Implications of the electrostatic approximation in the beam frame on the nonlinear Vlasov–Maxwell equations for intense beam propagation

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(Received 7 September 2001; accepted 18 October 2001)

This paper develops a clear procedure for solving the nonlinear Vlasov–Maxwell equations for a one-component intense charged particle beam or finite-length charge bunch propagating through a cylindrical conducting pipe (radius $r=r_w=\text{const}$), and confined by an applied focusing force $\mathbf{F}_{\text{foc}}$. In particular, the nonlinear Vlasov–Maxwell equations are Lorentz transformed to the beam frame ("primed" variables) moving with axial velocity $V_b=\beta_b c=\text{const}$ relative to the laboratory. In the beam frame, the particle motions are nonrelativistic for the applications of practical interest, already a major simplification. Then, in the beam frame, the electrostatic approximation ($\mathbf{E}'_s=-\nabla'\phi'$, $\mathbf{E}'_T\simeq 0\simeq\mathbf{B}'_s$) is made, which fully incorporates beam space-charge effects, but neglects any fast electromagnetic processes with transverse polarization (e.g., light waves). The resulting Vlasov–Maxwell equations are then Lorentz transformed back to the laboratory frame, and properties of the self-generated fields and resulting nonlinear Vlasov–Maxwell equations in the laboratory frame are discussed. [DOI: 10.1063/1.1427023]

I. INTRODUCTION

Periodic focusing accelerators and transport systems[1–7] have a wide range of applications ranging from basic scientific research in high energy and nuclear physics, to applications such as coherent radiation sources, heavy ion fusion, tritium production, nuclear waste transmutation, and spallation neutron sources for materials and biological research.[8,9] At the high beam currents and charge densities of practical interest, of particular importance are the effects of the intense self-fields produced by the beam space charge and current on determining the detailed equilibrium, stability and transport properties, and the nonlinear dynamics of the system. Through analytical studies based on the nonlinear Vlasov–Maxwell equations for the distribution function $f_b(\mathbf{x},\mathbf{p},t)$ and the self-generated electric and fields $\mathbf{E}_s(\mathbf{x},t)$ and $\mathbf{B}_s(\mathbf{x},t)$, and numerical simulations using particle-in-cell models and nonlinear perturbative simulation techniques, considerable progress has been made in developing an improved understanding of the collective processes and nonlinear beam dynamics characteristic of high-intensity beam propagation in periodic focusing and uniform focusing transport systems.[10–28] In almost all applications of the Vlasov–Maxwell equations to intense beam propagation, the analysis is carried out in the laboratory frame, and various simplifying approximations are made, ranging from the electrostatic–magnetostatic approximation[29] to the Darwin-model approximation,[30–35] which neglects fast transverse electromagnetic perturbations.

Given the general importance of model assumptions in affecting the detailed outcome of calculations, in this paper we develop a clear procedure for solving the nonlinear Vlasov–Maxwell equations for a one-component intense charged particle beam or finite-length charge bunch propagating through a cylindrical conducting pipe (radius $r=r_w=\text{const}$), and confined by an applied focusing force $\mathbf{F}_{\text{foc}}$. In particular, the nonlinear Vlasov–Maxwell equations are Lorentz transformed to the beam frame ("primed" variables) moving with axial velocity $V_b=\beta_b c=\text{const}$ relative to the laboratory.[24] In the beam frame, the particle motions are nonrelativistic for the applications of practical interest, already a major simplification. Then, in the beam frame, we make the electrostatic approximation ($\mathbf{E}'_s=-\nabla'\phi'$, $\mathbf{E}'_T\simeq 0\simeq\mathbf{B}'_s$) which fully incorporates beam space-charge effects, but neglects any fast electromagnetic processes with transverse polarization (e.g., light waves). The resulting Vlasov–Maxwell equations are then Lorentz transformed back to the laboratory frame, and properties of the self-generated fields and resulting nonlinear Vlasov–Maxwell equations in the laboratory frame are discussed.

