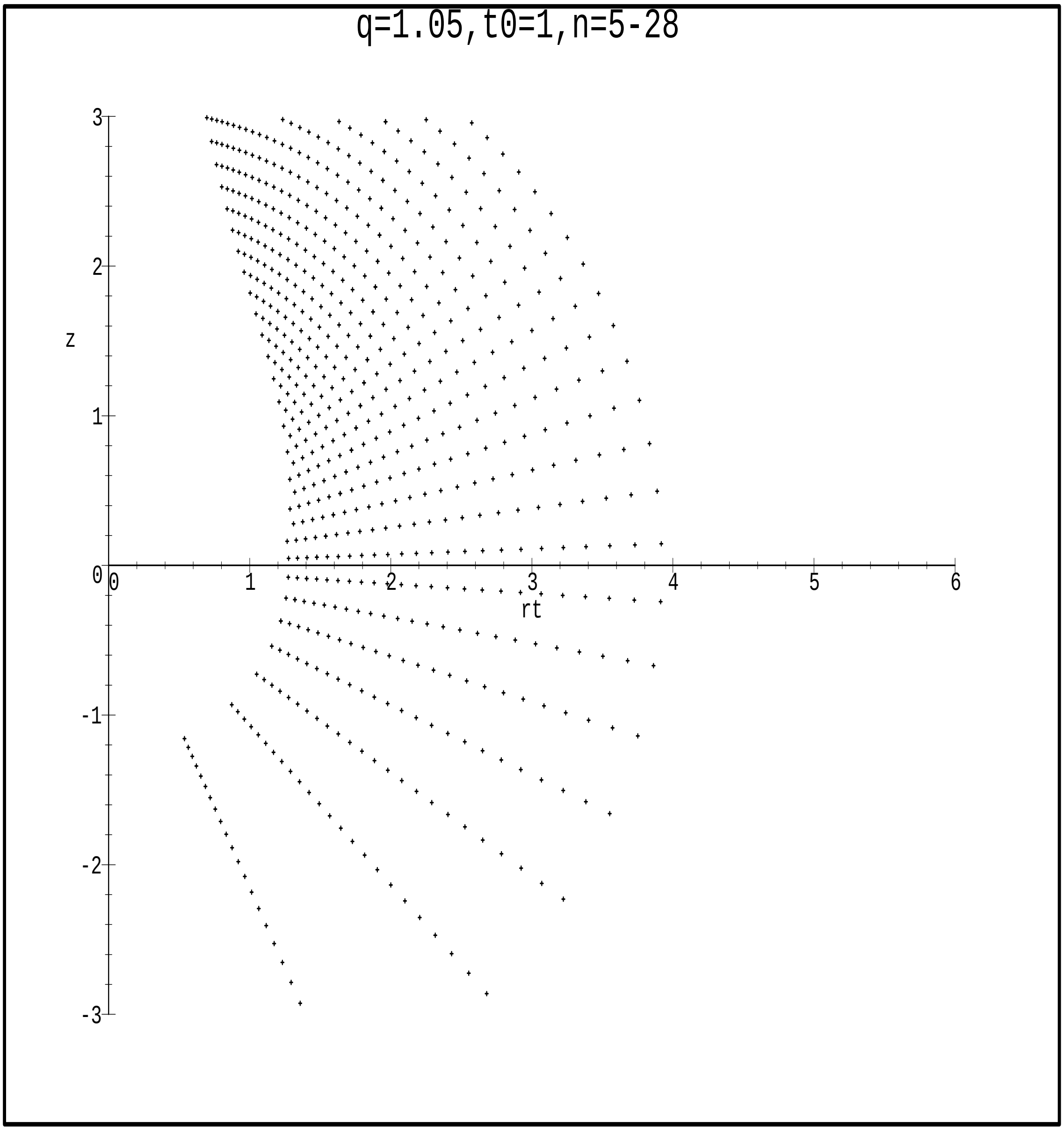

Figure 3.7: Eigenvalues of X^3 versus those of X_T for $q = 1.05$ in the case $s^2 = 0$.

Now, consider the case s^2 time-like. In this case X^3 can be diagonalized.

The condition $j \leq n$ implies that $|j = n, m = n, M, n\rangle$ is an eigenfunction of X^3, because from equation (3.8) the only other values to which j could jump are $n \pm 1$. But the condition $j \leq n$ prohibits jumping to $n + 1$ and the condition $m \leq j$ prohibits jumping to $n - 1$. By means of (3.8) and (3.54):

$$X^3|j = n, m = n, M, n\rangle = -\frac{t_0 q^{M-1}}{[2]}(q^n - q^{-n})|j = n, m = n, M, n\rangle \qquad (3.87)$$

Now, with a similar reasoning (3.8) gives:

$$\begin{aligned} X^3\,|j=n,m=n-1,M,n\rangle &\sim \alpha\,|j=n-1,m=n-1,M,n\rangle \\ &\quad +\beta\,|j=n,m=n-1,M,n\rangle \qquad (3.88) \\ X^3\,|j=n-1,m=n-1,M,n\rangle &\sim \gamma\,|j=n-1,m=n-1,M,n\rangle \\ &\quad +\alpha\,|j=n,m=n-1,M,n\rangle \end{aligned}$$

with α, β and γ some coefficients. More in general, if $m \geq 0$ and n, M are fixed, then the matrix describing X^3 is block diagonal and the states $|j,m,M,n\rangle$ with $j = m, \dots n$ constitute one block.

In complete analogy, if $m < 0$ the states $|j,m,M,n\rangle$ with $j = |m|, \dots n$ constitute one block. In particular, for $m = -n$ it holds:

$$X^3|j=n,m=-n,M,n\rangle = \frac{t_0 q^{M+1}}{[2]}(q^n - q^{-n})|j=n,m=-n,M,n\rangle \qquad (3.89)$$

In each of these blocks X^3 can be separately diagonalized and the eigenvalues of X^3 are:

$$z_{\kappa,M,n} = -\frac{t_0 q^M}{[2]}(q^{2\kappa-n+1} + q^{2\kappa-n-1} - q^{n+1} - q^{-n-1}) \qquad (3.90)$$

with $\kappa \in \mathbb{Z}$, $0 \leq \kappa \leq n$ and each eigenvalue has a multiplicity $n+1$. This result coincides with the previous one (3.73), but now also the multiplicity is known, and in principle it is possible to explicitly determine the eigenvectors.

Let $M_{n,m}$ be the matrix giving X^3 in the block determined by m and n.

Then $M_{n,m}$ has the eigenvalues $z_{\kappa,M,n}$ with $m \leq \kappa \leq n$ for $m \geq 0$ and $0 \leq \kappa \leq n+m$ for $m < 0$.

The proof is by induction on n and m.

As the eigenvalues have already been determined with the algebraic method, and because this diagonalization procedure cannot be performed in the case s^2 space-like (there is no lower bound for κ) not the whole proof is given here.

Sketch of a proof:

Case $n = 0$: It must be $j = m = 0$ and (3.8) implies that $z_{0,0,0} = 0$

Case $n = 1$:

For $m = 1$ it is enough to apply (3.87). Then the eigenvalue is $z_{1,M,1} = -\frac{t_0 q^{M-1}(q-q^{-1})}{[2]}$. In a similar way, for $m = -1$ from (3.87) follows that another eigenvalue is $z_{0,M,1} = \frac{t_0 q^{M+1}(q-q^{-1})}{[2]}$

For $m = 0$ the matrix describing X^3 is the 2×2 matrix:

$$M_{n=1,m=0} = \begin{pmatrix} \frac{t_0 q^M (q-q^{-1})^2}{[2]} & \frac{t_0 q^M (q-q^{-1})}{[2]} \\ \frac{t_0 q^M (q-q^{-1})}{[2]} & 0 \end{pmatrix} \qquad (3.91)$$

It can easily be diagonalized, e.g. by calculating the characteristic polynomial, and the result is that the eigenvalues are $z_{1,M,1}$ and $z_{0,M,1}$. Therefore both eigenvalues $z_{1,M,1}$ and $z_{0,M,1}$ have multiplicity 2.

This proves the assumption for $n = 0, 1$.
Case of general n:

For $m = n$ eqn. (3.87) immediately provides the eigenvalue $z_{n,M,n}$ and for $m = -n$ eqn. (3.89) the eigenvalue $z_{0,M,n}$.

When $m = n-1$ the 2×2 matrix describing X^3 is given by:

$$M_{n,n-1} = \begin{pmatrix} -\frac{t_0 q^M q^{2n}(q^{-1}+q^{-3}-q)-q^{-2n-1}}{(q^n+q^{-n})(q+q^{-1})} & \frac{t_0 q^{M+n-1}(q-q^{-1})}{(q^n+q^{-n})} \\ \frac{t_0 q^{M+n-1}(q-q^{-1})}{(q^n+q^{-n})} & -\frac{t_0 q^{M-1}(q^{n-1}+q^{-n+1}-q)(q^{n+1}+q^{-n-1}))}{(q^n+q^{-n})(q+q^{-1})} \end{pmatrix}$$

The corresponding eigenvalues are $z_{n,M,n}$ and $z_{n-1,M,n}$.

A similar calculation proves that for $m = -(n-1)$ the eigenvalues of $M_{n,-(n-1)}$ are given by $z_{0,M,n}$ and $z_{1,M,n}$.

In the case of general m, as already stated, not the whole proof is given here. To provide a hint that it is true, it is only shown that $z_{m,M,n}$ for $m \geq 0$ and $z_{n+m,M,n}$ for $m < 0$ is an eigenvalue of $M_{n,m}$ if it is assumed that the other eigenvalues are $z_{\kappa,M,n}$ with $m+1 \leq \kappa \leq n$ for $m \geq 0$ and $0 \leq \kappa \leq n+m-1$ for $m < 0$.

If m is fixed, the trace of $M_{n,m}$ is:

$$\text{Trace}(M_{n,m}) = \sum_{j=m}^{n} \frac{t_0 q^M (q^{n+1}+q^{-n-1})}{(q^j+q^{-j})(q^{j+1}-q^{-j-1})}\left(q^{2m} - \frac{q^{2j+1}+q^{-2j-1}}{[2]}\right) \tag{3.92}$$

for $m \geq 0$ and

$$\text{Trace}(M_{n,m}) = \sum_{j=0}^{m+n} \frac{t_0 q^M (q^{n+1}+q^{-n-1})}{(q^j+q^{-j})(q^{j+1}-q^{-j-1})}\left(q^{2m} - \frac{q^{2j+1}+q^{-2j-1}}{[2]}\right) \tag{3.93}$$

for $m < 0$.

By induction on n, if the eigenvalues for $n-1$ are $z_{\kappa,M,n}$ with $m \leq \kappa \leq n-1$ for $m \geq 0$ then:

$$\begin{aligned} \text{Trace}(M_{n-1,m}) &= \sum_{\kappa=m}^{n-1} z_{\kappa,M,n-1} \qquad (3.94)\\ &= \frac{1}{(q^2-1)[2]}\left\{q^{n-1}\left(q^3(n-m-1)+q(m-n-1)\right)\right.\\ &\quad \left. +q^{-n+1}(n-m)(q-q^{-1})+q^{2m-n}(q+q^{-1})\right\} \end{aligned}$$

Analogously, if the eigenvalues are $z_{\kappa,M,n}$ with $0 \leq \kappa \leq n+m$ for $m < 0$ then:

$$\begin{aligned} \text{Trace}(M_{n-1,m}) &= \sum_{\kappa=0}^{m+n-1} z_{\kappa,M,n-1} \qquad (3.95)\\ &= -\frac{(q^{n+2m+1}-1)(q+q^{-1})-(q^n+q^{-n})(m+n)}{(q^2-1)[2]} \end{aligned}$$

By expressing Trace(M_n) in terms of Trace(M_{n-1}) it is possible to prove that for $m \geq 0$:

$$\mathrm{Trace}(M_{n,m}) = \sum_{\kappa=m}^{n} z_{\kappa,M,n} \tag{3.96}$$

and for $m < 0$:

$$\mathrm{Trace}(M_{n,m}) = \sum_{\kappa=0}^{m+n} z_{\kappa,M,n} \tag{3.97}$$

□

Notice that the result (3.73) for the eigenvalues of X^3 is the same as the result obtained in [31, 32].

Each eigenvalue $z_{\kappa,M,n}$ has multiplicity $n+1$, because it appears $\kappa+1$ times for $m \geq 0$ and $n-\kappa$ times for $m < 0$.

Chapter 4

The matrix elements of R, S, U

In this chapter the matrix elements of the generators R^A, S^A, U of the q-deformed Lorentz algebra are calculated.

The generators of the q-deformed Lorentz algebra commute with $X \cdot X$ and therefore do not change the value of s^2 which is labeled by M. As already noticed in chap. 2, from the commutation relations with the rotations follows that R and S transform like vectors under the action of the q-deformed rotations, and hence the m-dependence of the matrix elements is the same as for the X-matrix elements (3.6,3.7,3.8).

From the conjugation properties of R, it immediately follows:

$$\begin{aligned} \langle j, M, n \| S^- \| j, M, n' \rangle &= -\overline{\langle j, M, n' \| R^- \| j, M, n \rangle} \\ \langle j+1, M, n \| S^- \| j, M, n' \rangle &= q^{2j+2}\overline{\langle j, M, n' \| R^- \| j+1, M, n \rangle} \\ \langle j-1, M, n \| S^- \| j, M, n' \rangle &= q^{-2j}\overline{\langle j, M, n' \| R^- \| j-1, M, n \rangle} \end{aligned} \tag{4.1}$$

Therefore the reduced matrix elements of S^- can be calculated from the reduced matrix elements of R^-.

The reduced matrix elements of R^+,S^+ and R^3,S^3 can be obtained from those of R^-,S^- respectively, by means of the Wigner-Eckart theorem (3.10),(3.11) and (3.12).

Also, it should be remarked that U behaves like a scalar under the action of $SU_q(2)$. Therefore U does not change j nor m and

$$\langle j, m, s^2, t' | U | j, m, s^2, t \rangle = \langle j, s^2, t' \| U \| j, s^2, t \rangle \tag{4.2}$$

4.1 The j-dependence of the R-matrix elements

In this section the j-dependence of the reduced matrix elements of R^- is determined.

Eqns. (3.18) and (3.19) yield (3.22) and this condition implies that the only non-vanishing matrix elements of $R \circ X$ and U are those between n and $n \pm 1$.

By means of (3.17) it is possible to see that non-vanishing matrix elements of X^- and R^- lead to non-vanishing matrix elements of $X \circ R$. Therefore, the only non-vanishing matrix elements of R^- are those between n and $n \pm 1$.

The j-dependence of the reduced matrix elements of R^- can be computed by means of the RX-relations (2.23).

The first step is to use the relation:

$$R^+X^+ = qX^+R^+ \tag{4.3}$$

to find a recursion relation in j.

Taking the matrix elements between $j+2$ and j of (4.3) yields:

$$\frac{\langle j+2, s^2, t' \| R^- \| j+1, s^2, t\rangle}{\langle j+1, s^2, t' \| R^- \| j, s^2, t\rangle} = q\frac{\langle j+2, s^2, t' \| X^- \| j+1, s^2, t'\rangle}{\langle j+1, s^2, t \| X^- \| j, s^2, t\rangle} \tag{4.4}$$

and in a similar way taking them between $j-2$ and j yields:

$$\frac{\langle j-2, s^2, t', \| R^- \| j-1, s^2, t\rangle}{\langle j-1, s^2, t' \| R^- \| j, s^2, t\rangle} = q\frac{\langle j-2, s^2, t' \| X^- \| j-1, s^2, t'\rangle}{\langle j-1, s^2, t \| X^- \| j, s^2, t\rangle} \tag{4.5}$$

The recursion relation (4.4) allows to express the reduced matrix element $\langle j+1, s^2, t' \| R^- \| j, s^2, t\rangle$ in terms of $\langle 1, s^2, t' \| R^- \| 0, s^2, t\rangle$, while (4.5) allows to express $\langle j-1, s^2, t' \| R^- \| j, s^2, t\rangle$ in terms of $\langle 0, s^2, t' \| R^- \| 1, s^2, t\rangle$.

