# $K^{\pm}_{\ell 3}$ AND $K^{0}_{\ell 3}$ FORM FACTORS

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Assuming that only the vector current contributes to $K \rightarrow \pi\ell\nu$ decays, we write the matrix element as

$$M \propto f_+(t) \left[(P_K + P_\pi)_\mu \overline{\ell}\gamma_\mu(1+\gamma_5)\nu\right] \\ + f_-(t) \left[m_\ell \overline{\ell}(1+\gamma_5)\nu\right] \ , \tag{1}$$

where $P_K$ and $P_\pi$ are the four-momenta of the $K$ and $\pi$ mesons, $m_\ell$ is the lepton mass, and $f_+$ and $f_-$ are dimensionless form factors which can depend only on $t = (P_K - P_\pi)^2$, the square of the four-momentum transfer to the leptons. If time-reversal invariance holds, $f_+$ and $f_-$ are relatively real. $K_{\mu 3}$ experiments, discussed immediately below, measure $f_+$ and $f_-$, while $K_{e3}$ experiments, discussed further below, are sensitive only to $f_+$ because the small electron mass makes the $f_-$ term negligible.

***$K_{\mu 3}$ Experiments***. Analyses of $K_{\mu 3}$ data frequently assume a linear dependence of $f_+$ and $f_-$ on $t$, *i.e.*,

$$f_\pm(t) = f_\pm(0)\left[1 + \lambda_\pm(t/m^2_{\pi^+})\right] \ . \tag{2}$$

Most $K_{\mu 3}$ data are adequately described by Eq. (2) for $f_+$ and a constant $f_-$ (*i.e.*, $\lambda_- = 0$).

There are two equivalent parametrizations commonly used in these analyses:

***(1) $\lambda_+, \xi(0)$ parametrization***. Older analyses of $K_{\mu 3}$ data often introduce the ratio of the two form factors

$$\xi(t) = f_-(t)/f_+(t) \ . \tag{3}$$

The $K_{\mu 3}$ decay distribution is then described by the two parameters $\lambda_+$ and $\xi(0)$ (assuming time reversal invariance and $\lambda_- = 0$).

***(2) $\lambda_+, \lambda_0$ parametrization***. More recent $K_{\mu 3}$ analyses have parametrized in terms of the form factors $f_+$ and $f_0$, which are associated with vector and scalar exchange, respectively, to the lepton pair. $f_0$ is related to $f_+$ and $f_-$ by

$$f_0(t) = f_+(t) + \left[t/(m^2_K - m^2_\pi)\right] f_-(t) \ . \tag{4}$$

Here $f_0(0)$ must equal $f_+(0)$ unless $f_-(t)$ diverges at $t = 0$. The earlier assumption that $f_+$ is linear in $t$ and $f_-$ is constant leads to $f_0$ linear in $t$:

$$f_0(t) = f_0(0)\left[1 + \lambda_0(t/m^2_{\pi^+})\right] \ . \tag{5}$$

With the assumption that $f_0(0) = f_+(0)$, the two parametrizations, $(\lambda_+, \xi(0))$ and $(\lambda_+, \lambda_0)$ are equivalent as long as correlation information is retained. $(\lambda_+, \lambda_0)$ correlations tend to be less strong than $(\lambda_+, \xi(0))$ correlations.

Since the 2006 edition of the *Review* [4], we no longer quote results in the $(\lambda_+, \xi(0))$ parametrization. We have removed many older low statistics results from the Listings. See the 2004 version of this note [5] for these older results, and the 1982 version [6] for additional discussion of the $K^0_{\mu 3}$ parameters, correlations, and conversion between parametrizations.

***Quadratic Parametrization***. More recent high-statistics experiments have included a quadratic term in the expansion of $f_+(t)$,

$$f_+(t) = f_+(0)\left[1 + \lambda^{'}_+(t/m^2_{\pi^+}) + \frac{\lambda^{''}_+}{2}(t/m^2_{\pi^+})^2\right] \ . \tag{6}$$

If there is a non-vanishing quadratic term, then $\lambda_+$ of Eq. (2) represents the average slope, which is then different from $\lambda^{'}_+$. Our convention is to include the factor $\frac{1}{2}$ in the quadratic term, and to use $m_{\pi^+}$ even for $K^+_{e3}$ and $K^+_{\mu 3}$ decays. We have converted other's parametrizations to match our conventions, as noted in the beginning of the "$K^{\pm}_{\ell 3}$ and $K^0_{\ell 3}$ Form Factors" sections of the Listings.