II. VLASOV–MAXWELL EQUATIONS AND TRANSFORMATION TO THE BEAM FRAME

In the present analysis, we consider an intense charged particle beam with characteristic transverse dimensions a and b propagating in the z direction with average axial velocity $V_b=\beta_b c=\text{const}$ and characteristic directed kinetic energy $(\gamma_b-1)m_b c^2$. Here, c is the speed of light *in vacuo*, $\gamma_b=(1-\beta_b^2)^{-1/2}$ is the relativistic mass factor, and e_b and m_b are the charge and rest mass, respectively, of a beam particle. A perfectly conducting cylindrical wall is located at radius $r=r_w$, where $r=(x^2+y^2)^{1/2}$ is the radial distance from the beam axis. The particle motion in the beam frame ("primed" coordinates) is assumed to be nonrelativistic with $|\mathbf{v}'|\ll c$. Furthermore, the beam current density and charge density are allowed to be arbitrarily large, subject only to the requirement that the beam particles be confined by the applied focusing fields $\mathbf{E}_{\text{foc}}(\mathbf{x},t)$ and $\mathbf{B}_{\text{foc}}(\mathbf{x},t)$. The specific forms of $\mathbf{E}_{\text{foc}}(\mathbf{x},t)$ and $\mathbf{B}_{\text{foc}}(\mathbf{x},t)$ of course depend on the particular focusing field configuration under consideration (quadrupole, solenoidal, rf, etc.). Finally, in the present analysis, the beam can be continuous in the z direction, or correspond to a finite-length charge bunch.

Within the context of the above-mentioned assumptions, a complete description of the collective processes and nonlinear dynamics of the charged particle beam is provided by the nonlinear Vlasov–Maxwell equations,[1] which describe the evolution of the distribution function $f_b(\mathbf{x},\mathbf{p},t)$ in the six-dimensional laboratory-frame phase space $(\mathbf{x},\mathbf{p})$, and the corresponding self fields, $\mathbf{E}_s(\mathbf{x},t)$ and $\mathbf{B}_s(\mathbf{x},t)$, generated self-consistently by the beam space charge and current. In laboratory-frame variables, the nonlinear Vlasov–Maxwell equations describing the self-consistent evolution of $f_b(\mathbf{x},\mathbf{p},t)$, $\mathbf{E}_s(\mathbf{x},t)$, and $\mathbf{B}_s(\mathbf{x},t)$ can be expressed as

$$\frac{\partial f_b}{\partial t}+\mathbf{v}\cdot\frac{\partial f_b}{\partial \mathbf{x}}+\left[\mathbf{F}_{\mathrm{foc}}+e_b\left(\mathbf{E}_s+\frac{1}{c}\mathbf{v}\times\mathbf{B}_s\right)\right]\cdot\frac{\partial f_b}{\partial \mathbf{p}}=0, \quad (1)$$

and

$$\nabla\cdot\mathbf{E}_s=4\pi e_b\int d^3p f_b, \quad (2)$$

$$\nabla\times\mathbf{B}_s=\frac{1}{c}4\pi e_b\int d^3p\,\mathbf{v} f_b+\frac{1}{c}\frac{\partial \mathbf{E}_s}{\partial t}, \quad (3)$$

$$\nabla\times\mathbf{E}_s=-\frac{1}{c}\frac{\partial \mathbf{B}_s}{\partial t}, \quad (4)$$

$$\nabla\cdot\mathbf{B}_s=0. \quad (5)$$

Here, $\mathbf{F}_{\mathrm{foc}}=e_b(\mathbf{E}_{\mathrm{foc}}+c^{-1}\mathbf{v}\times\mathbf{B}_{\mathrm{foc}})$ is the applied focusing force in the laboratory frame, and the velocity $\mathbf{v}$ and momentum $\mathbf{p}$ are related by $\mathbf{p}=\gamma m_b\mathbf{v}$, where $\gamma=(1+\mathbf{p}^2/m_b^2c^2)^{1/2}$. The Vlasov equation (1) describes the incompressible evolution of the distribution function $f_b(\mathbf{x},\mathbf{p},t)$ in the six-dimensional phase space $(\mathbf{x},\mathbf{p})$ as the beam particles interact with the applied focusing fields, $\mathbf{E}_{\mathrm{foc}}(\mathbf{x},t)$ and $\mathbf{B}_{\mathrm{foc}}(\mathbf{x},t)$, and the average self-fields, $\mathbf{E}_s(\mathbf{x},t)$ and $\mathbf{B}_s(\mathbf{x},t)$, generated by the beam particles. Note that the Vlasov equation (1) is highly nonlinear because $\mathbf{E}_s(\mathbf{x},t)$ and $\mathbf{B}_s(\mathbf{x},t)$ are determined self-consistently in terms of the beam charge density, $\rho_b(\mathbf{x},t)=e_b\int d^3p f_b(\mathbf{x},\mathbf{p},t)$, and current density, $\mathbf{J}_b(\mathbf{x},t)=e_b\int d^3p\,\mathbf{v} f_b(\mathbf{x},\mathbf{p},t)$, from Maxwell's equations (2)–(5). Here, $n_b(\mathbf{x},t)=\int d^3p f_b(\mathbf{x},\mathbf{p},t)$ is the number density of beam particles.