The result is:

$$\begin{aligned}\langle j+1, M, n' \| R^- \| j, M.n\rangle &= (\delta_{n',n+1} + \delta_{n',n-1})\langle 1, M, n' \| R^- \| 0, M, n\rangle q^j \\ &\cdot \prod_{k=1}^{j} \frac{\langle k+1, M, n' \| X^- \| k, M, n'\rangle}{\langle k, M, n \| X^- \| k-1, M, n\rangle}\end{aligned} \tag{4.6}$$

which is valid for $j \geq 1$,

$$\begin{aligned}\langle j-1, M, n' \| R^- \| j, M, n\rangle &= (\delta_{n',n+1} + \delta_{n',n-1})\langle 0, s^2, t' \| R^- \| 1, s^2, t\rangle q^{-j-1} \\ &\cdot \prod_{k=2}^{j} \frac{\langle k-1, M, n \| X^- \| k, M, n\rangle}{\langle k-2, M, n' \| X^- \| k-1, M, n'\rangle}\end{aligned} \tag{4.7}$$

which is valid for $j \geq 2$.

By means of the expressions (3.69) for $\langle j+1, M, n \| X^- \| j, M, n\rangle$ and (3.70) for $\langle j-1, M, n \| X^- \| j, M, n\rangle$ these equations can be written more explicitly:
In the case $s^2 = 0$, $n, n' \in \mathbb{Z}$

$$\begin{aligned}\langle j+1, n' \| R^- \| j, n\rangle &= (\delta_{n',n+1} + \delta_{n',n-1})\langle 1, n' \| R^- \| 0, n\rangle \\ &\cdot \frac{q^{(n'-n+2)j}[2]\sqrt{[3]}}{\{j+1\}\sqrt{[2j+3][2j+1]}}\end{aligned} \tag{4.8}$$

$$\begin{aligned}\langle j-1, n' \| R^- \| j, n\rangle &= (\delta_{n',n+1} + \delta_{n',n-1})\langle 0, n' \| R^- \| 1, n\rangle \\ &\cdot \frac{q^{(n-n'-2)(j-1)}[2]\sqrt{[3]}}{\{j\}\sqrt{[2j-1][2j+1]}}\end{aligned}$$

In the case s^2 time-like, $M \in \mathbb{Z}$, $n' \geq j+1$, $n \geq j$, $j = 0, 1, \ldots$

$$\langle j+1, M, n' \| R^- \| j, M, n \rangle = (\delta_{n',n+1} + \delta_{n',n-1}) \langle 1, M, n' \| R^- \| 0, M, n \rangle \tag{4.9}$$
$$\cdot \frac{q^{2j}[2]\sqrt{[3]}}{\{j+1\}\sqrt{[2j+3][2j+1]}} \sqrt{\frac{[(n'-n)(j+1)+n'+1][(n'-n)j+n'+1]}{[2n'-n+1][n'+1]}}$$

while in the case s^2 time-like, $M \in \mathbb{Z}$, $n' \geq j-1$, $n \geq j$, $j = 1, 2, \ldots$

$$\langle j-1, M, n' \| R^- \| j, M, n \rangle = (\delta_{n',n+1} + \delta_{n',n-1}) \langle 0, M, n' \| R^- \| 1, M, n \rangle$$
$$\cdot \frac{q^{-2j+2}[2]\sqrt{[3]}}{\{j\}\sqrt{[2j-1][2j+1]}} \sqrt{\frac{[n'-(n'-n)j+1][n-(n'-n)j+1]}{[2n-n'+1][n+1]}}$$

In the case s^2 space-like, $M, n, n' \in \mathbb{Z}$, $j = 0, 1, \ldots$

$$\langle j+1, M, n' \| R^- \| j, M, n \rangle = (\delta_{n',n+1} + \delta_{n',n-1}) \langle 1, M, n' \| R^- \| 0, M, n \rangle \tag{4.10}$$
$$\frac{q^{2j}[2]\sqrt{[3]}}{\{j+1\}\sqrt{[2j+3][2j+1]}} \sqrt{\frac{\{(n'-n)(j+1)+n'\}\{(n'-n)j+n'\}}{\{2n'-n\}\{n'\}}}$$

For s^2 space-like, $M, n, n' \in \mathbb{Z}$, $j = 1, 2, \ldots$

$$\langle j-1, M, n' \| R^- \| j, M, n \rangle = (\delta_{n',n+1} + \delta_{n',n-1}) \langle 0, M, n' \| R^- \| 1, M, n \rangle$$
$$\frac{q^{-2j+2}[2]\sqrt{[3]}}{\{j\}\sqrt{[2j-1][2j+1]}} \sqrt{\frac{\{n'-(n'-n)j\}\{n-(n'-n)j\}}{\{2n-n'\}\{n\}}}$$

In order to calculate the reduced matrix elements $\langle j, M, n' \| R^- \| j, M, n \rangle$, the relation (4.3) is not enough, it is necessary to use also:

$$R^+(X^3 - X^0) = \frac{1}{q}(X^3 - X^0)R^+ \tag{4.11}$$

By taking the matrix elements of (4.11) between $j+1$ and j and dividing through $q^{m-j}\sqrt{[j+m+1][j+m+2]}$, it follows:

$$-q^{m+2}\sqrt{[2]}[j-m+1]\langle j+1, s^2, t' \| R^- \| j+1, s^2, t \rangle \langle j+1, s^2, t \| X^- \| j, s^2, t \rangle$$
$$+\langle j+1, s^2, t' \| R^- \| j, s^2, t \rangle \frac{[2]q^{m-j}}{\{j+1\}} \left(\frac{t'\{m+1\}}{\{j+2\}} - \frac{t\{m\}}{\{j\}} \right) \tag{4.12}$$
$$+q^{m+2}\sqrt{[2]}[j-m]\langle j, s^2, t' \| R^- \| j, s^2, t \rangle \langle j+1, s^2, t' \| X^- \| j, s^2, t' \rangle = 0$$

Notice that this equation only holds for $j \geq 1$, because the expression (3.54) has been used for $\langle j, s^2, t \| X^- \| j, s^2, t \rangle$, which holds only for $j \geq 1$.

By combining (4.12) and the relation which follows from (4.3) when taking the matrix elements between $j+1$ and j:

$$\begin{gathered}\langle j+1,s^2,t'\|R^-\|j+1,s^2,t\rangle\langle j+1,s^2,t\|X^-\|j,s^2,t\rangle \\ +\langle j+1,s^2,t'\|R^-\|j,s^2,t\rangle\langle j,s^2,t\|X^-\|j,s^2,t\rangle = \\ q\langle j+1,s^2,t'\|X^-\|j+1,s^2,t'\rangle\langle j+1,s^2,t'\|R^-\|j,s^2,t\rangle \\ +q\langle j+1,s^2,t'\|X^-\|j,s^2,t'\rangle\langle j,s^2,t'\|R^-\|j,s^2,t\rangle\end{gathered} \tag{4.13}$$

one finds that for $j \geq 1$:

$$\begin{gathered}\frac{\langle j+1,s^2,t'\|R^-\|j+1,s^2,t\rangle}{\langle j,s^2,t'\|R^-\|j,s^2,t\rangle} = \\ \frac{\langle j+1,s^2,t'\|X^-\|j,s^2,t'\rangle}{\langle j+1,s^2,t\|X^-\|j,s^2,t\rangle}\frac{(t'\{j+1\}-t\{j+2\})\{j\}}{(t'\{j\}-t\{j+1\})\{j+2\}}\end{gathered} \tag{4.14}$$

$$\begin{gathered}\langle j+1,s^2,t'\|R^-\|j,s^2,t\rangle\sqrt{1+q^2}\left(\frac{t'\{j\}-t\{j+1\}}{\{j\}\{j+1\}}\right) \\ = \langle j,s^2,t'\|R^-\|j,s^2,t\rangle\langle j+1,s^2,t'\|X^-\|j,s^2,t'\rangle q^{j+\frac{5}{2}}\end{gathered} \tag{4.15}$$

For $j \geq 0$

$$\begin{gathered}\langle j+1,s^2,t'\|R^-\|j,s^2,t\rangle\sqrt{1+q^2}\left(\frac{t'\{j+1\}-t\{j+2\}}{\{j+1\}\{j+2\}}\right) \\ = \langle j+1,s^2,t'\|R^-\|j+1,s^2,t\rangle\langle j+1,s^2,t\|X^-\|j,s^2,t\rangle q^{j+\frac{5}{2}}\end{gathered} \tag{4.16}$$

In a similar way, it can be proved that for $j \geq 2$:

$$\begin{gathered}\langle j-1,s^2,t'\|R^-\|j,s^2,t\rangle\sqrt{1+q^2}\left(\frac{t'\{j\}-t\{j-1\}}{\{j-1\}\{j\}}\right) \\ = \langle j-1,s^2,t'\|R^-\|j-1,s^2,t\rangle\langle j-1,s^2,t\|X^-\|j,s^2,t\rangle q^{-j+\frac{3}{2}}\end{gathered} \tag{4.17}$$

For $j \geq 1$

$$\begin{gathered}\langle j-1,s^2,t'\|R^-\|j,s^2,t\rangle\sqrt{1+q^2}\left(\frac{t'\{j+1\}-t\{j\}}{\{j\}\{j+1\}}\right) \\ = \langle j,s^2,t'\|R^-\|j,s^2,t\rangle\langle j-1,s^2,t'\|X^-\|j,s^2,t'\rangle q^{-j+\frac{3}{2}}\end{gathered} \tag{4.18}$$

Using (4.15) and the expression (4.6) for $\langle j+1,M,n'\|R^-\|j,M,n\rangle$ it is possible to express $\langle j,M,n'\|R^-\|j,M,n\rangle$ in terms of $\langle 1,M,n'\|R^-\|0,M,n\rangle$. In complete analogy (4.18) allows to express $\langle j,M,n'\|R^-\|j,M,n\rangle$ in terms of $\langle 0,M,n'\|R^-\|1,M,n\rangle$.

The result is:

$$\begin{aligned}\langle j, M, n' \| R^- \| j, M, n\rangle &= (\delta_{n',n+1} + \delta_{n',n-1})\langle 1, M, n' \| R^- \| 0, M, n\rangle \frac{q^{-2}}{r}[2]\sqrt{[3]} \\ &\cdot \left(\frac{t'\{j\} - t\{j+1\}}{\{j\}\{j+1\}}\right) \prod_{k=2}^{j} \left(\frac{\langle k, M, n' \| X^- \| k-1, M, n'\rangle}{\langle k, M, n \| X^- \| k-1, M, n\rangle}\right) \\ &= (\delta_{n',n+1}\delta_{n',n-1})\langle 0, M, n' \| R^- \| 1, M, n\rangle \frac{(-q^2)}{r'}[2]\sqrt{[3]} \qquad (4.19) \\ &\cdot \left(\frac{t'\{j+1\} - t\{j\}}{\{j\}\{j+1\}}\right) \prod_{k=2}^{j} \left(\frac{\langle k-1, M, n \| X^- \| k, M, n\rangle}{\langle k-1, M, n' \| X^- \| k, M, n'\rangle}\right)\end{aligned}$$

These formulas can be written out more explicitly, by taking into account the expressions (3.69), (3.70):
For $s^2 = 0$, $n', n \in \mathbb{Z}$, $j = 1, 2, \ldots$,

$$\langle j, n' \| R^- \| j, n\rangle = (\delta_{n',n+1} + \delta_{n',n-1})\langle 1, n' \| R^- \| 0, n\rangle \frac{(1 - q^{-2(n'-n)})[2]\sqrt{[3]}}{q^2(q^{j+1} + q^{-j-1})(q^j + q^{-j})}$$

For s^2 time-like, $M \in \mathbb{Z}$, $n' \geq j$, $n \geq j$, $j = 1, 2, \ldots$

$$\langle j, M, n' \| R^- \| j, M, n\rangle = (n' - n)(\delta_{n',n+1} + \delta_{n',n-1})\langle 1, M, n' \| R^- \| 0, M, n\rangle$$
$$\frac{q^{-2}[2]\sqrt{[3]}(q - q^{-1})}{(q^{j+1} + q^{-j-1})(q^j + q^{-j})} \sqrt{\frac{[(n'-n)j + n' + 1][n - (n'-n)j + 1]}{[2n' - n + 1][n' + 1]}}$$

For s^2 space-like, $M \in \mathbb{Z}$, $n, n' \in \mathbb{Z}$, $j = 1, 2, \ldots$

$$\langle j, M, n' \| R^- \| j, M, n\rangle = (n' - n)(\delta_{n',n+1} + \delta_{n',n-1})\langle 1, M, n' \| R^- \| 0, M, n\rangle$$
$$\frac{q^{-2}[2]\sqrt{[3]}(q - q^{-1})}{(q^{j+1} + q^{-j-1})(q^j + q^{-j})} \sqrt{\frac{\{(n'-n)j + n'\}\{n - (n'-n)j\}}{\{2n' - n\}\{n'\}}}$$

Notice that (4.19) induces a relation between $\langle 1, M, n' \| R^- \| 0, M, n\rangle$ and $\langle 0, M, n' \| R^- \| 1, M, n\rangle$, because the two expressions of $\langle j, M, n' \| R^- \| j, M, n\rangle$, one in terms of $\langle 1, M, n' \| R^- \| 0, M, n\rangle$ and one in terms of $\langle 0, M, n' \| R^- \| 1, M, n\rangle$, must provide the same result. This relation is independent of j as can be seen by induction. Writing it out e.g. for $j = 2$ gives:

$$\langle 1, s^2, t' \| R^- \| 0, s^2, t\rangle = \langle 0, s^2, t' \| R^- \| 1, s^2, t\rangle(-q^4)\frac{r}{r'}\frac{t'\{3\} - t\{2\}}{t'\{2\} - t\{3\}}\frac{\rho_{r,t}(2)}{\rho_{r',t'}(2)} \qquad (4.20)$$

More explicitly:
For $s^2 = 0$

$$\langle 1, n' \| R^- \| 0, n\rangle = -\langle 0, n' \| R^- \| 1, n\rangle q^{-2(n-n'-2)} \qquad (4.21)$$

For s^2 time-like,$n' \geq 1$,$n \geq 0$, $M \in \mathbb{Z}$

$$\langle 1, M, n' \| R^- \| 0, M, n\rangle = -\langle 0, M, n' \| R^- \| 1, M, n\rangle q^4 \sqrt{\frac{[2n'-n+1][n'+1]}{[2n-n'+1][n+1]}}$$

For s^2 space-like, $M, n, n' \in \mathbb{Z}$

$$\langle 1, M, n' \| R^- \| 0, M, n\rangle = -\langle 0, M, n' \| R^- \| 1, M, n\rangle q^4 \sqrt{\frac{\{2n'-n\}\{n'\}}{\{2n-n'\}\{n\}}}$$

where $n' = n \pm 1$.

The same relation can be obtained in a more straightforward way by taking the matrix elements between $j = 0$ and $j = 0$ of (B.9) and substituting (3.18) or (3.19) for the matrix elements of U.

4.2 Computation of $\langle 1, M, n' \| R^- \| 0, M, n\rangle$

To determine the reduced matrix elements of R^- it is necessary to compute $\langle 1, M, n' \| R^- \| 0, M, n\rangle$.