***Pole Parametrization***: The pole model describes the $t$-dependence of $f_+(t)$ and $f_0(t)$ in terms of the exchange of the lightest vector and scalar $K^*$ mesons with masses $M_v$ and $M_s$, respectively:

$$f_+(t) = f_+(0)\left[\frac{M^2_v}{M^2_v - t}\right] \ , \qquad f_0(t) = f_0(0)\left[\frac{M^2_s}{M^2_s - t}\right] \ . \tag{7}$$

***Dispersive Parametrization*** [7,8]. This approach uses dispersive techniques and the known low-energy K-$\pi$ phases to parametrize the vector and scalar form factors:

$$f_+(t) = f_+(0)\mathrm{exp}\left[\frac{\mathrm{t}}{\mathrm{m}^2_\pi}(\Lambda_+ + \mathrm{H(t)})\right] ; \tag{8}$$

$$f_0(t) = f_+(0)\mathrm{exp}\left[\frac{\mathrm{t}}{(\mathrm{m}_{\mathrm{K}}^2 - \mathrm{m}_\pi^2)}\left(\ln[\mathrm{C}] - \mathrm{G(t)}\right)\right], \qquad (9)$$

where $\Lambda_+$ is the slope of the vector form factor, and $\ln[\mathrm{C}]=\ln[\mathrm{f}_0(\mathrm{m}_{\mathrm{K}}^2 - \mathrm{m}_\pi^2)]$ is the logarithm of the scalar form factor at the Callan-Treiman point. The functions H(t) and G(t) are dispersive integrals.

***$K_{e3}$* Experiments**: Analysis of $K_{e3}$ data is simpler than that of $K_{\mu3}$ because the second term of the matrix element assuming a pure vector current [Eq. (1) above] can be neglected. Here $f_+$ can be assumed to be linear in $t$, in which case the linear coefficient $\lambda_+$ of Eq. (2) is determined, or quadratic, in which case the linear coefficient $\lambda_+^{'}$ and quadratic coefficient $\lambda_+^{''}$ of Eq. (6) are determined.

If we remove the assumption of a pure vector current, then the matrix element for the decay, in addition to the terms in Eq. (1), would contain

$$\begin{aligned} &+2m_K\, f_S\, \overline{\ell}(1+\gamma_5)\nu \\ &\qquad +(2f_T/m_K)(P_K)_\lambda(P_\pi)_\mu\, \overline{\ell}\, \sigma_{\lambda\mu}(1+\gamma_5)\nu\ , \end{aligned} \qquad (10)$$

where $f_S$ is the scalar form factor, and $f_T$ is the tensor form factor. In the case of the $K_{e3}$ decays where the $f_-$ term can be neglected, experiments have yielded limits on $|f_S/f_+|$ and $|f_T/f_+|$.

***Fits for $K_{\ell 3}$ FormFactors***. For $K_{e3}$ data, we determine best values for the three parametrizations: linear ($\lambda_+$), quadratic ($\lambda_+^{'}$, $\lambda_+^{''}$) and pole ($M_v$). For $K_{\mu3}$ data, we determine best values for the three parametrizations: linear ($\lambda_+$, $\lambda_0$), quadratic ($\lambda_+^{'}, \lambda_+^{''}$, $\lambda_0$) and pole ($M_v$, $M_s$). We then assume $\mu - e$ universality so that we can combine $K_{e3}$ and $K_{\mu3}$ data, and again determine best values for the three parametrizations: linear ($\lambda_+$, $\lambda_0$), quadratic ($\lambda_+^{'}, \lambda_+^{''}$, $\lambda_0$), and pole ($M_v$, $M_s$). When there is more than one parameter, fits are done including input correlations. Simple averages suffice in the two $K_{e3}$ cases where there is only one parameter: linear ($\lambda_+$) and pole ($M_v$).

Both KTeV and KLOE see an improvement in the quality of their fits relative to linear fits when a quadratic term is introduced, as well as when the pole parametrization is used.

The quadratic parametrization has the disadvantage that the quadratic parameter $\lambda_+^{''}$ is highly correlated with the linear parameter $\lambda_+^{'}$, in the neighborhood of 95%, and that neither parameter is very well determined. The pole fit has the same number of parameters as the linear fit, but yields slightly better fit probabilities, so that it would be advisable for all experiments to include the pole parametrization as one of their choices [9].

The "Kaon Particle Listings" show the results with and without assuming $\mu$-$e$ universality. The "Meson Summary Tables" show all of the results assuming $\mu$-$e$ universality, but most results not assuming $\mu$-$e$ universality are given only in the Listings.

**References**


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9. We thank P. Franzini (Rome U. and Frascati) for useful discussions on this point.