The Vlasov–Maxwell equations (1)–(5) can of course be analyzed directly in laboratory-frame variables. However, for a beam consisting of a single charge component (the case considered here), there is considerable advantage to transforming to the beam frame moving with axial velocity $V_b=\beta_b c$ relative to the laboratory. In the beam frame the particle motion is nonrelativistic ($|\mathbf{v}'|\ll c$) by assumption, which results in a welcome simplification of the corresponding Vlasov–Maxwell equations in the beam frame. The Lorentz transformation[36,37] relating the primed variables $(\mathbf{x}',\mathbf{p}',t')$ in the beam frame to the unprimed variables $(\mathbf{x},\mathbf{p},t)$ in the laboratory frame is given by

$$\begin{aligned} &x'=x,\quad y'=y,\quad z'=\gamma_b(z-V_bt),\\ &t'=\gamma_b(t-V_bz/c^2),\\ &p'_x=p_x,\quad p'_y=p_y,\quad p'_z=\gamma_b(p_z-\gamma m_bV_b),\\ &\gamma'=\gamma_b(\gamma-V_bp_z/m_bc^2), \end{aligned} \quad (6)$$

where $\gamma_b=(1-V_b^2/c^2)^{-1/2}$. Here, the particle momentum and velocity are related by $\mathbf{p}=\gamma m_b\mathbf{v}$ and $\mathbf{p}'=\gamma' m_b\mathbf{v}'$, where $\gamma=(1+\mathbf{p}^2/m_b^2c^2)^{1/2}$ and $\gamma'=(1+\mathbf{p}'^2/m_b^2c^2)^{1/2}$ are the kinematic mass factors. In the beam frame, the nonlinear Vlasov equation for the distribution function $f'_b(\mathbf{x}',\mathbf{p}',t')$ can be expressed as[24]

$$\frac{\partial f'_b}{\partial t'}+\mathbf{v}'\cdot\frac{\partial f'_b}{\partial \mathbf{x}'}+\left[\mathbf{F}'_{\mathrm{foc}}+e_b\left(\mathbf{E}'_s+\frac{1}{c}\mathbf{v}'\times\mathbf{B}'_s\right)\right]\cdot\frac{\partial f'_b}{\partial \mathbf{p}'}=0. \quad (7)$$

In Eq. (7), $\mathbf{E}'_s(\mathbf{x}',t)$ and $\mathbf{B}'_s(\mathbf{x}',t')$ are the self-generated fields in the beam frame, and we approximate $\gamma'=1+\mathbf{p}'^2/2m_b^2c^2$ and $\mathbf{p}'=m_b\mathbf{v}'$ because the particle motion in the beam frame is assumed to be nonrelativistic. Furthermore, $\mathbf{F}'_{\mathrm{foc}}=e_b(\mathbf{E}'_{\mathrm{foc}}+c^{-1}\mathbf{v}'\times\mathbf{B}'_{\mathrm{foc}})$ is the applied focusing force on a particle in the beam frame. Here, the applied electric and magnetic fields transform according to $\mathbf{E}'_{\mathrm{foc}}=[E_z\hat{\mathbf{e}}_z+\gamma_b(E_x\hat{\mathbf{e}}_x+E_y\hat{\mathbf{e}}_y)+\gamma_bc^{-1}V_b\hat{\mathbf{e}}_z\times\mathbf{B}]_{\mathrm{foc}}$ and $\mathbf{B}'_{\mathrm{foc}}=[B_z\hat{\mathbf{e}}_z+\gamma_b(B_x\hat{\mathbf{e}}_x+B_y\hat{\mathbf{e}}_y)-\gamma_bc^{-1}V_b\hat{\mathbf{e}}_z\times\mathbf{E}]_{\mathrm{foc}}$, which determines $\mathbf{E}'_{\mathrm{foc}}$ and $\mathbf{B}'_{\mathrm{foc}}$ directly in terms of the applied focusing fields in the laboratory frame and the Lorentz transformation in Eq. (6).