As a first step (3.18) is used to find a connection between the matrix elements $\langle 0, 0, s^2, t' | U | 0, 0, s^2, t\rangle$ and $\langle 1, s^2, t' \| R^- \| 0, s^2, t\rangle$:

$$\begin{aligned} \langle 0, 0, s^2, t' | U | 0, 0, s^2, t\rangle &= -\frac{(2t' - (q+q^{-1})t)\, q^3 [2]^2 [3]}{r'^2} \\ &\quad \cdot \langle 0, s^2, t' \| X^- \| 1, s^2, t'\rangle \langle 1, s^2, t' \| R^- \| 0, s^2, t\rangle \end{aligned} \tag{4.22}$$

The result is:
For $s^2 = 0$

$$\langle 0, 0, n | U | 0, 0, n-1\rangle = [2]^{\frac{3}{2}} \sqrt{[3]} (q - \frac{1}{q}) \langle 1, n \| R^- \| 0, n-1\rangle \tag{4.23}$$

$$\langle 0, 0, n | U | 0, 0, n+1\rangle = -q^2 [2]^{\frac{3}{2}} \sqrt{[3]} (q - \frac{1}{q}) \langle 1, n \| R^- \| 0, n+1\rangle$$

For s^2 time-like, $n \geq 1$

$$\begin{aligned} &\langle 0, 0, M, n | U | 0, 0, M, n-1\rangle = \\ &\quad [2]^{\frac{3}{2}} \sqrt{[3]} (q^2 - 1) \sqrt{\frac{[n]}{[n+2]}} \langle 1, M, n \| R^- \| 0, M, n-1\rangle \end{aligned} \tag{4.24}$$

For s^2 time-like, $n \geq 0$

$$\begin{aligned} &\langle 0, 0, M, n | U | 0, 0, M, n+1\rangle = \\ &\quad -[2]^{\frac{3}{2}} \sqrt{[3]} (q^2 - 1) \sqrt{\frac{[n+2]}{[n]}} \langle 1, M, n \| R^- \| 0, M, n+1\rangle \end{aligned} \tag{4.25}$$

For s^2 space-like

$$\langle 0,0,M,n|U|0,0,M,n-1\rangle = \\ [2]^{\frac{3}{2}}\sqrt{[3]}(q^2-1)\sqrt{\frac{\{n-1\}}{\{n+1\}}}\langle 1,M,n\|R^-\|0,M,n-1\rangle \tag{4.26}$$

$$\langle 0,0,M,n|U|0,0,M,n+1\rangle = \\ -[2]^{\frac{3}{2}}\sqrt{[3]}(q^2-1)\sqrt{\frac{\{n+1\}}{\{n-1\}}}\langle 1,M,n\|R^-\|0,M,n+1\rangle \tag{4.27}$$

U is a symmetric operator, and therefore:

$$\overline{\langle 0,0,M,n'|U|0,0,M,n\rangle} = \langle 0,0,M,n|U|0,0,M,n'\rangle \tag{4.28}$$

The following notation can be introduced:

$$\begin{aligned}\Gamma_M(n) &= \langle 0,0,M,n|U|0,0,M,n+1\rangle\langle 0,0,M,n+1|U|0,0,M,n\rangle \\ &= |\langle 0,0,M,n|U|0,0,M,n+1\rangle|^2 \end{aligned}\tag{4.29}$$

By taking the matrix elements between $j=0$ and $j=0$ of (2.22):

$$U^2 - 1 = (q^4-1)^2 R\circ R \tag{4.30}$$

the recursion formula follows:
For $s^2 = 0$

$$\Gamma(n)(1+q^2) + \Gamma(n-1)(1+q^{-2}) = 1$$

For s^2 time-like

$$\frac{[2]}{[n+1]}\left(\Gamma_M(n)[n+2] + \Gamma_M(n-1)[n]\right) = 1 \tag{4.31}$$

For s^2 space-like

$$\frac{[2]}{(q^n+q^{-n})}\left(\Gamma_M(n)(q^{n+1}+q^{-n-1}) + \Gamma_M(n-1)(q^{n-1}+q^{-n+1})]\right) = 1$$

The solution of these recursion relations is:
For $s^2 = 0$

$$\Gamma(n) = \frac{1+(-1)^{n+1}q^{-2n}}{[2]^2} + (-1)^n q^{-2n}\Gamma(0)$$

For s^2 time-like

$$\Gamma_M(n) = \frac{(-1)^n}{[n+1][n+2]}\left(\Gamma_M(0)[2] + \frac{1}{[2]}(-1)^n\frac{[n+1][n+2]}{[2]} - \frac{1}{[2]}\right) \tag{4.32}$$

For s^2 space-like

$$\begin{aligned}\Gamma_M(n) &= \frac{(-1)^n}{\{n\}\{n+1\}}\left(\Gamma_M(0)2[2]\right.\\ &\left.+\frac{1}{[2]^4}(-1)^n\{n\}\{n+1\}-\frac{2}{[2]^3}\right)\end{aligned}$$

These equations also fix $\Gamma(0)$: it can be found by just putting $n = 0$.

An alternative way to find an expression for $\Gamma_M(0)$, is to observe that $\Gamma_M(n)$ has to satisfy also the recursion relation following from the UR-commutation relation:

$$UR^+ = R^+U \tag{4.33}$$

and (4.33) implies:

$$\Gamma_M(n) = \Gamma_M(n-1) \tag{4.34}$$

In other words, $\Gamma_M(n)$ does not depend on n at all, and inserting (4.34) into (4.32) fixes its value:

$$\Gamma_M(n) = \Gamma_M(0) = \frac{1}{[2]^2} \tag{4.35}$$

This is the same result which is obtained by putting $n = 0$ in (4.32), so that everything is consistent.

Again, the algebra alone does not fix the phase of the U-matrix element. From the definition (4.29) of Γ_M, by choosing $\langle 0, 0, M, n'|U|0, 0, M, n\rangle$ to be real and positive, it follows:

$$\langle 0, 0, M, n|U|0, 0, M, n+1\rangle = \langle 0, 0, M, n+1|U|0, 0, M, n+1\rangle = \frac{1}{[2]} \tag{4.36}$$

Notice that there is no way to see from this relation that in the case s^2 time-like the condition $n \geq 0$ must be satisfied. This is a problem for $j = 0$. This condition can be obtained in the following way.

In the case $r' = 0$ it is not possible to use eqn. (3.22). But one immediately sees that for $r' = 0$ eqn. (3.19) is enough to fix t. It can be rewritten as:

$$\langle j, m, s^2 = -t'^2, t'|U|j, m, s^2 = -t'^2, t\rangle \left(t - \frac{1}{q}\frac{q^4+1}{q^2+1}t'\right) = 0 \tag{4.37}$$

Here the fact is used that $r^2 = 0$ implies:

$$\langle j', s^2 = -t^2, t\|X^-\|j, t, s^2 = -t^2, t\rangle = 0 \quad \text{for any } j' \tag{4.38}$$

and by the Wigner-Eckart theorem (3.11,3.12) also the reduced matrix elements of X^+ and X^3 vanish.

For this reason $\langle j, m, s^2 = -t'^2, t'|X \circ R|j, m, s^2 = -t'^2, t\rangle = 0$

From (4.37) immediately follows that for $r' = 0$, $t' = t_0 q^M$ if the matrix element $\langle j, m, s^2 = -t'^2, t'|U|j, m, s^2 = -t'^2, t\rangle$ is required not to vanish the condition has to be satisfied:

$$t = t_0 q^M \frac{q^2 + q^{-2}}{q + q^{-1}} \tag{4.39}$$

At last, it is possible to compute $\langle 1, M, n'\|R^-\|0, M, n\rangle$ through (4.22) :
For $s^2 = 0$

$$\begin{aligned}\langle 1, n\|R^-\|0, n-1\rangle &= \frac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}(q-\frac{1}{q})} \\ \langle 1, n\|R^-\|0, n+1\rangle &= -\frac{1}{q^2}\frac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}(q-\frac{1}{q})}\end{aligned} \tag{4.40}$$

For s^2 time-like

$$\begin{aligned}\langle 1, M, n\|R^-\|0, M, n-1\rangle &= \frac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}q(q-\frac{1}{q})}\sqrt{\frac{[n+2]}{[n]}} \\ \langle 1, M, n\|R^-\|0, M, n+1\rangle &= -\frac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}q(q-\frac{1}{q})}\sqrt{\frac{[n]}{[n+2]}}\end{aligned} \tag{4.41}$$

For s^2 space-like

$$\begin{aligned}\langle 1, M, n\|R^-\|0, M, n-1\rangle &= \frac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}q(q-\frac{1}{q})}\sqrt{\frac{\{n+1\}}{\{n-1\}}} \\ \langle 1, M, n\|R^-\|0, M, n+1\rangle &= -\frac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}q(q-\frac{1}{q})}\sqrt{\frac{\{n-1\}}{\{n+1\}}}\end{aligned} \tag{4.42}$$

4.3 Final results for R, S and U

Now, $\langle 1, M, n'\|R^-\|0, M, n\rangle$ can be inserted into the expressions (4.6),(4.7) and (4.19) which have been found in section 4.1 for $\langle j \pm 1, M, n'\|R^-\|j, M, n\rangle$ and $\langle j, M, n'\|R^-\|j, M, n\rangle$. In all the following expressions $M \in \mathbb{Z}$.

For $s^2 = 0$

$$\begin{aligned}\langle j+1, n'\|R^-\|j, n\rangle &= (\delta_{n',n+1} + \delta_{n',n-1})\frac{(n'-n)q^{(n'-n+2)(j+1)-3}}{\{j+1\}(q-\frac{1}{q})[2]^{\frac{3}{2}}\sqrt{[2j+1][2j+3]}} \\ &\quad \text{for } n, n' \in \mathbb{Z},\ j = 0, 1, \ldots\end{aligned}$$

$$\begin{aligned}
\langle j,n'\|R^-\|j,n\rangle &= (\delta_{n',n+1}+\delta_{n',n-1}\frac{q^{-3}}{\{j\}\{j+1\}[2]^{\frac{3}{2}}} \\
&\quad \text{for } n,n'\in\mathbb{Z},\ j=1,2,\ldots \\
\langle j-1,n'\|R^-\|j,n\rangle &= -(\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{(n-n'-2)j-3}}{\{j\}(q-\frac{1}{q})[2]^{\frac{3}{2}}\sqrt{[2j+1][2j-1]}} \\
&\quad \text{for } n,n'\in\mathbb{Z},\ j=1,2,\ldots
\end{aligned}$$

For s^2 time-like

$$\begin{aligned}
&\langle j+1,M,n'\|R^-\|j,M,n\rangle = (\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{2j-1}}{\{j+1\}[2]^{\frac{3}{2}}\sqrt{[2j+1][2j+3]}} \\
&\quad\cdot\frac{1}{q-q^{-1}}\sqrt{\frac{[(n'-n)(j+1)+n'+1][(n'-n)j+n'+1]}{[n+1][n'+1]}} \\
&\quad \text{for } n'\geq j+1,\ n\geq j,\ j=0,1,\ldots \\
&\langle j,M,n'\|R^-\|j,M,n\rangle = (\delta_{n',n+1}+\delta_{n',n-1})\frac{q^{-3}}{\{j+1\}\{j\}[2]^{\frac{3}{2}}} \\
&\quad\cdot\sqrt{\frac{[(n'-n)j+n'+1][n-(n'-n)j+1]}{[n+1][n'+1]}} \qquad (4.43) \\
&\quad \text{for } n'\geq j,\ n\geq j,\ j=1,2\ldots \\
&\langle j-1,M,n'\|R^-\|j,M,n\rangle = -(\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{-2j-3}}{\{j\}[2]^{\frac{3}{2}}\sqrt{[2j+1][2j-1]}} \\
&\quad\cdot\frac{1}{q-q^{-1}}\sqrt{\frac{[n'-(n'-n)j+1][n-(n'-n)j+1]}{[n+1][n'+1]}} \\
&\quad \text{for } n'\geq j-1,\ n\geq j,\ j=1,2\ldots
\end{aligned}$$

For s^2 space-like

$$\begin{aligned}
&\langle j+1,M,n'\|R^-\|j,M,n\rangle = (\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{2j-1}}{\{j+1\}[2]^{\frac{3}{2}}\sqrt{[2j+1][2j+3]}} \\
&\quad\cdot\frac{1}{q-q^{-1}}\sqrt{\frac{\{(n'-n)(j+1)+n'\}\{(n'-n)j+n'\}}{\{n\}\{n'\}}} \\
&\quad \text{for } n',n\in\mathbb{Z},\ j=0,1,\ldots \\
&\langle j,M,n'\|R^-\|j,M,n\rangle = (\delta_{n',n+1}+\delta_{n',n-1})\frac{q^{-3}}{\{j+1\}\{j\}[2]^{\frac{3}{2}}} \\
&\quad\cdot\sqrt{\frac{\{(n'-n)j+n'\}\{n-(n'-n)j\}}{\{n\}\{n'\}}} \\
&\quad \text{for } n',n\in\mathbb{Z},\ j=1,2,\ldots
\end{aligned}$$

$$\langle j-1,M,n'\|R^-\|j,M,n\rangle = -(\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{-2j-3}}{\{j\}[2]^{\frac{3}{2}}\sqrt{[2j+1][2j-1]}}$$
$$\cdot\frac{1}{q-q^{-1}}\sqrt{\frac{\{n'-(n'-n)j\}\{n-(n'-n)j\}}{\{n\}\{n'\}}}$$
$$\text{for } n',n\in\mathbb{Z},\ j=1,2,\dots$$

The conjugation properties (4.1) allow to compute the non-vanishing reduced matrix elements of S^-:

For $s^2=0$, $n,n'\in\mathbb{Z}$

$$\begin{aligned}
\langle j+1,n'\|S^-\|j,n\rangle &= (\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{(n'-n)(j+1)-3}}{\{j+1\}(q-\frac{1}{q})[2]^{\frac{3}{2}}\sqrt{[2j+1][2j+3]}}\\
&\quad \text{for } j=0,1,\dots\\
\langle j,n'\|S^-\|j,n\rangle &= -(\delta_{n',n+1}+\delta_{n',n-1}\frac{q^{-3}}{\{j\}\{j+1\}[2]^{\frac{3}{2}}}\\
&\quad \text{for } j=1,2,\dots\\
\langle j-1,n'\|S^-\|j,n\rangle &= -(\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{(n-n')j-3}}{\{j\}(q-\frac{1}{q})[2]^{\frac{3}{2}}\sqrt{[2j+1][2j-1]}}\\
&\quad \text{for } j=1,2,\dots
\end{aligned}$$

For s^2 time-like

$$\begin{aligned}
&\langle j+1,M,n'\|S^-\|j,M,n\rangle = (\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{-3}}{\{j+1\}[2]^{\frac{3}{2}}\sqrt{[2j+1][2j+3]}}\\
&\quad\cdot\frac{1}{q-q^{-1}}\sqrt{\frac{[(n'-n)(j+1)+n'+1][(n'-n)j+n'+1]}{[n+1][n'+1]}}\\
&\quad\text{for } n'\geq j+1,\ n\geq j,\ j=0,1,\dots\\
&\langle j,M,n'\|S^-\|j,M,n\rangle = -(\delta_{n',n+1}+\delta_{n',n-1})\frac{q^{-3}}{\{j+1\}\{j\}[2]^{\frac{3}{2}}}\\
&\quad\cdot\sqrt{\frac{[(n'-n)j+n'+1][n-(n'-n)j+1]}{[n+1][n'+1]}}\\
&\quad\text{for } n'\geq j,\ n\geq j,\ j=1,2,\dots\\
&\langle j-1,M,n'\|S^-\|j,M,n\rangle = -(\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{-3}}{\{j\}[2]^{\frac{3}{2}}\sqrt{[2j+1][2j-1]}}\\
&\quad\cdot\frac{1}{q-q^{-1}}\sqrt{\frac{[n'-(n'-n)j+1][n-(n'-n)j+1]}{[n+1][n'+1]}}\\
&\quad\text{for } n'\geq j-1,\ n\geq j,\ j=1,2,\dots
\end{aligned} \tag{4.44}$$

For s^2 space-like, $n, n' \in \mathbb{Z}$

$$\langle j+1, M, n' \| S^- \| j, M, n\rangle = (\delta_{n',n+1} + \delta_{n',n-1}) \frac{(n'-n)q^{-3}}{\{j+1\}[2]^{\frac{3}{2}}\sqrt{[2j+1][2j+3]}}$$
$$\cdot \frac{1}{q-q^{-1}} \sqrt{\frac{\{(n'-n)(j+1)+n'\}\{(n'-n)j+n'\}}{\{n\}\{n'\}}}$$
for $j = 0, 1, \ldots$
$$\langle j, M, n' \| S^- \| j, M, n\rangle = -(\delta_{n',n+1} + \delta_{n',n-1}) \frac{q^{-3}}{\{j+1\}\{j\}[2]^{\frac{3}{2}}}$$
$$\cdot \sqrt{\frac{\{(n'-n)j+n'\}\{n-(n'-n)j\}}{\{n\}\{n'\}}}$$
for $j = 1, 2, \ldots$
$$\langle j-1, M, n' \| S^- \| j, M, n\rangle = -(\delta_{n',n+1} + \delta_{n',n-1}) \frac{(n'-n)q^{-3}}{\{j\}[2]^{\frac{3}{2}}\sqrt{[2j+1][2j-1]}}$$
$$\cdot \frac{1}{q-q^{-1}} \sqrt{\frac{\{n'-(n'-n)j\}\{n-(n'-n)j\}}{\{n\}\{n'\}}}$$
for $j = 1, 2, \ldots$

It is consistent to set:

$$\langle 0, M, n' \| R^- \| 0, M, n\rangle = \langle 0, M, n' \| S^- \| 0, M, n\rangle = 0 \tag{4.45}$$

because R^- and S^- would change m to -1 and this is not possible for $j = 0$.

