Bose-Einstein Condensation in the Relativistic Ideal Bose Gas

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Currently there is considerable interest in the results being produced by the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory. There, high density quantum chromodynamics (QCD) is very important. In particular the behavior of π mesons and diquarks, which are massive bosons, is relevant. In our just-published paper [1] we study the behavior of the Bose-Einstein condensation for an ideal Bose gas in this regime.

Since its theoretical prediction by Einstein in 1925 based on the 1924 work by Bose on photons, and after many decades languishing as a mere academic exercise in textbooks, Bose-Einstein condensation (BEC) has been observed in the laboratory in laser-cooled, magnetically trapped ultra-cold bosonic atomic clouds. We study instead the conditions in high-density QCD where BEC may occur.

In early papers on the relativistic ideal Boson gas, critical transition temperature T_c-formulae were derived for both relativistic and ultra-relativistic cases, but the production of anti-bosons was not considered. At high temperature, such as would be expected at RHIC, there will be very considerable production of anti-bosons.

In our work we exhibit, as a function of boson number density, exact BEC transition temperatures for the relativistic ideal boson gas (RIBG) system with and without anti-bosons in 3D. The system with both kinds of bosons always has the higher T_c, i.e., is the system with the first BEC singularity that appears as it is cooled. This suggests that the Helmholtz free energy might be lower and thus correspond thermodynamically to the *stable* system as opposed to a *metastable* system for the lower-T_c system. It is then calculated and indeed found to be lower, for all densities for the complete problem with both bosons and anti-bosons, when compared with the problem without anti-bosons. This implies that the omission of anti-bosons will not lead to stable states.

The number of bosons N of mass m that make up an ideal boson gas in d dimensions (*without* anti-bosons) is given by

$$N = \sum_{\mathbf{k}} [e^{\beta(|E_k|-\mu)} - 1]^{-1} \tag{1}$$

where $\beta = 1/k_BT$; k_B is the Boltzmann constant, and $\mu(T)$ is the boson chemical potential. Here, the total energy of each boson is

$$\begin{aligned} |E_k| &\equiv \sqrt{c^2\hbar^2k^2 + m^2c^4} && (2)\\ &= mc^2 + \hbar^2k^2/2m + O(k^4) \\ &\quad \text{if } c\hbar k \ll mc^2 \quad \mathbf{NR} && (3)\\ &= c\hbar k[1 + \tfrac{1}{2}(mc/\hbar k)^2 + O(k^{-4})] \\ &\quad \text{if } c\hbar k \gg mc^2 \quad \mathbf{UR} && (4)\end{aligned}$$

where k is the particle wave number, m refers to the boson rest mass, and c is the speed of light. The two limits refer to the nonrelativistic *(NR)* and ultrarelativistic *(UR)* extremes. For a cubic box of side length L in the continuous limit the sum in (1) over the d-dimensional wave vector $\mathbf{k}$ becomes an integral, namely

$$\sum_{\mathbf{k}} \to (L/2\pi)^d \int d^dk. \tag{5}$$

At the BEC critical transition temperature T_c, $\mu(T_c) = mc^2$ and the boson number, density can be expressed as

$$n \equiv \frac{N}{L^d} = \frac{1}{(2\pi)^d} \int d^dk \, \frac{1}{\exp[\beta_c(|E_k| - mc^2)] - 1} \tag{6}$$

where $\beta_c \equiv 1/k_BT_c$.

At sufficiently high temperatures such that $k_BT >> mc^2$ boson–anti-boson pair-production will occur abundantly. In fact, the total energy E_k of each particle satisfies $E_k^2 = c^2h^2k^2 + m^2c^4$ so that

$$E_k = \pm\sqrt{c^2\hbar^2k^2 + m^2c^4} \equiv \pm |E_k| \tag{7}$$

with the + sign referring to bosons and the – sign to anti-bosons. The *complete* number equation is now

$$\begin{aligned} N - \overline{N} &\equiv \sum_{\mathbf{k}} (n_{\mathbf{k}} - \overline{n}_{\mathbf{k}}) && (8)\\ &= \sum_{\mathbf{k}} \left[\frac{1}{\exp[\beta(|E_k| - \mu)] - 1} \right. \\ &\quad \left. - \frac{1}{\exp[\beta(|E_k| + \mu)] - 1}\right] \end{aligned}$$

where $n_{\mathbf{k}}$, $(\bar{n}_{\mathbf{k}})$ is the average number of bosons (anti-bosons) in the state of energy $\pm| E_{\mathbf{k}} |$, respectively, at a given temperature T and N $(\bar{N})$ is their respective total number at that temperature. Since $n_{\mathbf{k}}, \bar{n}_{\mathbf{k}} > 0$ for all $\mathbf{k}$ and $E_0 = mc^2$, the chemical potential must be bounded by

$$-mc^2 \leq \mu \leq mc^2. \tag{9}$$

Instead of N constant we must now impose the constancy of $N - \bar{N}$ to extract the correct BEC critical temperature, which we call $T_c^{B\bar{B}}$ as it involves both bosons (B) and anti-bosons $(\bar{B})$. At $T = T_c^{B\bar{B}}$ one has $|\mu(T_c^{B\bar{B}})| = mc^2$. Figure 1 shows our results.

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Reference
[1] M. Grether, M. de Llano, and G.A. Baker, Jr., *Phys. Rev. Lett.* **90,** 200406 (2007), and the references therein.

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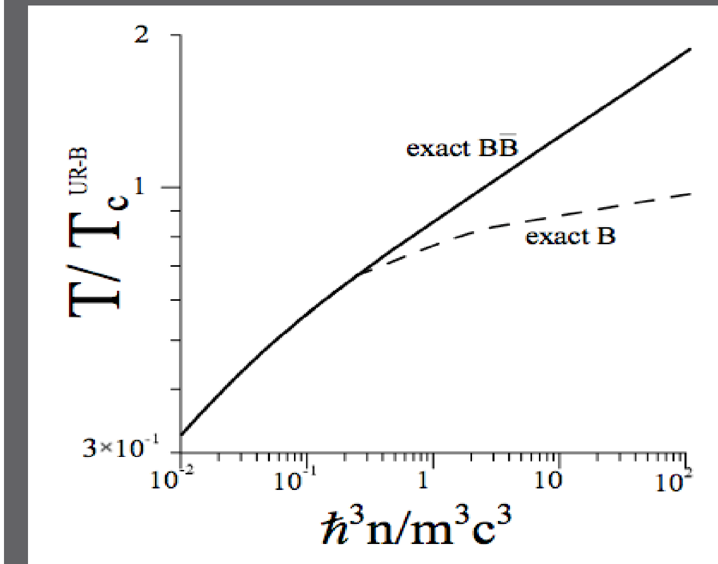

Fig. 1. BEC T_c (in units = $k_B T^{UR-B} = \hbar c \pi^{2/3} [n\zeta(3)]^{1/3}$) as a function of boson number-density n expressed in dimensionless form as $\hbar^3 n/m^3c^3$. The solid line corresponds to BEC T_c in a RIBG with both bosons B and anti-bosons $\bar{B}$. The dashed line is the exact line for bosons B only. ζ is Riemann's zeta function.