Maxwell's equations in the beam frame of course are similar in form to Eqs. (2)–(5), and relate the self-generated electric and magnetic fields, $\mathbf{E}'_s(\mathbf{x}',t')$ and $\mathbf{B}'_s(\mathbf{x}',t')$, to the distribution function $f'_b(\mathbf{x}',\mathbf{p},t')$. For present purposes, it is convenient to introduce the scalar and vector potentials, $\phi'(\mathbf{x}',t')$ and $\mathbf{A}'(\mathbf{x}',t')$, and express

$$\begin{aligned} &\mathbf{E}'_s=\mathbf{E}'_L+\mathbf{E}'_T=-\frac{\partial}{\partial \mathbf{x}'}\phi'-\frac{1}{c}\frac{\partial}{\partial t'}\mathbf{A}',\\ &\mathbf{B}'_s=\frac{\partial}{\partial \mathbf{x}'}\times\mathbf{A}', \end{aligned} \quad (8)$$

where $\mathbf{E}'_L=-\nabla'\phi'$ is the longitudinal electric field, $\mathbf{E}'_T=c^{-1}\partial\mathbf{A}'/\partial t'$ is the transverse electric field, and the Coulomb gauge condition with $\nabla'\cdot\mathbf{A}'=0$ is assumed. From Eq. (8), the Maxwell equations $\nabla'\cdot\mathbf{B}'_s=0$ and $\nabla'\times\mathbf{E}'_s=-c^{-1}\partial\mathbf{B}'_s/\partial t'$ are automatically satisfied, and Poisson's equation and the $\nabla'\times\mathbf{B}'_s$ Maxwell equation in the beam frame are readily expressed as[24]

$$\nabla'^2\phi'=-4\pi e_b\int d^3p' f'_b, \quad (9)$$

$$\nabla'^2\mathbf{A}'=-\frac{1}{c}4\pi e_b\int d^3p'\,\mathbf{v}' f'_b+\frac{1}{c^2}\frac{\partial^2\mathbf{A}'}{\partial t'^2}+\frac{1}{c}\nabla'\frac{\partial\phi'}{\partial t'}, \quad (10)$$

where use has been made of $\nabla'\cdot\mathbf{A}'=0$. In Eqs. (9) and (10), note that the electrostatic potential $\phi'(\mathbf{x}',t')$ is determined self-consistently in terms of the beam charge density $\rho'_b(\mathbf{x}',t')=e_bn'_b(\mathbf{x}',t')=e_b\int d^3p' f'_b(\mathbf{x}',\mathbf{p}',t')$ from Eq. (9), and $\mathbf{A}'(\mathbf{x}',t')$ is determined in terms of the beam current density $\mathbf{J}'_b(\mathbf{x}',t')=e_bn'_b(\mathbf{x}',t')\mathbf{V}'_b(\mathbf{x}',t')=e_b\int d^3p'\,\mathbf{v}' f'_b(\mathbf{x}',\mathbf{p}',t')$ from Eq. (10). Here, $n'_b(\mathbf{x}',t')$ is the local number density and $\mathbf{V}'_b(\mathbf{x}',t')$ is the local average flow velocity of particles in the beam frame, and $\mathbf{v}'$

$=\mathbf{p}'/m_b$ is the (nonrelativistic) particle velocity. Note that Poisson's equation (9) can be viewed as an initial condition to Eq. (10), which if true at $t'=0$ remains true at all subsequent t'. This follows upon taking the divergence of Eq. (10) and making use of the Coulomb gauge condition $\nabla'\cdot\mathbf{A}'=0$. This readily gives

$$\frac{1}{c}\frac{\partial}{\partial t'}\nabla'^2\phi' - \frac{1}{c}4\pi e_b\nabla'\cdot(n_b'\mathbf{V}_b') = \frac{1}{c}\frac{\partial}{\partial t'}[\nabla'^2\phi' + 4\pi e_b n_b'] = 0, \tag{11}$$

where use has been made of $\partial n_b'/\partial t' + \nabla'\cdot(n_b'\mathbf{V}_b')=0$. It follows trivially from Eq. (11) that if Poisson's equation (9) is satisfied initially at $t'=0$, then it remains true at all subsequent times t'.

For the boundary conditions at the perfectly conducting cylindrical wall at radius $r=r'=r_w$, we impose the requirements that the tangential electric field and the normal magnetic field vanish. That is, $[E_z]_{r=r_w}=[E_\theta]_{r=r_w}=[B_r]_{r=r_w}=0$, where E_z, E_θ, and B_r denote field components in cylindrical polar coordinates (r,θ,z) in the laboratory frame. In the beam frame, the corresponding field components are given by $E_z'=E_z$, $B_r'=\gamma_b(B_r+V_bE_\theta/c)$, and $E_\theta'=\gamma_b(E_\theta+V_bB_r/c)$. Therefore, the corresponding boundary conditions at the conducting wall $r'=r_w$ in the beam frame are also given by $[E_z']_{r'=r_w}=[E_\theta']_{r'=r_w}=[B_r']_{r'=r_w}=0$. Expressed in terms of the scalar and vector potentials, $\phi'(\mathbf{x}',t')$ and $\mathbf{A}'(\mathbf{x}',t')$, these boundary conditions can be expressed as