Finally, $\langle j \pm 1, M, n' \| R^- \| j, M, n\rangle$ and $\langle j, M, n' \| R^- \| j, M, n\rangle$ can be used to get $\langle j, m, M, n' | U | j, m, M, n\rangle$ by means of (3.18). It is a lengthy calculation. By observing that:

$$(q^{2j+1} - q^{-2j-1})(q^j + q^{-j})(q^{j+1} + q^{-j-1}) = q^{4j+2} - q^{-4j-2} + (q^{2j+1} - q^{-2j-1})(q + \frac{1}{q})$$

it turns out that the only non-vanishing matrix elements of U are:
For $s^2 = 0$, $n \in \mathbb{Z}$, $j = 0, 1, \ldots$

$$\langle j, m, n | U | j, m, n+1\rangle = \langle j, m, n+1 | U | j, m, n\rangle = \frac{1}{[2]} \tag{4.46}$$

For s^2 time-like, $n \geq j$, $j = 0, 1, \ldots$

$$\langle j, m, M, n | U | j, m, M, n+1\rangle = \langle j, m, M, n+1 | U | j, m, M, n\rangle =$$
$$\frac{1}{[2]} \sqrt{\frac{[n-j+1][n+j+2]}{[n+1][n+2]}}$$

For s^2 space-like, $n \in \mathbb{Z}$, $j = 0, 1, \ldots$

$$\langle j, m, M, n|U|j, m, M, n+1\rangle = \langle j, m, M, n+1|U|j, m, M, n\rangle = \frac{1}{[2]}\sqrt{\frac{\{n-j\}\{n+j+1\}}{\{n\}\{n+1\}}}$$

Now, all the matrix elements of the generators of the q-deformed Lorentz algebra are known.

Chapter 5

The matrix elements of the momentum

Now, the matrix elements of the momentum coordinates can be calculated. As a first step the matrix elements of P^0 are determined and then the reduced matrix elements of P^-. The spatial coordinates P^A form a vector, therefore this is enough to get all the matrix elements of P^A.

5.1 The energy

To compute the matrix elements of P^0, the X^0P^0-commutation relation (B.19) is used:

$$P^0X^0 - X^0P^0 = \frac{i}{2}(q^4\Lambda^{-\frac{1}{2}} + \Lambda^{\frac{1}{2}})U \tag{5.1}$$

A preliminary step is to find the action of Λ on a state $|j,m,M,n\rangle$. As already observed in chap. 3, the ΛX-commutation relations (2.9) imply that if $|j,m,s^2,t\rangle$ is an eigenfunction of X^0 and $X \cdot X$ with eigenvalues t and s^2 respectively, then

$$X^0(\Lambda^{-\frac{1}{2}}|j,m,s^2,t\rangle) = \frac{1}{q}\Lambda^{-\frac{1}{2}}X^0|j,m,s^2,t\rangle = \frac{1}{q}t(\Lambda^{-\frac{1}{2}}|j,m,s^2,t\rangle) \tag{5.2}$$

$$X \cdot X(\Lambda^{-\frac{1}{2}}|j,m,s^2,t\rangle) = \frac{1}{q^2}\Lambda^{-\frac{1}{2}}X \cdot X|j,m,s^2,t\rangle = \frac{1}{q^2}s^2(\Lambda^{-\frac{1}{2}}|j,m,s^2,t\rangle) \tag{5.3}$$

and therefore also $\Lambda^{-\frac{1}{2}}|j,m,s^2,t\rangle$ is an eigenfunction of X^0 and $X \cdot X$ with eigenvalues $\frac{1}{q}t$ and $\frac{1}{q^2}s^2$ respectively. It holds:

$$\begin{aligned} &\text{For } s^2 \neq 0 & \Lambda^{\frac{1}{2}}\,|j,m,M,n\rangle &= \alpha_{j,m,M,n}\,|j,m,M+1,n\rangle \\ &\text{For } s^2 = 0 & \Lambda^{\frac{1}{2}}\,|j,m,n\rangle &= \alpha_{j,m,n}\,|j,m,n+1\rangle \end{aligned} \tag{5.4}$$

Here $\alpha_{j,m,M,n}$ is a normalization constant, which has to be determined.

To fix $\alpha_{j,m,M,n}$ remember the conjugation property (2.12) of Λ, namely:

$$\overline{\Lambda^{1/2}} = q^4 \Lambda^{-1/2}$$

This means that:

$$\begin{aligned} 1 &= \langle j,m,s^2,t|j,m,s^2,t\rangle = \langle j,m,s^2,t|\Lambda^{-1/2}\Lambda^{1/2}|j,m,s^2,t\rangle \\ &= \langle j,m,s^2,t|\frac{1}{q^4}\overline{\Lambda^{1/2}}\Lambda^{1/2}|j,m,s^2,t\rangle \\ &= \frac{1}{q^4}\alpha_{j,m,M,n}\ \overline{\alpha_{j,m,M,n}}\ \langle j,m,s^2q^2,tq \mid j,m,s^2q^2,tq\rangle \\ &= \frac{1}{q^4}|\alpha_{j,m,M,n}|^2 \end{aligned}$$

and hence that:

$$|\alpha_{j,m,M,n}|^2 = q^4 \tag{5.5}$$

Again, the algebra does not fix the phase, and a consistent choice is to take $\alpha_{j,m,M,n}$ real and positive:

$$\text{For } s^2 = 0 \qquad \Lambda^{\frac{1}{2}}\ |j,m,n\rangle = q^2\ |j,m,n+1\rangle \tag{5.6}$$

$$\Lambda^{-\frac{1}{2}}\ |j,m,n\rangle = \frac{1}{q^2}\ |j,m,n-1\rangle \tag{5.7}$$

$$\text{For } s^2 \neq 0 \qquad \Lambda^{\frac{1}{2}}\ |j,m,M,n\rangle = q^2\ |j,m,M+1,n\rangle \tag{5.8}$$

$$\Lambda^{-\frac{1}{2}}\ |j,m,M,n\rangle = \frac{1}{q^2}\ |j,m,M-1,n\rangle \tag{5.9}$$

P^0 commutes with the q-deformed rotations, exactly as X^0 does, and does not change the value of j and m.

Consider the case $s'^2 \neq 0, s^2 \neq 0$. Then eqn. (5.1) implies that if $t' \neq t$ the energy P^0 has non-vanishing matrix elements only between M, n and $M \pm 1, n \pm 1$, because $\Lambda^{\pm\frac{1}{2}}$ can change M to $M \pm 1$ and U has non-vanishing matrix elements only between n and $n \pm 1$. Through (5.1) these matrix elements can be determined.

The result is:

$$\begin{aligned} &(t-t')\langle j,m,M',n'|P^0|j,m,M,n\rangle = \\ &\quad \frac{iq^2}{2}(\langle j,m,M',n'|U|j,m,M-1,n\rangle + \langle j,m,M',n'|U|j,m,M+1,n\rangle) \end{aligned} \tag{5.10}$$

Inserting the expressions (4.46) for the matrix elements of U, one obtains:

For s^2 time-like (5.11)

$$
\begin{aligned}
\langle j,m,M+1,n+1|P^0|j,m,M,n\rangle &= -\frac{iq}{2}\frac{1}{t_0q^{M+n}}\frac{1}{q^2-1}\sqrt{\frac{[n-j+1][n+j+2]}{[n+1][n+2]}}\\
\langle j,m,M+1,n-1|P^0|j,m,M,n\rangle &= -\frac{i}{2}\frac{q^{n+3}}{t_0q^M}\frac{1}{q^2-1}\sqrt{\frac{[n-j][n+j+1]}{[n][n+1]}}\\
\langle j,m,M-1,n+1|P^0|j,m,M,n\rangle &= \frac{i}{2}\frac{q^{n+5}}{t_0q^M}\frac{1}{q^2-1}\sqrt{\frac{[n-j+1][n+j+2]}{[n+1][n+2]}}\\
\langle j,m,M-1,n-1|P^0|j,m,M,n\rangle &= \frac{i}{2}\frac{q^3}{t_0q^{M+n}}\frac{1}{q^2-1}\sqrt{\frac{[n-j][n+j+1]}{[n][n+1]}}
\end{aligned}
$$

Notice that for $s^2<0$ the matrix element $\langle j,m,M',n'|P^0|j,m,M,n\rangle$ vanishes if the condition $n',n\geq j$ is not satisfied.
For s^2 space-like (5.12)

$$
\begin{aligned}
\langle j,m,M+1,n+1|P^0|j,m,M,n\rangle &= -\frac{iq^2}{2}\frac{1}{l_0q^{M+n}}\frac{1}{q^2-1}\sqrt{\frac{\{n-j\}\{n+j+1\}}{\{n\}\{n+1\}}}\\
\langle j,m,M+1,n-1|P^0|j,m,M,n\rangle &= \frac{i}{2}\frac{q^{n+2}}{l_0q^M}\frac{1}{q^2-1}\sqrt{\frac{\{n-j-1\}\{n+j\}}{\{n\}\{n-1\}}}\\
\langle j,m,M-1,n+1|P^0|j,m,M,n\rangle &= -\frac{i}{2}\frac{q^{n+4}}{l_0q^M}\frac{1}{q^2-1}\sqrt{\frac{\{n-j\}\{n+j+1\}}{\{n\}\{n+1\}}}\\
\langle j,m,M-1,n-1|P^0|j,m,M,n\rangle &= \frac{i}{2}\frac{q^4}{l_0q^{M+n}}\frac{1}{q^2-1}\sqrt{\frac{\{n-j-1\}\{n+j\}}{\{n-1\}\{n\}}}
\end{aligned}
$$

Unfortunately, the commutation relation (5.1) does not provide any information about $\langle j,m,M,n|P^0|j,m,M,n\rangle$. But in this case an alternative method to compute the matrix elements of P^0 can be used. It consists in finding a connection between the matrix elements $\langle j,m,s'^2,t'|X\circ P|j,m,s^2,t\rangle$ and $\langle j,m,s'^2,t'|P^0|j,m,s^2,t\rangle$ by means of (B.17) and then to apply this relation to get $\langle j,m,s'^2,t'|P^0|j,m,s^2,t\rangle$ from (B.16).

In fact, taking the matrix elements of eqn.(B.17):

$$\frac{1}{q^2}\frac{(q^2-1)}{(q^2+1)}(X\circ P-X^0P^0)=\frac{i}{2}(\Lambda^{\frac{1}{2}}-\Lambda^{-\frac{1}{2}})U \tag{5.13}$$

yields:

$$\langle j,m,M',n' \mid X\circ P \mid j,m,M,n\rangle = t'\langle j,m,M',n' \mid P^0 \mid j,m,M,n\rangle \qquad (5.14)$$

$$+q^4\frac{(q^2+1)}{q^2-1}\frac{i}{2}\left\{\langle j,m,M',n'|U|j,m,M+1,n\rangle - \frac{1}{q^4}\langle j,m,M',n'|U|j,m,M-1,n\rangle\right\}$$

Then taking the matrix elements of (B.16):

$$X\circ PX^0 - \frac{2}{1+q^2}X^0X\circ P - \frac{q^2-1}{q^2+1}X\circ XP^0 = iq^2(q^4-1)X\circ R\Lambda^{-\frac{1}{2}} \qquad (5.15)$$

and applying (5.14) implies:

$$\begin{aligned}
&\frac{i}{2}\frac{q^4}{q^2-1}(t(1+q^2)-2t')\left[\langle j,m,M',n'|U|j,m,M+1,n\rangle\right.\\
&\left.-\frac{1}{q^4}\langle j,m,M',n'|U|j,m,M-1,n\rangle\right] \qquad (5.16)\\
&+\langle j,m,M',n' \mid P^0 \mid j,m,M,n\rangle\frac{1}{q^2+1}\left[t'(t(1+q^2)-2t')-(q^2-1)r'^2\right]\\
&= i(q^2-1)\frac{r'^2}{2t'-(1+q^{-2})t}\langle j,m,M',n'|U|j,m,M-1,n\rangle
\end{aligned}$$

Now, this equation shows that the matrix elements of P^0 for $t'=t$ vanish, because the U-matrix elements vanish in this case, but not the factor t^2-r^2.

Eqn. (5.16) can also be used to check the expressions (5.11) and (5.12).

Now the case $s^2=0$ has to be studied. In this case, when $t'=t$ the equations resulting from (5.1) on the one side and (5.13), (5.15) on the other side cause a contradiction.

On the one hand from (5.1) it follows:

$$\begin{aligned}
0 =\ & (t-t)\langle j,m,n|P^0|j,m,n\rangle\\
=\ & \tfrac{iq^2}{2}(\langle j,m,n|U|j,m,n+1\rangle + \langle j,m,n|U|j,m,n-1\rangle)
\end{aligned}$$

and hence:

$$\langle j,m,n|U|j,m,n+1\rangle + \langle j,m,n|U|j,m,n-1\rangle = 0 \qquad (5.17)$$

On the other side (5.13) gives:

$$\langle j,m,n \mid X\circ P \mid j,m,n\rangle = \tau_0 q^n\langle j,m,n \mid P^0 \mid j,m,n\rangle \qquad (5.18)$$

$$+q^2\frac{(q^2+1)}{q^2-1}\frac{i}{2}\left\{q^2\langle j,m,n|U|j,m,n+1\rangle - \frac{1}{q^2}\langle j,m,n'|U|j,m,n-1\rangle\right\}$$

Inserting (5.18) in the equation which follows by taking the matrix elements of (5.15) provides the equation:

$$\begin{aligned}\langle j,m,n \mid P^0 \mid j,m,n\rangle \tfrac{1}{q^2+1}\tau_0^2 q^{2n}(1+q^2-2-q^2+1) = 0 \\ = \tfrac{i\tau_0}{2}\langle j,m,n|U|j,m,n-1\rangle q^n(\tfrac{1+q^2-2}{q^2-1}+2\tfrac{q^2-1}{2-1-q^{-2}}) \\ -\tfrac{i}{2}\langle j,m,n|U|j,m,n+1\rangle q^{4+n}\tfrac{1+q^2-2}{q^2-1}\end{aligned} \tag{5.19}$$

and hence:

$$\langle j,m,n|U|j,m,n-1\rangle(\frac{1}{2}+q^2) - \frac{q^4}{2}\langle j,m,n|U|j,m,n+1\rangle = 0 \tag{5.20}$$

But the two conditions (5.17) and (5.20) are not compatible, because they would imply

$$\langle j,m,n|U|j,m,n-1\rangle = \langle j,m,n|U|j,m,n+1\rangle = 0 \tag{5.21}$$

which is in contrast with the result (4.35) of the previous section:

$$|\langle j,m,n+1|U|j,m,n\rangle|^2 = \frac{1}{[2]^2} \tag{5.22}$$

This shows that it is not possible to construct a representation on the light-cone.

Notice that eqn. (5.17) alone is not enough to prove that the representation does not exist. In fact, an appropriate choice of the free phase in the expression for $\langle 0,0,n+1|U|0,0,n\rangle$ in (4.36) would allow (5.17) to be satisfied.

5.2 The reduced matrix elements of P^-

As a next step it is possible to find the reduced matrix elements of P^-.

First, $\langle 1,M',n'\|P^-\|0,M,n\rangle$ can be determined by means of (5.13). The relation holds:

$$\begin{aligned}&-q^2[2][3]\langle 0,M',n'\|X^-\|1,M',n'\rangle\langle 1,M',n'\|P^-\|0,M,n\rangle \\ &-t'\langle 0,0,M',n'|P^0|0,0,M,n\rangle = \\ &\tfrac{i}{2}\tfrac{q^2(q^2+1)}{q^2-1}(q^2\langle 0,0,M',n'|U|0,0,M+1,n\rangle - \tfrac{1}{q^2}\langle 0,0,M',n'|U|0,0,M-1,n\rangle\end{aligned} \tag{5.23}$$

This fixes $\langle 1,M',n'\|P^-\|0,M,n\rangle$. The non-vanishing matrix elements are:
For s^2 time-like

$$\begin{aligned}\langle 1,M+1,n+1\|P^-\|0,M,n\rangle &= \frac{iq^2}{2}\frac{1}{t_0 q^{M+n}}\frac{1}{(q^2-1)\sqrt{[2][3]}}\sqrt{\frac{[n+3]}{[n+1]}} \\ \langle 1,M+1,n-1\|P^-\|0,M,n\rangle &= -\frac{iq^4}{2}\frac{1}{t_0 q^{M-n}}\frac{1}{(q^2-1)\sqrt{[2][3]}}\sqrt{\frac{[n-1]}{[n+1]}}\end{aligned} \tag{5.24}$$

$$\langle 1, M-1, n+1 \| P^- \| 0, M, n\rangle = \frac{iq^4}{2}\frac{1}{t_0 q^{M-n}}\frac{1}{(q^2-1)\sqrt{[2][3]}}\sqrt{\frac{[n+3]}{[n+1]}}$$
$$\langle 1, M-1, n-1 \| P^- \| 0, M, n\rangle = -\frac{iq^2}{2}\frac{1}{t_0 q^{M+n}}\frac{1}{(q^2-1)\sqrt{[2][3]}}\sqrt{\frac{[n-1]}{[n+1]}}$$

For the matrix element $\langle 1, M+1, n+1 \| P^- \| 0, M, n\rangle$ it must be $n' \geq 1$, $n \geq 0$. For s^2 space-like

$$\begin{aligned}
\langle 1, M+1, n+1 \| P^- \| 0, M, n\rangle &= \frac{iq^3}{2}\frac{1}{l_0 q^{M+n}}\frac{1}{(q^2-1)\sqrt{[2][3]}}\sqrt{\frac{\{n+2\}}{\{n\}}} \\
\langle 1, M+1, n-1 \| P^- \| 0, M, n\rangle &= \frac{iq^3}{2}\frac{1}{l_0 q^{M-n}}\frac{1}{(q^2-1)\sqrt{[2][3]}}\sqrt{\frac{\{n-2\}}{\{n\}}} \qquad (5.25)\\
\langle 1, M-1, n+1 \| P^- \| 0, M, n\rangle &= -\frac{iq^3}{2}\frac{1}{l_0 q^{M-n}}\frac{1}{(q^2-1)\sqrt{[2][3]}}\sqrt{\frac{\{n+2\}}{\{n\}}} \\
\langle 1, M-1, n-1 \| P^- \| 0, M, n\rangle &= -\frac{iq^3}{2}\frac{1}{l_0 q^{M+n}}\frac{1}{(q^2-1)\sqrt{[2][3]}}\sqrt{\frac{\{n-2\}}{\{n\}}}
\end{aligned}$$

$\langle j+1, M', n' \| P^- \| j, M, n\rangle$ for $j \geq 1$ is then obtained by solving the recursion relation following from:

$$P^+ X^+ = X^+ P^+ \qquad (5.26)$$

By taking the matrix elements of this equation between $j+1$ and j one has:

$$\frac{\langle j+2, M', n' \| P^- \| j+1, M, n\rangle}{\langle j+1, M', n' \| P^- \| j, M, n\rangle} = \frac{\langle j+2, M', n' \| X^- \| j+1, M', n'\rangle}{\langle j+1, M, n \| X^- \| j, M, n\rangle} \qquad (5.27)$$

This yields for $j \geq 1$:

$$\begin{aligned}
&\langle j+1, M', n' \| P^- \| j, M, n\rangle = (\delta_{n',n+1} + \delta_{n',n-1})(\delta_{M',M+1} + \delta_{M',M-1}) \\
&\cdot \langle 1, M', n' \| P^- \| 0, M, n\rangle \prod_{k=1}^{j} \frac{\langle k+1, M', n' \| X^- \| k, M', n'\rangle}{\langle k, M, n \| X^- \| k-1, M, n\rangle} \qquad (5.28)
\end{aligned}$$

and hence:
For s^2 time-like, $M \in \mathbb{Z}$, $j = 0, 1, \ldots$

$$\begin{aligned}
&\langle j+1, M', n+1 \| P^- \| j, M, n\rangle = \frac{i}{2}(\delta_{M',M+1} q^{2+2j-n} + \delta_{M',M-1} q^{4+n}) \\
&\cdot \frac{1}{t_0 q^M (q^2-1)\{j+1\}}\sqrt{\frac{[2]}{[2j+1][2j+3]}}\sqrt{\frac{[n+j+3][n+j+2]}{[n+2][n+1]}} \\
&\langle j+1, M', n-1 \| P^- \| j, M, n\rangle = -\frac{i}{2}(\delta_{M',M+1} q^{4+2j+n} + \delta_{M',M-1} q^{2-n}) \\
&\cdot \frac{1}{t_0 q^M (q^2-1)\{j+1\}}\sqrt{\frac{[2]}{[2j+1][2j+3]}}\sqrt{\frac{[n-j-1][n-j]}{[n][n+1]}}
\end{aligned}$$

For the matrix element $\langle j+1, M', n' \| P^- \| j, M, n\rangle$ it must be $n' \geq j+1$, $n \geq j$.
For s^2 space-like, $M, n \in \mathbb{Z}$, $j = 0, 1, \ldots$

$$\langle j+1, M', n+1 \| P^- \| j, M, n\rangle = \frac{i}{2}(\delta_{M',M+1} q^{3+2j-n} - \delta_{M',M-1} q^{3+n})$$
$$\cdot \frac{1}{(q^2-1)\{j+1\} l_0 q^M} \sqrt{\frac{[2]}{[2j+1][2j+3]}} \sqrt{\frac{\{n+j+2\}\{n+j+1\}}{\{n\}\{n+1\}}}$$
$$\langle j+1, M', n-1 \| P^- \| j, M, n\rangle = \frac{i}{2}(\delta_{M',M+1} q^{3+2j+n} - \delta_{M',M-1} q^{3-n})$$
$$\cdot \frac{1}{(q^2-1)\{j+1\} l_0 q^M} \sqrt{\frac{[2]}{[2j+1][2j+3]}} \sqrt{\frac{\{n-j-2\}\{n-j-1\}}{\{n\}\{n-1\}}}$$

The conjugation properties (3.13) allow to get $\langle j, M', n' \| P^- \| j+1, M, n\rangle$ for the values $M' = M \pm 1$ and $n' = n \pm 1$.

$$\begin{aligned} \langle j+1, s'^2, t' \| P^- \| j, s^2, t\rangle &= -q^{2j+2} \overline{\langle j, s^2, t \| P^- \| j+1, s'^2, t'\rangle} \\ \langle j-1, s'^2, t' \| P^- \| j, s^2, t\rangle &= -q^{-2j} \overline{\langle j, s^2, t \| P^- \| j-1, s'^2, t'\rangle} \end{aligned} \tag{5.29}$$

The result is:

For s^2 time-like, $M \in \mathbb{Z}$, $j = 0, 1, \ldots$

$$\begin{aligned} \langle j, M', n+1 \| P^- \| j+1, M, n\rangle &= -\frac{i}{2}(\delta_{M',M+1} q^{-2-2j-n} + \delta_{M',M-1} q^{4+n}) \\ &\cdot \frac{1}{(q^2-1)\{j+1\} t_0 q^M} \sqrt{\frac{[2]}{[2j+1][2j+3]}} \sqrt{\frac{[n-j+1][n-j]}{[n+2][n+1]}} \\ \langle j, M', n-1 \| P^- \| j+1, M, n\rangle &= \frac{i}{2}(\delta_{M',M+1} q^{-2j+n} + \delta_{M',M-1} q^{2-n}) \\ &\cdot \frac{1}{(q^2-1)\{j+1\} t_0 q^M} \sqrt{\frac{[2]}{[2j+1][2j+3]}} \sqrt{\frac{[n+j+2][n+j+1]}{[n][n+1]}} \end{aligned} \tag{5.30}$$

For the matrix element $\langle j, M', n' \| P^- \| j+1, M, n\rangle$ it must be $n' \geq j$, $n \geq j+1$.
For s^2 space-like, $M, n \in \mathbb{Z}$, $j = 0, 1 \ldots$

$$\begin{aligned} \langle j, M', n+1 \| P^- \| j+1, M, n\rangle &= -\frac{i}{2}(\delta_{M',M+1} q^{-1-2j-n} - \delta_{M',M-1} q^{3+n}) \\ &\cdot \frac{1}{(q^2-1)\{j+1\} l_0 q^M} \sqrt{\frac{[2]}{[2j+1][2j+3]}} \sqrt{\frac{\{n-j\}\{n-j-1\}}{\{n\}\{n+1\}}} \\ \langle j, M', n-1 \| P^- \| j+1, M, n\rangle &= -\frac{i}{2}(\delta_{M',M+1} q^{-1-2j+n} - \delta_{M',M-1} q^{3-n}) \\ &\cdot \frac{1}{(q^2-1)\{j+1\} l_0 q^M} \sqrt{\frac{[2]}{[2j+1][2j+3]}} \sqrt{\frac{\{n+j+1\}\{n+j\}}{\{n\}\{n-1\}}} \end{aligned} \tag{5.31}$$

An alternative way to calculate $\langle j, M', n' \| P^- \| j+1, M, n\rangle$ would be to conjugate (B.17) and to use it to get $\langle 0, M', n' \| P^- \| 1, M, n\rangle$. Then (5.26) can be used to find a recursion relation for $\langle j, M', n' \| P^- \| j+1, M, n\rangle$ for $j \geq 1$. This method can be applied to check the results (5.30) and (5.31).

Finally, knowing the $\langle j+1, M', n' \| P^- \| j, M, n\rangle$ and $\langle j-1, M', n' \| P^- \| j, M, n\rangle$, it is possible to use them to compute $\langle j, M', n' \| P^- \| j, M, n\rangle$ by means of (B.17) and (3.17).

For s^2 time-like, $M \in \mathbb{Z}$, $j = 1, 2, \ldots$

$$\langle j, M', n+1 \| P^- \| j, M, n\rangle = \frac{i(q-q^{-1})}{2\{j\}\{j+1\}\sqrt{[2]}t_0 q^M}$$
$$\cdot(\delta_{M',M+1} q^{-1-n} - \delta_{M',M-1} q^{3+n})\sqrt{\frac{[n+j+2][n-j+1]}{[n+2][n+1]}}$$
$$\langle j, M', n-1 \| P^- \| j, M, n\rangle = \frac{i(q-q^{-1})}{2\{j\}\{j+1\}\sqrt{[2]}t_0 q^M}$$
$$\cdot(\delta_{M',M+1} q^{1+n} - \delta_{M',M-1} q^{1-n})\sqrt{\frac{[n+j+1][n-j]}{[n][n+1]}}$$

For the matrix element $\langle j, M', n' \| P^- \| j, M, n\rangle$ it must be $n, n' \geq j$.

For s^2 space-like, $M, n \in \mathbb{Z}$, $j = 1, 2, \ldots$

$$\langle j, M', n+1 \| P^- \| j, M, n\rangle = \frac{i(q-q^{-1})}{2\{j\}\{j+1\}\sqrt{[2]}l_0 q^M}$$
$$\cdot(\delta_{M',M+1} q^{-n} + \delta_{M',M-1} q^{2+n})\sqrt{\frac{\{n+j+1\}\{n-j\}}{\{n\}\{n+1\}}}$$
$$\langle j, M', n-1 \| P^- \| j, M, n\rangle = -\frac{i(q-q^{-1})}{2\{j\}\{j+1\}\sqrt{[2]}l_0 q^M}$$
$$\cdot(\delta_{M',M+1} q^{n} + \delta_{M',M-1} q^{2-n})\sqrt{\frac{\{n-j-1\}\{n+j\}}{\{n\}\{n-1\}}}$$

This concludes the calculation of the matrix elements of the momentum four-vector.

Appendix A

R-matrices, metric and ε-tensor

The results and notations of [21], which are used throughout this thesis, are reviewed in this appendix. It is necessary to review here also some of the results regarding the Euclidean three-dimensional plane, as it is a section of the Minkowski phase-space.