$$\phi'(r'=r_w,\theta',z',t') = A_z'(r'=r_w,\theta',z',t') = A_\theta'(r'=r_w,\theta',z',t') = \text{const}, \tag{12}$$

where (r',θ',z') are cylindrical polar coordinates in the beam frame, with $x'=r'\cos\theta'$ and $y'=r'\sin\theta'$.

The nonlinear Vlasov–Maxwell equations (7), (9), and (10) in the beam frame, subject to the boundary conditions in Eq. (12), are fully equivalent to the Vlasov–Maxwell equations (1)–(5) in the laboratory frame, and provide a complete description of the collective processes and nonlinear dynamics of intense beam propagation. Equations (7), (9), and (10) can be used to investigate detailed equilibrium and stability properties in the beam frame for a wide range of system parameters and choices of applied field configurations. Moreover, as noted earlier, because the particle motion in the beam frame is nonrelativistic, a detailed investigation of Eqs. (7), (9), and (10) is more tractable analytically and numerically than the corresponding Vlasov–Maxwell equations (1)–(5) in the laboratory frame. Furthermore, once the solutions for $f_b'(\mathbf{x}',\mathbf{p}',t')$, $\phi'(\mathbf{x}',t')$, and $\mathbf{A}'(\mathbf{x}',t')$ are obtained in the beam frame, the corresponding solutions are readily obtained in the laboratory frame. For example, the variables $(\mathbf{x}',\mathbf{p}',t')$ are related to $(\mathbf{x},\mathbf{p},t)$ by the Lorentz transformation in Eq. (6). Furthermore, the scalar and vector potentials $\phi(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$ in the laboratory frame are related to the potentials $\phi'(\mathbf{x}',t')$ and $\mathbf{A}'(\mathbf{x}',t')$ in the beam frame by the transformation[24,36]

$$\begin{aligned}
&\phi = \gamma_b(\phi' + V_bA_z'/c),\\
&A_x = A_x', \quad A_y = A_y',\\
&A_z = \gamma_b(A_z' + V_b\phi'/c),
\end{aligned} \tag{13}$$

where the arguments $(\mathbf{x}',t')$ are related to $(\mathbf{x},t)$ by Eq. (6).

III. IMPLICATIONS OF ELECTROSTATIC APPROXIMATION IN THE BEAM FRAME

The introduction of the Coulomb gauge condition $(\nabla'\cdot\mathbf{A}'=0)$ and the resulting forms of Maxwell's equations (9) and (10) also make ancillary approximations more transparent in the beam frame. In the following analysis, we consider such a case corresponding to the electrostatic approximation in the beam frame. In particular, it is assumed that the electromagnetic field components with *transverse* polarization, $\mathbf{E}_T'=-c^{-1}\partial\mathbf{A}'/\partial t'$ and $\mathbf{B}_s'=\nabla'\times\mathbf{A}'$, are negligibly small in comparison with the *longitudinal* electric field, $\mathbf{E}_L'=-\nabla'\phi'$. In this case, we approximate

$$\begin{aligned}
&\mathbf{A}'=0,\\
&\mathbf{E}_T'=0=\mathbf{B}_s',
\end{aligned} \tag{14}$$

and the nonlinear Vlasov equation (7) in the beam frame becomes

$$\frac{\partial f_b'}{\partial t'} + \mathbf{v}'\cdot\frac{\partial f_b'}{\partial \mathbf{x}'} + [\mathbf{F}_{\text{foc}}' - e_b\nabla'\phi']\cdot\frac{\partial f_b'}{\partial \mathbf{p}'} = 0. \tag{15}$$

Of course the scalar potential $\phi'(\mathbf{x}',t')$ occurring in Eq. (15) is determined self-consistently in terms of the charge density $e_b\int d^3p' f_b'(\mathbf{x}',\mathbf{p}',t')$ from Poisson's equation (9). By virtue of Eq. (14), we have neglected any fast electromagnetic processes in the beam frame with transverse polarization (e.g., light waves), and it is assumed that the current carried by the particles in the beam frame is sufficiently small that the self-generated transverse field components $\mathbf{E}_T'$ and $\mathbf{B}_s'$ can be neglected. Equations (9) and (15) of course include the full influence of space-charge effects in the beam frame through the longitudinal electric field $\mathbf{E}_L'=-\nabla'\phi'$.