A.1 The Euclidean space

For the Euclidean space the $\hat{R}$-matrix is given by:

$$\hat{R}^{AB}{}_{CD}: \tag{A.1}$$

	++	−−	+3	3+	3−	−3	+−	3 3	−+
++	1	0	0	0	0	0	0	0	0
−−	0	1	0	0	0	0	0	0	0
+3	0	0	0	$\frac{1}{q^2}$	0	0	0	0	0
3+	0	0	$\frac{1}{q^2}$	$1-\frac{1}{q^4}$	0	0	0	0	0
3−	0	0	0	0	0	$\frac{1}{q^2}$	0	0	0
−3	0	0	0	0	$\frac{1}{q^2}$	$1-\frac{1}{q^4}$	0	0	0
+−	0	0	0	0	0	0	0	0	$\frac{1}{q^4}$
3 3	0	0	0	0	0	0	0	$\frac{1}{q^2}$	$\frac{1-q^{-4}}{q}$
−+	0	0	0	0	0	0	$\frac{1}{q^4}$	$\frac{1-q^{-4}}{q}$	$(1-\frac{1}{q^2})(1-\frac{1}{q^4})$

The metric tensor is defined as:

$$g_{AB}: \quad g_{+-} = -q, \quad g_{33} = 1, \quad g_{-+} = -\frac{1}{q} \tag{A.2}$$

$$g^{AB}: \quad g^{+-} = -q, \quad g^{33} = 1, \quad g^{-+} = -\frac{1}{q}$$

For the metric it is true that:

$$g_{AB} g^{BC} = \delta_A^C = g^{CB} g_{BA}$$

Through the metric indices can be raised and lowered:

$$X_A = g_{AB} X^B, X^A = g^{AB} X_B$$

and an invariant scalar product can be given:

$$X \circ Y = g_{AB} X^A Y^B = X^3 Y^3 - q X^+ Y^- - \frac{1}{q} X^- Y^+ \tag{A.3}$$

The ε-tensor is constructed as:

$$\begin{aligned} \varepsilon_{+-3} &= q, \quad \varepsilon_{-+3} = -q, \quad \varepsilon_{333} = 1 - q^2, \\ \varepsilon_{+3-} &= -\frac{1}{q}, \quad \varepsilon_{3+-} = q, \\ \varepsilon_{-3+} &= q^3, \quad \varepsilon_{3-+} = -q \end{aligned} \tag{A.4}$$

Indices of the ε-tensor can also be raised and lowered through the metric, e.g.:

$$\varepsilon_{ABC} = g_{CD} \varepsilon_{AB}{}^D$$

and cyclic permutations. Therefore:

$$\begin{aligned} \varepsilon_{+-}{}^3 &= q, \quad \varepsilon_{-+}{}^3 = -q, \quad \varepsilon_{33}{}^3 = 1 - q^2, \\ \varepsilon_{+3}{}^+ &= 1, \quad \varepsilon_{3+}{}^+ = -q^2, \\ \varepsilon_{-3}{}^- &= -q^2, \quad \varepsilon_{3-}{}^- = 1. \end{aligned} \tag{A.5}$$

Analogously:

$$\begin{aligned} \varepsilon_3{}^{33} &= 1 - q^2, \quad \varepsilon_3{}^{+-} = -q, \quad \varepsilon_3{}^{-+} = q, \\ \varepsilon_+{}^{+3} &= -q^2, \quad \varepsilon_+{}^{3+} = 1, \\ \varepsilon_-{}^{-3} &= 1, \quad \varepsilon_-{}^{3-} = -q^2. \end{aligned} \tag{A.6}$$

Some useful relations satisfied by the ε-tensor are the following:
Cyclic permutation of indices:

$$\varepsilon_{AB}{}^C = \varepsilon_{EAB} g^{EC}, \quad \varepsilon_{EAB} = \varepsilon_{AB}{}^C g_{CE} \tag{A.7}$$

Cyclic permutation of indices for two ε-tensors:

$$\varepsilon^{ABE} \varepsilon_{CDE} = \varepsilon_E{}^{AB} \varepsilon^E_{CD} = \varepsilon^{EAB} \varepsilon_{ECD} \tag{A.8}$$

Contraction of one index of two ε-tensors:

$$\varepsilon_{TSE}\varepsilon_{DC}{}^{E} = q^2(g_{CT}g_{DS} - g_{ST}g_{CD}) + g_{CB}g_{RT}\varepsilon^{BRE}\varepsilon_{SDE} \tag{A.9}$$

Contraction of the ε-tensor with the metric:

$$g^{BA}\varepsilon_{ABC} = 0, \quad g^{CB}\varepsilon_{ABC} = 0 \tag{A.10}$$

Contraction of two indices of two ε-tensors:

$$\varepsilon_{D}{}^{FE}\varepsilon_{EFR} = (1+q^4)g_{RD}, \quad \varepsilon^{BCF}\varepsilon_{CBA} = (1+q^4)\delta^F_A \tag{A.11}$$

By repeatedly applying these formulas it is possible to demonstrate the following identities.
Formulas containing three ε-tensors:

$$\begin{aligned}
\varepsilon^{ABE}\varepsilon_{RLE}\varepsilon_{MN}{}^{L} &= q^2\varepsilon_{N}{}^{AB}g_{MR} - q^2\varepsilon^{AB}_{R}g_{NM} + \varepsilon^{ABM}\varepsilon_{QNM}\varepsilon_{RM}{}^{Q} \\
\varepsilon_{LC}{}^{B}\varepsilon^{ACG}\varepsilon_{STG} &= q^2g^{AB}\varepsilon_{LST} - q^2\varepsilon_{ST}{}^{A}\delta^B_L + q^2\varepsilon_{N}{}^{AB}g_{SL} \\
&\quad -q^2\varepsilon^{AB}_{L}g_{TS} + \varepsilon^{ABM}\varepsilon_{QTM}\varepsilon_{LS}{}^{Q}
\end{aligned} \tag{A.12}$$

Formula containing four ε-tensors:

$$\begin{aligned}
\varepsilon_{LM}{}^{A}\varepsilon^{LBG}\varepsilon_{TSG}\varepsilon_{C}{}^{MS} &= q^2(1+q^4-q^2)g^{AB}g_{CT} + q^4\delta^A_C\delta^B_T \\
&\quad +(1+q^4-2q^2)\varepsilon_{M}{}^{AB}\varepsilon_{TC}{}^{M}
\end{aligned} \tag{A.13}$$

Formula containing five ε-tensors:

$$\begin{aligned}
\varepsilon_{LM}{}^{A}\varepsilon^{LBG}\varepsilon_{TSG}\varepsilon^{MSC}\varepsilon_{NRC} &= (q^6-q^4+q^2)g^{AB}\varepsilon_{TNR} + q^4\varepsilon_{NR}{}^{A}\delta^B_T \\
&\quad +(1+q^4-2q^2)\varepsilon^{ABG}\varepsilon_{NR}{}^{S}\varepsilon_{TSG}
\end{aligned} \tag{A.14}$$

The conjugation properties of the metric and the ε-tensor are:

$$\overline{\varepsilon_{BC}{}^{A}} = \varepsilon_{DEA}g^{EB}g^{DC} = \varepsilon^{CBK}g_{KA} \tag{A.15}$$

and

$$\overline{g_{AB}} = g^{AB}$$

In terms of the metric and of the ε the $\hat{R}$-matrix (A.1) can be written in the form:

$$\hat{R}^{AB}{}_{CD} = \delta^A_C\delta^B_D - q^{-4}\varepsilon^{FAB}\varepsilon_{FDC} - q^{-4}(q^2-1)g^{AB}g_{CD} \tag{A.16}$$

From this it follows that:

$$\begin{aligned}
g^{CB}\hat{R}^{AF}{}_{BD}g_{FE} &= q^{-4}\hat{R}^{-1CA}{}_{DE} \\
g^{AF}\hat{R}^{-1BE}{}_{FC}g_{ED} &= q^{+4}\hat{R}^{AB}{}_{CD} \\
g^{GC}g^{ED}\hat{R}^{BA}{}_{DC}g_{AF}g_{BK} &= \hat{R}^{GE}{}_{FK}
\end{aligned} \tag{A.17}$$

A.2 The Minkowski space

The q-deformed Minkowski space can be constructed as the tensor product of the fundamental representation of two $SL_q(2)$ [2]. Doing this it is possible to construct the corresponding R-matrices. It turns out that two different R-matrices are necessary. Their projector decomposition is given by:

$$\hat{R}_I = P_S + P_T - q^2 P_+ - q^{-2} P_- = 1\!\!1 - (1+q^2)P_+ - (1+\frac{1}{q^2})P_- \tag{A.18}$$

$$\hat{R}_{II} = q^{-2} P_S + q^2 P_T - P_+ - P_- = \frac{1}{q^2} 1\!\!1 + (q^2 - \frac{1}{q^2})P_T - (1+\frac{1}{q^2})P_A \tag{A.19}$$

In these definitions P_S,P_T,P_+,P_- are the projectors on the symmetric, trace, selfdual, antiselfdual eigenspaces respectively. This decomposition shows clearly that $\hat{R}_I$ cannot distinguish the symmetric while $\hat{R}_{II}$ cannot distinguish the antisymmetric eigenspaces, because they have the same eigenvalue, so that both matrices are necessary to distinguish all the spaces. The explicit expression of the projectors follows.

P_+:

	00	$C0$	$0D$	CD
00	0	0	0	0
$A0$	0	$\frac{q^2}{(1+q^2)^2}\delta^A_C$	$-\frac{1}{(1+q^2)^2}\delta^A_D$	$\frac{1}{(1+q^2)^2}\varepsilon_{DC}{}^A$
$0B$	0	$-\frac{q^4}{(1+q^2)^2}\delta^B_C$	$\frac{q^2}{(1+q^2)^2}\delta^B_D$	$-\frac{q^2}{(1+q^2)^2}\varepsilon_{DC}{}^B$
AB	0	$\frac{q^2 g^{EB} g^{FA}\varepsilon_{FEC}}{(1+q^2)^2}$	$-\frac{g^{EB} g^{FA}\varepsilon_{FED}}{(1+q^2)^2}$	$\frac{\varepsilon_{DC}{}^E g^{SB} g^{RA}\varepsilon_{RSE}}{(1+q^2)^2}$

(A.20)

P_-:

	00	$C0$	$0D$	CD
00	0	0	0	0
$A0$	0	$\frac{q^2}{(1+q^2)^2}\delta^A_C$	$-\frac{q^4}{(1+q^2)^2}\delta^A_D$	$-\frac{q^2}{(1+q^2)^2}\varepsilon_{DC}{}^A$
$0B$	0	$-\frac{1}{(1+q^2)^2}\delta^B_C$	$\frac{q^2}{(1+q^2)^2}\delta^B_D$	$\frac{1}{(1+q^2)^2}\varepsilon_{DC}{}^B$
AB	0	$-\frac{g^{EB}g^{FA}\varepsilon_{FEC}}{(1+q^2)^2}$	$\frac{q^2 g^{EB}g^{FA}\varepsilon_{FED}}{(1+q^2)^2}$	$\frac{\varepsilon_{DC}{}^E g^{SB}g^{RA}\varepsilon_{RSE}}{(1+q^2)^2}$

(A.21)

P_T:

	00	$C0$	$0D$	CD
00	$\frac{q^2}{(1+q^2)^2}$	0	0	$-\frac{q^2}{(1+q^2)^2}g_{CD}$
$A0$	0	0	0	0
$0B$	0	0	0	0
AB	$-\frac{q^2}{(1+q^2)^2}g^{AB}$	0	0	$\frac{q^2}{(1+q^2)^2}g^{AB}g_{CD}$

(A.22)

It holds:

$$\mathbb{1} = P_S + P_T + P_+ + P_- \tag{A.23}$$

Using P_T it is possible to construct a 4-dimensional metric:

$$\begin{array}{ll} \eta_{00} = -1, & \eta_{33} = 1 \\ \eta_{+-} = -q, & \eta_{-+} = -\frac{1}{q} \\ \eta^{ab} = \eta_{ab} & \end{array} \tag{A.24}$$

which enables to raise and lower indices:

$$X_A = \eta_{AB}X^B, X^A = \eta^{AB}X_B \tag{A.25}$$

and to define an invariant scalar product in 4 dimensions:

$$\begin{aligned} X \cdot Y &= -X^0Y^0 + X^3Y^3 - qX^+Y^- - \frac{1}{q}X^-Y^+ \\ &= \eta_{ab}X^aY^b \end{aligned} \tag{A.26}$$

The sum of the selfdual and antiselfdual projectors defines the q-deformed antisymmetrizer:

$$P_A = P_+ + P_- \tag{A.27}$$

while their difference defines the q-deformed 4-dimensional ε-tensor:

$$\varepsilon^{ab}{}_{cd} = P_{+}^{ab}{}_{cd} - P_{-}^{ab}{}_{cd} \tag{A.28}$$

Further, some interesting identities hold:

$$\eta_{sb}\eta_{ta}P^{ab}{}_{cd} = \eta_{ud}\eta_{vc}P^{uv}{}_{st} \qquad \text{for all projectors} \tag{A.29}$$

$$\eta_{ij}(P_A)^{jk}_{mp}\eta_{kl}\eta^{pl} = \frac{2(1+q^2+q^4)}{(1+q^2)^2}\eta_{im} \tag{A.30}$$

$$\eta^{ab}\hat{R}^{cd}_{II\,be}\eta_{cf} = q^{-2}\hat{R}^{-1ac}_{II\ \ \ ef} \tag{A.31}$$

Appendix B

Useful relations

In this appendix some relations which are used to explicitly perform some of the calculations are reported.

B.1 Relations defining the Minkowski space

Using the expression (A.5) for the ε-tensor the XX-relations can be written as:

$$\begin{aligned} X^3X^+ - q^2X^+X^3 &= (1-q^2)X^0X^+ \\ X^-X^3 - q^2X^3X^- &= (1-q^2)X^0X^- \\ qX^-X^+ - qX^+X^- + (1-q^2)X^3X^3 &= (1-q^2)X^0X^3 \end{aligned} \tag{B.1}$$

Analogously for the RR and SS-relations it holds:

$$\begin{aligned} R^3R^+ - q^2R^+R^3 &= \frac{1}{q^2+1}UR^+ \\ R^-R^3 - q^2R^3R^- &= \frac{1}{q^2+1}UR^- \\ qR^-R^+ - qR^+R^- + (1-q^2)R^3R^3 &= \frac{1}{q^2+1}UR^3 \end{aligned} \tag{B.2}$$

$$\begin{aligned} S^3S^+ - q^2S^+S^3 &= -\frac{1}{q^2+1}US^+ \\ S^-S^3 - q^2S^3S^- &= -\frac{1}{q^2+1}US^- \\ qS^-S^+ - qS^+S^- + (1-q^2)S^3S^3 &= -\frac{1}{q^2+1}US^3 \end{aligned} \tag{B.3}$$

Using the R-matrix (A.1) the RS-relations become:

$$R^+S^+ = q^2S^+R^+$$

$$\begin{aligned}
R^+S^3 &= S^3R^+ \\
R^+S^- &= \frac{1}{q^2}S^-R^+ \\
R^3\,S^+ &= S^+R^3 + (q^2 - \frac{1}{q^2})S^3R^+ \\
R^3\,S^3 &= S^3R^3 + q(1 - \frac{1}{q^4})S^-R^+ \\
R^3\,S^- &= S^-R^3 \\
R^-S^+ &= \frac{1}{q^2}S^+R^- + q(q - \frac{1}{q})(1 - \frac{1}{q^4})S^-R^+ + q(1 - \frac{1}{q^4})S^3R^3 \\
R^-S^3 &= S^3R^- + q^2(1 - \frac{1}{q^4})S^-R^3 \\
R^-S^- &= q^2S^-R^-
\end{aligned} \tag{B.4}$$

The RX-relations (2.23) read:

$$\begin{aligned}
R^+X^0 &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^0R^+ + \frac{1}{q}\frac{q^2-1}{q^2+1}X^3R^+ - \frac{q(q^2-1)}{q^2+1}X^+R^3 \\
&\quad -\frac{q}{(q^2+1)^2}X^+U \\
R^+X^+ &= qX^+R^+ \\
R^+X^- &= \frac{1}{q}X^-R^+ + \frac{(q^2-1)}{(q^2+1)}X^0R^3 + \frac{1}{(q^2+1)^2}X^0U - \frac{q^2-1}{q^2+1}X^3R^3 \\
&\quad -\frac{1}{(q^2+1)^2}X^3U \\
R^+X^3 &= -\frac{q(q^2-1)}{q^2+1}X^+R^3 - \frac{q}{(q^2+1)^2}X^+U + \frac{2q}{1+q^2}X^3R^+ \\
&\quad +q\frac{q^2-1}{q^2+1}X^0R^+ \\
R^-X^0 &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^0R^- - \frac{q(q^2-1)}{q^2+1}X^3R^- + \frac{(q^2-1)}{q(q^2+1)}X^-R^3 \\
&\quad -\frac{q}{(q^2+1)^2}X^-U \\
R^-X^+ &= \frac{1}{q^2+1}\Bigg((q^3 - q - \frac{1}{q} + \frac{1}{q^3})X^-R^+ + (q + \frac{1}{q})X^+R^- \\
&\quad +(2q^2 - 1 - \frac{1}{q^2})X^3R^3 - (q^2-1)X^0R^3 + \frac{1}{q^2(q^2+1)}X^0U \\
&\quad +\frac{1}{q^2+1}X^3U\Bigg) \\
R^-X^- &= qX^-R^-
\end{aligned} \tag{B.5}$$

$$
\begin{aligned}
R^-X^3 &= \frac{1}{q^2+1}\left((q^3+q-\frac{2}{q})X^-R^3+2qX^3R^- - \frac{1}{q}(q^2-1)X^0R^-\right.\\
&\left.+\frac{1}{q(q^2+1)}X^-U\right)\\
R^3X^0 &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^0R^3+\frac{q^2-1}{q^2+1}X^-R^+ - \frac{q^2-1}{q^2+1}X^+R^- - \frac{1}{q}\frac{(q^2-1)^2}{q^2+1}X^3R^3\\
&-\frac{q}{(q^2+1)^2}X^3U\\
R^3X^+ &= \frac{1}{q^2+1}\left(\frac{1}{q}(q^4+q^2-2)X^3R^+ - \frac{1}{q}(q^2-1)X^0R^+ + 2qX^+R^3\right.\\
&\left.+\frac{1}{q}\frac{1}{q^2+1}X^+U\right)\\
R^3X^- &= \frac{1}{q^2+1}\left(q(1-q^2)X^3R^- + q(q^2-1)X^0R^-\right.\\
&\left.+2qX^-R^3 - \frac{q}{q^2+1}X^-U\right)\\
R^3X^3 &= \frac{1}{q^2+1}\left(\frac{1}{q}(-q^4+4q^2-1)X^3R^3+\frac{1}{q}(q^2-1)^2X^0R^3\right.\\
&-(q^2-1)X^+R^- + \frac{1}{q^2}(q^2-1)(1+2q^2)X^-R^+ - \frac{1}{q}\frac{1}{q^2+1}X^0U\\
&\left.-\frac{1}{q}\frac{q^2-1}{q^2+1}X^3U\right)
\end{aligned}
$$

The SX-relations (2.24) are:

$$
\begin{aligned}
S^+X^0 &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^0S^+ + \frac{q^2-1}{q(q^2+1)}X^3S^+ - \frac{q(q^2-1)}{q^2+1}X^+S^3\\
&-\frac{1}{q(q^2+1)^2}X^+U\\
S^+X^+ &= \frac{1}{q}X^+S^+\\
S^+X^- &= \frac{1}{q^2+1}\left(\frac{1}{q}(q^2-1)^2(q^2+1)X^+S^- + q(q^2+1)X^-S^+\right.\\
&-(q^2-1)(q^2+2)X^3S^3 + (q^2-1)X^0S^3 + \frac{q^2}{(q^2+1)}X^0U\\
&\left.+\frac{1}{(q^2+1)}X^3U\right)\\
S^+X^3 &= \frac{1}{q^2+1}\left((\frac{1}{q}+q-2q^3)X^+S^3 + 2qX^3S^+ + q(q^2-1)X^0S^+\right.\\
&\left.+\frac{q}{(q^2+1)}X^+U\right)
\end{aligned}
$$

$$
\begin{aligned}
S^- X^0 &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^0S^- + \frac{q^2-1}{q(q^2+1)}X^-S^3 - \frac{q(q^2-1)}{q^2+1}X^3S^- \\
&\quad -\frac{1}{q(q^2+1)^2}X^-U \\
S^- X^+ &= \frac{1}{q^2+1}\Big((q^2-1)X^3S^3 + q(q^2+1)X^+S^- - (q^2-1)X^0S^3 \\
&\quad +\frac{1}{(q^2+1)}X^0U - \frac{1}{(q^2+1)}X^3U\Big) \\
S^- X^- &= \frac{1}{q}X^-S^- \\
S^- X^3 &= \frac{1}{q^2+1}\left(\frac{1}{q}(q^2-1)X^-S^3 + 2qX^3S^- - \frac{1}{q}(q^2-1)X^0S^-\right. \\
&\quad \left.-\frac{1}{q}\frac{1}{(q^2+1)}X^-U\right) \\
S^3 X^0 &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^0S^3 + \frac{q^2-1}{q^2+1}X^-S^+ - \frac{q^2-1}{q^2+1}X^+S^- \\
&\quad -\frac{1}{q}\frac{(q^2-1)^2}{q^2+1}X^3S^3 - \frac{1}{q(q^2+1)^2}X^3U \\
S^3 X^+ &= \frac{1}{q^2+1}\left(\frac{1}{q}(q^2-1)X^3S^+ + 2qX^+S^3 - \frac{1}{q}(q^2-1)X^0S^+\right. \\
&\quad \left.-\frac{1}{q}\frac{1}{(q^2+1)}X^+U\right) \\
S^3 X^- &= \frac{1}{q^2+1}\left((\frac{1}{q}+q-2q^3)X^3S^- + 2qX^-S^3 + q(q^2-1)X^0S^-\right. \\
&\quad \left.+\frac{q}{(q^2+1)}X^-U\right) \\
S^3 X^3 &= \frac{1}{q^2+1}\left(\frac{1}{q}(-q^4+4q^2-1)X^3S^3 + \frac{1}{q}(q^2-1)^2X^0S^3 - \frac{q}{(q^2+1)}X^0U\right. \\
&\quad \left.-(q^2+2)(q^2-1)X^+S^- + (q^2-1)X^-S^+ + \frac{1}{q}\frac{q^2-1}{q^2+1}X^3U\right)
\end{aligned} \tag{B.6}
$$

The UX^A-relations (2.25) expressed in terms of R are:

$$
\begin{aligned}
UX^+ &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^+U - q(q^2-1)^2X^0R^+ \\
&\quad -\frac{1}{q}(q^2-1)^2X^3R^+ + q(q^2-1)^2X^+R^3
\end{aligned}
$$

$$
\begin{aligned}
UX^- &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^-U - q(q^2-1)^2X^0R^- \\
&\quad -\frac{1}{q}(q^2-1)^2X^-R^3 + q(q^2-1)^2X^3R^- \\
UX^3 &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^3U - q(q^2-1)^2X^0R^+ \\
&\quad -(q^2-1)^2X^-R^+ + (q^2-1)^2X^+R^- + \frac{1}{q}(q^2-1)^3X^3R^3
\end{aligned} \tag{B.7}
$$

while when expressed in terms of S they become (2.26):

$$
\begin{aligned}
UX^+ &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^+U - \frac{1}{q}(q^2-1)^2X^0S^+ \\
&\quad +\frac{1}{q}(q^2-1)^2X^3S^+ - q(q^2-1)^2X^+S^3 \\
UX^- &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^-U - \frac{1}{q}(q^2-1)^2X^0S^- \\
&\quad +\frac{1}{q}(q^2-1)^2X^-S^3 - q(q^2-1)^2X^3S^- \\
UX^3 &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^3U - \frac{1}{q}(q^2-1)^2X^0S^+ \\
&\quad +(q^2-1)^2X^-S^+ - (q^2-1)^2X^+S^- - \frac{1}{q}(q^2-1)^3X^3S^3
\end{aligned} \tag{B.8}
$$

Interesting commutation relations which can be derived from the action of R, S, U on the coordinates are the following:

$$
\begin{aligned}
U\,X\circ X &= 2\frac{q^4+1}{(q^2+1)^2}X\circ XU + \frac{1}{q^2}\frac{(q^2-1)^2(q^4+q^2+1)}{(q^2+1)^2}X^0X^0U \\
&\quad -\frac{1}{q^2}(q^2-1)^2(q^2+1)X^0\,X\circ R \\
UX^0X^0 &= \frac{1}{q^2}\frac{(q^4+1)^2}{(q^2+1)^2}X^0X^0U - \frac{1}{q^2}(q^2-1)^2q^2+1X^0X\circ R \\
&\quad +\frac{(q^2-1)^2}{(q^2+1)^2}X\circ XU \\
R^AX\circ X &= 2\frac{q^4+1}{(q^2+1)^2}X\circ XR^A - \frac{1}{q^2+1}X^AX^0U \\
&\quad +\frac{1}{q^2}\frac{(q^2-1)^2(q^4+q^2+1)}{(q^2+1)^2}X^0X^0R^A + \frac{1}{q^2}(q^2-1)\varepsilon_{CD}{}^AX^0X^DR^C
\end{aligned}
$$

$$
\begin{aligned}
S^A X\circ X &= 2\frac{q^4+1}{(q^2+1)^2}X\circ X S^A - \frac{1}{q^2}\frac{1}{q^2+1}X^A X^0 U \\
&\quad + \frac{1}{q^2}\frac{(q^2-1)^2(q^4+q^2+1)}{(q^2+1)^2}X^0X^0S^A + \frac{1}{q^2}(q^2-1)\varepsilon_{CD}{}^A X^0 X^D S^C
\end{aligned}
$$

Further by contracting the $R^A X^B$-relation (2.23) with g_{AB} follows:

$$
R\circ X = \frac{1}{q}\frac{q^2+q^{-2}}{1+q^2}X\circ R - \frac{1}{q}\frac{1+q^2+q^{-2}}{(1+q^2)^2}X^0 U \qquad \text{(B.9)}
$$

and by commuting $X\circ R$ with X^0:

$$
X\circ R\, X^0 = \frac{2q}{1+q^2}X^0\; X\circ R - \frac{q}{(1+q^2)^2}X\circ X\; U \qquad \text{(B.10)}
$$

Substituting the expression (A.19) for the $\hat{R}_{II}$-matrix in the PX-relations (2.8) it can be verified that the Heisenberg relations are:

$$
\begin{aligned}
P^0X^0 - \tfrac{q^2+q^{-2}}{1+q^2}X^0P^0 - \tfrac{1}{q^2}\tfrac{1-q^2}{1+q^2}X\circ P \\
= \tfrac{i}{2}\Lambda^{-\frac{1}{2}}(1+q^4)U \qquad \text{(B.11)} \\
P^AX^0 - \tfrac{q^2-1}{q^2+1}X^AP^0 - \tfrac{q^2+q^{-2}}{1+q^2}X^0P^A + \tfrac{1-q^2}{q^2(1+q^2)}\epsilon_{DC}{}^A X^C P^D \\
= -\tfrac{i}{2}q^2(1-q^4)\Lambda^{-\frac{1}{2}}(R^A+q^2S^A) \qquad \text{(B.12)}
\end{aligned}
$$

$$
\begin{aligned}
P^0X^A - \tfrac{q^2-1}{q^2+1}X^0P^A - \tfrac{q^2+q^{-2}}{1+q^2}X^AP^0 + \tfrac{1-q^2}{q^2(1+q^2)}\epsilon_{DC}{}^A X^C P^D \\
= \tfrac{i}{2}q^2(1-q^4)\Lambda^{-\frac{1}{2}}(q^2R^A+S^A) \qquad \text{(B.13)} \\
P^AX^B - X^AP^B - \tfrac{q^2-1}{q^2(1+q^2)}g^{AB}X^0P^0 + \tfrac{q^2-1}{q^2(1+q^2)}g^{AB}X\circ P \\
+\tfrac{q^2-1}{q^2(q^2+1)}\epsilon_C{}^{AB}X^CP^0 + \tfrac{q^2-1}{q^2(q^2+1)}\epsilon_C{}^{AB}X^0P^C + \tfrac{2}{q^2(q^2+1)}\epsilon_{DC}{}^E\epsilon_E{}^{AB}X^CP^D \\
= -\tfrac{i}{2}\Lambda^{-\frac{1}{2}}\left((1+q^4)g^{AB}U + q^2(1-q^4)\epsilon_C{}^{AB}(R^C-S^C)\right) \qquad \text{(B.14)}
\end{aligned}
$$

By contracting (B.14) with the metric g_{ab} one has:

$$
P\circ X - \frac{1}{q^2}\frac{q^2+q^{-2}}{1+q^2}X\circ P - q^2\frac{1-q^6}{1+q^2}X^0P^0 = -\frac{i}{2}\Lambda^{-\frac{1}{2}}(1+q^4)(1+q^2+\frac{1}{q^2})U \qquad \text{(B.15)}
$$

By multiplying (B.11) with X^B on the left and contracting the result with g_{BA} it can be verified that:

$$
X\circ PX^0 - \frac{2}{1+q^2}X^0X\circ P - \frac{q^2-1}{q^2+1}X\circ XP^0 = iq^2(q^4-1)X\circ R\Lambda^{-\frac{1}{2}} \qquad \text{(B.16)}
$$

By using the relation (B.15), (B.11) and their conjugates follows:

$$\frac{1}{q^2}\frac{(q^2-1)}{(q^2+1)}(X\circ P - X^0P^0) = \frac{i}{2}(\Lambda^{\frac{1}{2}} - \Lambda^{-\frac{1}{2}})U \tag{B.17}$$

$$P\circ X - X\circ P = -\frac{i}{2}\frac{(1-q^6)}{(1-q^2)}(q^2\Lambda^{-\frac{1}{2}} + \frac{1}{q^2}\Lambda^{\frac{1}{2}})U \tag{B.18}$$

$$P^0X^0 - X^0P^0 = \frac{i}{2}(q^4\Lambda^{-\frac{1}{2}} + \Lambda^{\frac{1}{2}})U \tag{B.19}$$

The last equation is particularly important, because it is the commutation relation between the time-component X^0 and the energy P^0. It can be seen, that it is expressed in terms of a commutator, and not in terms of a q-commutator, but it contains the scalar Λ in the right hand-side.

Notice that (B.18) and (B.19) are invariant under conjugation, whereas by conjugating (B.17) one gets:

$$\frac{1}{q^2}\frac{(q^2-1)}{(q^2+1)}(P\circ X - P^0X^0) = -\frac{i}{2}(q^4\Lambda^{-\frac{1}{2}} - q^{-4}\Lambda^{\frac{1}{2}})U \tag{B.20}$$

Nevertheless (B.20) does not provide any further information, because it is a linear combination of the equations (B.17), (B.18) and (B.19).

Using the eqns. (B.11)-(B.14) the commutation relation between the four-dimensional invariant length $X\cdot X$ and P^0 can be calculated.

$$\begin{aligned} &P^0(-X^0X^0 + X\circ X) - \tfrac{1}{q^2}(-X^0X^0 + X\circ X)P^0 \\ &= -\tfrac{i}{2}(q^2+1)^2X^0\Lambda^{-\frac{1}{2}}U + \tfrac{i}{2}(q^2+1)^2(1-q^4)X\circ R\,\Lambda^{-\frac{1}{2}} \end{aligned} \tag{B.21}$$

It is important to notice that they do not commute.