The nonlinear Vlasov-Poisson equations (9) and (15) constitute a closed description of the collective processes and nonlinear dynamics of the distribution function $f_b'(\mathbf{x}',\mathbf{p}',t')$ and space-charge potential $\phi'(\mathbf{x}',t')$ in the beam frame, valid in the electrostatic approximation. As such Eqs. (9) and (15) can be used to investigate detailed equilibrium and electrostatic stability properties for a wide range of system parameters and choices of focusing field configurations. The purpose of this paper is not to solve Eqs. (9) and (15) in detail. Rather, let us assume that the solutions to Eqs. (9) and (15) have been obtained in the beam frame (these could be analytical or numerical solutions), and pose the question: What are the properties of the corresponding solutions in the laboratory frame? The variables $(\mathbf{x}',\mathbf{p}',t')$ and $(\mathbf{x},\mathbf{p},t)$ of course transform according to Eq. (6). Furthermore, once $\phi'(\mathbf{x}',t')$ is determined in the beam frame, field quantities in the laboratory frame are readily obtained by making use of Eqs. (13) and (14). Substituting $\mathbf{A}'=0$ into Eq. (13) readily gives

$$\phi=\gamma_b\phi',$$
$$A_x=0=A_y, \tag{16}$$
$$A_z=\gamma_b\beta_b\phi'=\beta_b\phi,$$

where $\beta_b=V_b/c$. In Eq. (16), the arguments $(\mathbf{x}',t')$ and $(\mathbf{x},t)$ are related by Eq. (6), so Eq. (16) gives directly

$$\phi(x,y,z,t)=\gamma_b\phi'[x,y,\gamma_b(z-V_bt),\gamma_b(t-V_bz/c^2)], \tag{17}$$

as well as $A_x=A_y=0$, and

$$A_z(x,y,z,t)=\beta_b\phi(x,y,z,t). \tag{18}$$

Note from Eqs. (16) and (17) that the (x',y') and (x,y) dependencies of ϕ' and ϕ are identical, whereas the (z',t') and (z,t) dependences transform according to Eq. (6). In terms of Fourier–Laplace variables (k_z,ω) and (k_z',ω'), Eq. (17) leads directly to the familiar relations[36,37]

$$k_z=\gamma_b\left(k_z'+\frac{V_b}{c^2}\omega'\right),$$
$$\omega=\gamma_b(\omega'+k_z'V_b). \tag{19}$$

That is, if the potential $\phi'(\mathbf{x}',t')$ has axial wave number k_z' and frequency ω' in the beam frame, then the corresponding axial wave number k_z and frequency ω in the laboratory frame are given by Eq. (19). Of course the inverse transformation to Eq. (19) is obtained by interchanging (k_z,ω) and (k_z',ω'), and making the replacement $V_b\rightarrow -V_b$.

Equations (16)–(18) allow us to determine the self-generated fields, $\mathbf{E}_s=-\nabla\phi-c^{-1}\partial\mathbf{A}/\partial t$ and $\mathbf{B}_s=\nabla\times\mathbf{A}$, in the laboratory frame, consistent with the electrostatic approximation, $\mathbf{E}_s'=\mathbf{E}_L'=-\nabla'\phi'$, in the beam frame. It follows directly that $\mathbf{B}_s=\nabla\times A_z\hat{\mathbf{e}}_z$ has components

$$B_x=\frac{\partial A_z}{\partial y}=\beta_b\frac{\partial\phi}{\partial y},$$
$$B_y=-\frac{\partial A_z}{\partial x}=-\beta_b\frac{\partial\phi}{\partial x}, \tag{20}$$
$$B_z=0.$$

Furthermore, the self-generated electric field $\mathbf{E}_s=-\nabla\phi-c^{-1}(\partial A_z/\partial t)\hat{\mathbf{e}}_z$ can be expressed as

$$E_x=-\frac{\partial\phi}{\partial x},\quad E_y=-\frac{\partial\phi}{\partial y},$$
$$E_z=-\frac{\partial\phi}{\partial z}-\frac{1}{c}\frac{\partial A_z}{\partial t}=-\frac{\partial\phi}{\partial z}-\frac{\beta_b}{c}\frac{\partial\phi}{\partial t}. \tag{21}$$