B.2 Some relations involving Z,R and S

By contracting (2.17) with g_{AB} it can be verified that:

$$R\circ S = \frac{1}{q^4}S\circ R \tag{B.22}$$

$R\circ S$ is symmetric:

$$\overline{R\circ S} = R\circ S \tag{B.23}$$

The following definition is very handy:

$$Z^A = \varepsilon_{CB}{}^{A}R^BS^C = -\frac{1}{q^2}\varepsilon_{CB}{}^{A}S^CR^B \tag{B.24}$$

The conjugation properties of Z^A can be obtained by applying (2.12) and (A.15) and it turns out that:

$$\overline{Z^A} = Z_A = g_{AB}Z^B \tag{B.25}$$

Useful formulas involving Z which can be verified by using the RR,SS and RS commutation relations are:
Scalar products:

$$\begin{aligned} R\circ Z &= \frac{1}{q^2+1} U\, R\circ S \\ Z\circ R &= -\frac{q^2}{q^2+1} U\, R\circ S \\ S\circ Z &= \frac{q^2}{q^2+1} U\, R\circ S \\ Z\circ S &= -\frac{1}{q^2+1} U\, R\circ S \\ Z\circ Z &= -(R\circ R)\,(S\circ S) + q^4 (R\circ S)^2 - \frac{1}{(q^2+1)^2} U^2\, R\circ S \end{aligned} \tag{B.26}$$

Contraction with the ε-tensor:

$$\begin{aligned} \varepsilon_{BA}{}^{K} Z^A R^B &= -S^K R\circ R + q^4 R\circ S\, R^K + \frac{U}{q^2+1} Z^K \\ \varepsilon_{BA}{}^{K} R^A Z^B &= q^2 R\circ R\, S^K + \frac{1}{q^2+1} U Z^K - q^2 R^K R\circ S \\ \varepsilon_{BA}{}^{K} Z^A S^B &= q^2 R^K\, S\circ S - q^2\, R\circ S\, S^K - \frac{U}{q^2+1} Z^K \\ \varepsilon_{BA}{}^{K} S^A Z^B &= -S\circ S R^K + q^4 S^K\, R\circ S - \frac{1}{q^2+1} U Z^K \\ \varepsilon_{BA}{}^{K} Z^A Z^B &= -q^4 \frac{1}{q^2+1} U\, R\circ S S^C + q^2(q^2-1)\, R\circ S\, Z^C + \frac{q^2-1}{q^2+1} U R^C S\circ S \\ &\quad + \frac{1}{q^2+1} U\, R\circ S\, R^C + \frac{1}{q^2}\frac{q^2-1}{q^2+1} U\, R\circ R\, S^C - \frac{1}{q^2}\frac{q^2-1}{(q^2+1)^2} U^2 Z^C \end{aligned} \tag{B.27}$$

The commutation relations of Z with R and S are:

$$\begin{aligned} Z^A R^B &= \frac{1}{q^2}\frac{1}{q^2+1} U R^A S^B + (q^2-1)\varepsilon_D{}^{AB} R^D\, R\circ S - \frac{1}{q^2}\varepsilon_{CD}{}^{A}\varepsilon_L{}^{CB} R^D Z^L \\ &= R^A Z^B - \varepsilon_R{}^{AB}\, R\circ R S^R + \frac{1}{q^2}\frac{1}{q^2+1} U R^A S^B - \frac{1}{q^2}\frac{1}{q^2+1}\varepsilon_R{}^{AB} U Z^R \\ &\quad - g^{AB}\frac{1}{q^2+1} U\, R\circ S + q^2 \varepsilon_D{}^{AB} R^D\, R\circ S \\ Z^A S^B &= \frac{1}{q^2+1} U S^A R^B + (q^2-1)\varepsilon_D{}^{AB} S^D\, R\circ S - \frac{1}{q^2}\varepsilon_{CD}{}^{A}\varepsilon_L{}^{CB} S^D Z^L \end{aligned} \tag{B.28}$$

The commutation relations of $R\circ S$ with R,S and Z are given by:

$$R\circ S\, R^C = \frac{1}{q^2}(1-q^2)\, R\circ R\, S^C + q^2 R^C\, R\circ S - \frac{1}{q^2}\frac{1}{q^2+1} U Z^C$$

$$R\circ S\, S^C = \frac{1}{q^4}(q^2-1)\, S\circ S\, R^C + \frac{1}{q^2} S^C\, R\circ S - \frac{1}{q^4}\frac{1}{q^2+1} U Z^C \tag{B.29}$$

$$\begin{aligned} R\circ S\, Z^C &= Z^C\, R\circ S - \frac{1}{q^2}\frac{1}{q^2+1} U\, R\circ R\, S^C - \frac{1}{q^4}\frac{1}{q^2+1} U R^C\, S\circ S \\ &\quad + \frac{1}{q^2}\frac{1}{q^2+1} U S^C\, R\circ S + \frac{1}{q^2+1} U R^C\, R\circ S - \frac{1}{q^4}\frac{1}{(q^2+1)^2} U^2 Z^C \end{aligned}$$

Appendix C

The different forms of $SU_q(2)$

There are different ways of defining the algebra $SU_q(2)$. Here they are listed explicitly.
Jimbo [17]:
Generators: e, f, k, k^{-1}
Relations:

$$\begin{aligned} kk^{-1} &= 1 = k^{-1}k \\ ke &= q^2 ek \\ fk &= q^2 kf \\ ef - fe &= \frac{k - k^{-1}}{q - q^{-1}} \end{aligned} \tag{C.1}$$

T-algebra [22] first version:
Generators: T^+, T^-, T^3
Relations:

$$\begin{aligned} \frac{1}{q}T^+T^- - qT^-T^+ &= T^3 \\ q^2T^3T^+ - \frac{1}{q^2}T^+T^3 &= (q + \frac{1}{q})T^+ \\ q^2T^-T^3 - \frac{1}{q^2}T^3T^- &= (q + \frac{1}{q})T^- \end{aligned} \tag{C.2}$$

T-algebra second version:
Generators: $T^+, T^-, \tau^{\frac{1}{2}}, \tau^{-\frac{1}{2}}$
Relations:

$$\begin{aligned} \frac{1}{q}T^+T^- - qT^-T^+ &= \frac{1 - \tau}{q - \frac{1}{q}} \\ \tau^{\frac{1}{2}}T^+ &= \frac{1}{q^2}T^+\tau^{\frac{1}{2}} \\ \tau^{\frac{1}{2}}T^- &= q^2T^-\tau^{\frac{1}{2}} \end{aligned} \tag{C.3}$$

The second version of the T-algebra is obtained from the first by putting:

$$\tau = 1 - (q - \frac{1}{q})T^3 \quad \text{and its inverse} \quad T^3 = \frac{1-\tau}{q - \frac{1}{q}} \tag{C.4}$$

Notice that the two versions of the algebra are not completely equivalent, because the mapping defines τ, but not its square root $\tau^{\frac{1}{2}}$, which has to be explicitly added as further generator to the algebra.

The T-algebra admits the following realization in the Jimbo-algebra:

$$\begin{aligned} T^+ &= \tau^{\frac{1}{2}} e \\ T^- &= \tau^{\frac{1}{2}} f \\ \tau^{\frac{1}{2}} &= k^{-1} \end{aligned} \tag{C.5}$$

The first version of the T-algebra is useful, because all the relations are expressed in terms of q-commutators.

L-algebra

Generators: L^+, L^-, L^3, W

Relations:

$$\begin{aligned} L^3L^+ - q^2L^+L^3 &= -\frac{W}{q^2}L^+ \\ L^-L^3 - q^2L^3L^- &= -\frac{W}{q^2}L^- \\ qL^-L^+ - qL^+L^- + (1-q^2)L^3L^3 &= -\frac{W}{q^2}L^3 \\ L^+W &= WL^+ \\ L^-W &= WL^- \\ L^3W &= WL^3 \\ q^4(q^2-1)^2(L^3L^3 - qL^+L^- - \frac{1}{q}L^-L^+) &= W^2 - 1 \end{aligned} \tag{C.6}$$

or in a more compact way:

$$\begin{aligned} \varepsilon_{BC}{}^{A} L^C L^B &= -\frac{W}{q^2}L^A \\ L^A W &= W L^A. \\ q^4(q^2-1)^2 L \circ L &= W^2 - 1 \end{aligned} \tag{C.7}$$

As a consequence of the last relation W can be understood as a square root of the Casimir $L \circ L$.

The T-algebra admits the following realization in the L-algebra [19, 21]:

$$\begin{aligned} \tau^{-\frac{1}{2}} &= W + q^2(1-q^2)L^3 \\ T^+ &= q^2\sqrt{1+q^2} \quad \tau^{1/2}L^+ \\ T^- &= -q^3\sqrt{1+q^2} \quad \tau^{1/2}L^-. \end{aligned} \tag{C.8}$$

The T-algebra has a Casimir operator. It is given by:

$$\vec{T}^2 = qT^-T^+ + \frac{q}{(q-\frac{1}{q})^2}\tau^{-\frac{1}{2}} + \frac{1}{q}\frac{1}{(q-\frac{1}{q})^2}\left(\tau^{\frac{1}{2}} - q^2 - 1\right) \tag{C.9}$$

In this definition the constant term is added to obtain the correct classical limit for $q \to 1$.

The conjugation properties of the T-algebra are:

$$\begin{aligned} \overline{T^+} = \frac{1}{q^2}T^-, &\qquad \overline{T^-} = q^2T^+ \\ \overline{T^3} = T^3, &\qquad \overline{\tau} = \tau \end{aligned} \tag{C.10}$$

whereas the conjugation properties of the L,W-algebra are:

$$\begin{aligned} \overline{L^+} &= -qL^- \\ \overline{L^-} &= -\frac{1}{q}L^+ \\ \overline{L^3} &= L^3 \\ \overline{W} &= W \end{aligned} \tag{C.11}$$

Appendix D

Basis transformation of the coordinates

Throughout this thesis the basis $X^a = (X^0, X^A) = (X^0, X^+, X^-, X^3)$ is used for the coordinates. This basis explicitly separates the time coordinate from the spatial ones and shows the $SO_q(3)$-symmetric structure of the latter. Therefore it is particularly handy when studying the representation theory and when comparing the corresponding results with the results already known for the three-dimensional Euclidean case.

In the previous work [27], however, light-cone coordinates A, B, C, D were used, because the construction of the Minkowski space by means of product of spinors automatically leads to such kind of basis. In this appendix the basis transformation between these two sets of coordinates is given. This is necessary to compare the results of this thesis with the previous papers, such as [27, 2, 4, 39, 26, 31].

The q-deformed Minkowski space can be defined in light-cone coordinates as the algebra generated by A, B, C, D with the relations:

$$\begin{aligned} AB &= BA - \tfrac{1}{q}\lambda CD + q\lambda D^2 & BC &= CB - \tfrac{1}{q}\lambda BD \\ AC &= CA + q\lambda AD & BD &= q^2 DB \\ AD &= \tfrac{1}{q^2} DA & CD &= DC \end{aligned} \tag{D.1}$$

In the basis (A, B, C, D) the metric is defined as: [1]

$$\tilde{\eta}^{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{q^2} & 0 & 0 & 0 \\ 0 & 0 & -q\lambda & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \tilde{\eta}_{ij} = \begin{pmatrix} 0 & q^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & q\lambda \end{pmatrix} \tag{D.2}$$

[1] The indices i, j assume the values A, B, C, D

The invariant scalar product is defined as:

$$\widetilde{X \cdot X} = \frac{1}{q^2+1}\tilde{\eta}_{ij} X^i X^j = AB - q^{-2}CD \tag{D.3}$$

The basis transformation which leads to the X^a [2] is:

$$\begin{aligned} X^0 &= \frac{1}{\sqrt{q^2+1}}(C+D) \\ X^+ &= qA \\ X^- &= -B \\ X^3 &= \frac{1}{\sqrt{q^2+1}}(C-q^2D) \end{aligned} \tag{D.4}$$

and its inverse is:

$$\begin{aligned} A &= \frac{1}{q}X^+ \\ B &= -X^- \\ C &= \frac{1}{\sqrt{q^2+1}}(q^2X^0+X^3) \\ D &= \frac{1}{\sqrt{q^2+1}}(X^0-X^3) \end{aligned} \tag{D.5}$$

It can be immediately verified that the metrics $\tilde{\eta}^{ij}$ and η^{ab} are related by the similarity transformation induced by (D.4). The same is true for the $\hat{R}$-matrices and the projectors, which however are not explicitly listed here in the A, B, C, D-basis.

The scalar product has a different normalization than the one used in this thesis. By writing $\widetilde{X \cdot X}$ defined by (D.3) in the X^A-basis it turns out that:

$$\widetilde{X \cdot X} = \frac{1}{q^2+1}(X^0X^0 - X \circ X) = -\frac{1}{q^2+1}X \cdot X \tag{D.6}$$

Notice that the two expressions also differ in sign.

Appendix E

Maple programs drawing the plots

Program drawing the eigenvalues of X^3 versus those of r_o in the case $s^2 < 0$:

```
> # Sets definitions
> a:=5:
> b:=28:
> c:=b^2+3*b-(a-1)^2-3*(a-1):
> n:=a:
> q:=1.1:
> rt:=  k -> sqrt((q^(k+1)+q^(-2*n+k-1))*2*q/(q^2+1)-(q^(-2*n+2*k)+1)):
> z:= k -> -(q^(-n+k)-(q^(n+1)+q^(-n-1))*q/(q^2+1)):
```
```
> # Calculates circles for n from a to b
> arr:=array(1..c):
> for n from a to b do:
>     for k from 0 to n do:
>         arr[(n-1)^2+3*(n-1)-(a-1)^2-3*(a-1)+2*k+1]:=rt(2*k):
>         arr[(n-1)^2+3*(n-1)-(a-1)^2-3*(a-1)+2*k+2]:=z(2*k):
>     od:
> od:
```
```
> # Draws circles;
> plot(arr,x=0..12,y=-6..6,style=POINT,title=‘q=1.1,t0=1,n=5-28‘,labels=[‘rt‘,‘z‘])
  ;
```

Program drawing the eigenvalues of X^3 versus those of r_o in the case $s^2 > 0$:

```
> # Sets definitions
> a:=5:
> b:=28:
> c:=b^2+b-(a-1)^2-(a-1):
> n:=a:
> q:=1.05:
> rt:=  k -> sqrt((q^(k+1)-q^(-2*n+k+1))*2*q/(q^2+1)-(q^(-2*n+2*k+2)-1)):
> z:= k -> -(q^(-n+k+1)-(q^n-q^(-n))*q/(q^2+1)):
> # Calculates circles for n from a to b
> arr:=array(1..c):
> for n from a to b do:
>     for k from 0 to n-1 do:
>          arr[(n-1)^2+(n-1)-(a-1)^2-(a-1)+2*k+1]:=rt(2*k):
>          arr[(n-1)^2+(n-1)-(a-1)^2-(a-1)+2*k+2]:=z(2*k):
>     od:
> od:
> # Calculates circles for n from a to b
> arr2:=array(1..c):
> for n from a to b do:
>     for k from 0 to n-1 do:
>          arr2[(n-1)^2+(n-1)-(a-1)^2-(a-1)+2*k+1]:=rt(-2*k):
>          arr2[(n-1)^2+(n-1)-(a-1)^2-(a-1)+2*k+2]:=z(-2*k):
>     od:
> od:
> # Draws circles;
> plot({arr,arr2},x=0..4,y=-2..2,color=white,style=POINT,title=‘q=1.05,t0=1,n=5-
  28‘,labels=[‘rt‘,‘z‘]);
```

Program drawing the eigenvalues of X^0 versus those of r:

```
> # Fixes the parameters
> q:=1.1:
> t0:=1:
> # Defines the functions t,r in the case s^2  time-like
> t1:= n -> t0*(q^(n+1)+q^(-n-1)):
> r1:= n -> abs(sqrt(t1(n)^2-t0^2*(1+q^2)^2*q^(-2))):
> # Defines the functions t,r in the case s^2  on the light-cone
> v:= n -> t0*q^n:
> # Defines the functions t,r in the case s^2  space-like
> t2:= n -> t0*(q^n-q^(-n)):
> r2:= n -> abs(sqrt(t2(n)^2+t0^2*(1+q^2)^2*q^(-2))):
> # Loop to calculate t,r in the case s^2  time-like
> i:=60:
> j:=60:
> arr:=array(1..j,1..4*i):
> for l from 1 to j do:
>    for k from 1 to i do:
>       arr[l,2*k-1]:=r1(k-1)*q^(l-j/2):
>       arr[l,2*k]:=t1(k-1)*q^(l-j/2):
>    od:
> od:
> # Loop to calculate t,r in the case s^2  space-like
> for l from 1 to j do:
>    for k from 1 to i do:
>       arr[l,2*(k+i)-1]:=r2(k-1)*q^(l-j/2):
>       arr[l,2*(k+i)]:=t2(k-1)*q^(l-j/2):
>    od:
> od:
> # Loop to prepare the array to be plotted
> a:=4*i*j:
> arr2:=array(1..a+2*i):
> m:=1:
> for k from 1 to 2*i do:
>    for l from 1 to j do:
>       arr2[m]:=arr[l,2*k-1]:
>       arr2[m+1]:=arr[l,2*k]:
>       m:=m+2:
>    od:
> od:
> # Draws plot
> plot(arr2,x=0..12,y=0..12,style=POINT,title=‘q=1.1,t0=1‘,labels=[‘r‘,‘t‘]);
```

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