As would be expected, even though there is no magnetic field in the beam frame ($\mathbf{B}_s'=0$) by assumption, in the laboratory frame the beam carries an axial current, and there is a transverse magnetic field generated with components, $B_x=\partial A_z/\partial y$ and $B_y=-\partial A_z/\partial x$, where $A_z=\beta_b\phi$ according to Eq. (18). Furthermore, in addition to the longitudinal electric field $\mathbf{E}_L=-\nabla\phi$, it follows from Eq. (21) that there is an inductive electric field $E_{Tz}=-c^{-1}\partial A_z/\partial t$ in the laboratory frame, where $A_z=\beta_b\phi$. As noted earlier, the nonlinear Vlasov–Poisson equations (9) and (15) can be solved in the beam frame, and then Eqs. (6), (16), (20), and (21) can be used to determine the corresponding distribution function and self-generated fields in the laboratory frame.

While Eqs. (9) and (15) provide a complete description of the system in the electrostatic approximation in the beam frame, it is nonetheless interesting to examine the implications of the electrostatic approximation in the beam frame for the corresponding Vlasov–Maxwell equations in the laboratory frame. First, making use of Eqs. (20) and (21), it follows that the force on a beam particle in the laboratory frame due to the self-generated fields, $\mathbf{F}=e_b(\mathbf{E}_s+c^{-1}\mathbf{v}\times\mathbf{B}_s)$, can be expressed as $\mathbf{F}=\mathbf{F}_\perp+F_z\hat{\mathbf{e}}_z$, where

$$\mathbf{F}_\perp=e_b\left(-\nabla_\perp\phi+\frac{1}{c}[\mathbf{v}\times(\nabla A_z\times\hat{e}_z)]_\perp\right)$$
$$=-e_b\nabla_\perp\left(\phi-\frac{1}{c}v_zA_z\right) \tag{22}$$

denotes the perpendicular force in the $x-y$ plane, and

$$F_z=e_b\left(-\frac{\partial\phi}{\partial z}-\frac{1}{c}\frac{\partial A_z}{\partial t}-\frac{1}{c}\mathbf{v}_\perp\cdot\nabla_\perp A_z\right)$$
$$=-e_b\left[\frac{\partial}{\partial z}\left(\phi-\frac{1}{c}v_zA_z\right)+\frac{1}{c}\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)A_z\right] \tag{23}$$

is the axial force. Here, $\nabla_\perp=\hat{\mathbf{e}}_x\partial/\partial x+\hat{\mathbf{e}}_y\partial/\partial y$ is the perpendicular spatial gradient, and $A_z=\beta_b\phi$ follows from Eq. (18). Equations (22) and (23) can be substituted into the laboratory-frame Vlasov equation (1). One important simplification occurs in this regard. The characteristics of the Vlasov equation (1) are the single-particle orbits in the self-generated fields. For example, the coefficient of $\partial f_b/\partial p_z$ is $dp_z/dt=F_z$. From Eq. (23), introducing the axial canonical momentum[30] $P_z=p_z+(e_b/c)A_z$, and making use of $(d/dt)A_z=(\partial/\partial t+\mathbf{v}\cdot\nabla)A_z$, it follows that $dP_z/dt=-e_b(\partial/\partial z)(\phi-v_zA_z/c)$. Therefore, if we change variables from (x,y,z,p_x,p_y,p_z,t) to (x,y,z,P_x,P_y,P_z,t), where $P_x=p_x$ and $P_y=p_y$ (because $A_x=0=A_y$) and $P_z=p_z+(e_b/c)A_z$, it follows from Eqs. (1), (22), and (23) that the nonlinear Vlasov equation for the distribution function $f_b(\mathbf{x},\mathbf{P},t)$ in the laboratory frame can be expressed in the compact form

$$\frac{\partial f_b}{\partial t}+\mathbf{v}\cdot\frac{\partial f_b}{\partial\mathbf{x}}+\left[\mathbf{F}_{\rm foc}-e_b\left(1-\frac{v_z}{c}\beta_b\right)\nabla\phi\right]\cdot\frac{\partial f_b}{\partial\mathbf{P}}=0. \tag{24}$$

In obtaining Eq. (24), use has been made of $A_z=\beta_b\phi$ to express $\nabla(\phi-v_zA_z/c)=(1-\beta_bv_z/c)\nabla\phi$. Furthermore, from Eqs. (2) and (21), Poisson's equation can be expressed as

$$\nabla_\perp^2\phi+\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}+\frac{1}{c}\beta_b\frac{\partial\phi}{\partial t}\right)=-4\pi e_b\int d^3P f_b(\mathbf{x},\mathbf{P},t), \tag{25}$$

where $\nabla_\perp^2=\partial^2/\partial x^2+\partial^2/\partial y^2$. The Vlasov–Poisson equations (24) and (25), valid in the laboratory frame, are fully equivalent to the Vlasov–Poisson equations (9) and (15) obtained in the beam frame in the electrostatic approximation. Of course when the two frames coincide ($\beta_b=0$, $\gamma_b=1$, $\phi=\phi'$ and $A_z=A_z'=0$), Eqs. (24) and (25) are identical in form to Eqs. (9) and (15), as expected.

As noted earlier, because the particle motion is nonrelativistic in the beam frame, it is often advantageous to solve Eqs. (9) and (15) directly, rather than Eqs. (24) and (25). Nonetheless, with some ancillary approximations, the laboratory-frame Vlasov–Poisson equations also simplify further. For example, if the axial velocity spread around $v_z=V_b=\beta_b c$ is very small ($|\Delta v_z|\gamma_b^2\beta_b \ll c$), then we approximate $1-v_z\beta_b c \simeq 1-\beta_b^2=1/\gamma_b^2$, and Eq. (24) reduces to

$$\frac{\partial f_b}{\partial t}+\mathbf{v}\cdot\frac{\partial f_b}{\partial \mathbf{x}}+\left[\mathbf{F}_{\mathrm{foc}}-\frac{e_b}{\gamma_b^2}\nabla\phi\right]\cdot\frac{\partial f_b}{\partial \mathbf{P}}=0. \tag{26}$$

Equation (26) shows clearly that the magnetic pinching force due to the self-magnetic field in the laboratory frame reduces the electric force by the factor $1/\gamma_b^2$.[1] A further simplification occurs in Eq. (25) in circumstances corresponding to an axially continuous beam or very long charge bunch. In Eq. (25), we denote $\partial/\partial x \sim 1/a$ and $\partial/\partial y \sim 1/b$, where $a\sim b$ are the transverse beam dimensions; $\partial/\partial z \sim 1/L$, where L is the characteristic length scale of axial variations; and $\partial/\partial t \sim \omega$, where ω^{-1} is the characteristic time scale of variations in $\phi(\mathbf{x},t)$. Then, for $a\sim b$, the terms on the left-hand side of Eq. (25) stand in the ratio

$$1:\frac{a^2}{L^2}:\frac{a^2}{L^2}\frac{\omega L}{c}\beta_b. \tag{27}$$

Then for $a^2 \ll L^2$, and even for moderately high frequency with $|\omega L/c|\sim 1$ or $|\omega L/c|\sim \beta_b$, the second and third terms on the left-hand side of Eq. (25) are negligibly small, and Poisson's equation can be approximated by

$$\nabla_\perp^2\phi=-4\pi e_b\int d^3P f_b(\mathbf{x},\mathbf{P},t), \tag{28}$$

where $\nabla_\perp^2=\partial^2/\partial x^2+\partial^2/\partial y^2$. Equations (26) and (28) are similar to the laboratory-frame Vlasov–Poisson equations widely used in the literature[29] to describe thin beam propagation in the paraxial approximation. Approximate Vlasov–Maxwell equations similar to Eqs. (26) and (28) can also be derived using the Darwin-approximation model[31–35] developed by Lee *et al.*[30] for intense beam propagation.

IV. CONCLUSIONS

In summary, in this paper we have developed a clear procedure for solving the nonlinear Vlasov–Maxwell equations for a one-component intense charged particle beam or finite-length charge bunch propagating through a cylindrical conducting pipe (radius $r=r_w=$const), and confined by an applied focusing force $\mathbf{F}_{\mathrm{foc}}$. In particular, the nonlinear Vlasov–Maxwell equations were Lorentz transformed to be beam frame ("primed" variables) moving with axial velocity $V_b=\beta_b c=$const relative to the laboratory. In the beam frame, the particle motions are nonrelativistic for the applications of practical interest, already a major simplification. Then, in the beam frame, we made the electrostatic approximation ($\mathbf{E}_s'=-\nabla'\phi'$, $\mathbf{E}_T'\simeq 0\simeq \mathbf{B}_s'$) which fully incorporates beam space-charge effects, but neglects any fast electromagnetic processes with transverse polarization (e.g., light waves). The resulting Vlasov–Maxwell equations were then Lorentz transformed back to the laboratory frame, and properties of the self-generated fields and resulting nonlinear Vlasov–Maxwell equations in the laboratory frame were discussed.

ACKNOWLEDGMENT

This research was supported by the U.S. Department of Energy.

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