The analytic solutions of the equations of motion of GRB afterglow and the analytic expressions of the beaming angle – arrival time relation

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RTN Summer School "Gamma-ray bursts: The first three hours"
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Afterglow equations of motion

$$\begin{cases} dE_{\rm int} = (\gamma - 1)\, dM_{\rm ism} c^2 \\ d\gamma = -\frac{\gamma^2-1}{M} dM_{\rm ism} \\ dM = \frac{1-\varepsilon}{c^2} dE_{\rm int} + dM_{\rm ism} \\ dM_{\rm ism} = 4\pi m_p n_{\rm ism} r^2 dr \end{cases}$$

Fully radiative condition: ε = 1

$$\gamma = \frac{1 + (M_{\rm ism}/M_B)\left(1+\gamma_\circ^{-1}\right)[1+(1/2)(M_{\rm ism}/M_B)]}{\gamma_\circ^{-1} + (M_{\rm ism}/M_B)\left(1+\gamma_\circ^{-1}\right)[1+(1/2)(M_{\rm ism}/M_B)]}$$

$\gamma_0 >> \gamma >> 1$ approximation

$$\gamma \propto r^{-3}$$

$$t = \frac{r}{c}\left(1 + \frac{1}{14\gamma^2}\right)$$

$$C = {M_B}^2 (\gamma_\circ - 1) / (\gamma_\circ + 1)$$

$$m_i^\circ = (4/3)\,\pi m_p n_{\rm ism} r_\circ^3$$

$$A = \sqrt[3]{\left(M_B - m_i^\circ\right)/m_i^\circ}$$

Analytic integration

$$t = \frac{M_B - m_i^\circ}{2c\sqrt{C}}(r - r_\circ) + \frac{m_i^\circ r_\circ}{8c\sqrt{C}}\left[\left(\frac{r}{r_\circ}\right)^4 - 1\right]$$
$$+ \frac{r_\circ\sqrt{C}}{12 c m_i^\circ A^2} \ln\left\{\frac{[A + (r/r_\circ)]^3\left(A^3+1\right)}{\left[A^3 + (r/r_\circ)^3\right](A+1)^3}\right\} + t_\circ$$
$$+ \frac{r_\circ\sqrt{3C}}{6 c m_i^\circ A^2}\left[\arctan\frac{2(r/r_\circ) - A}{A\sqrt{3}} - \arctan\frac{2-A}{A\sqrt{3}}\right]$$

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Ruffini, Bianco, Chardonnet, Fraschetti, Vitagliano, Xue, "Cosmology and Gravitation", AIP vol. 668, (2003)

Bianco, Ruffini, *ApJ*, 620, L23, (2005)

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Fully adiabatic condition: ε = 0

$$\gamma^2 = \frac{\gamma_\circ^2 + 2\gamma_\circ \left(M_{\rm ism}/M_B\right) + \left(M_{\rm ism}/M_B\right)^2}{1 + 2\gamma_\circ \left(M_{\rm ism}/M_B\right) + \left(M_{\rm ism}/M_B\right)^2}$$

$\gamma_0^2 >> \gamma^2 >> 1$ **approximation**

$$\gamma \propto r^{-3/2}$$

$$t = \frac{r}{c}\left[1 + \frac{1}{8\gamma^2(r)}\right]$$

Analytic integration

$$t(r) = \left(\gamma_\circ - \frac{m_i^\circ}{M_B}\right)\frac{r - r_\circ}{c\sqrt{\gamma_\circ^2 - 1}} + \frac{m_i^\circ}{4 M_B r_\circ^3}\frac{r^4 - r_\circ^4}{c\sqrt{\gamma_\circ^2 - 1}} + t_\circ$$

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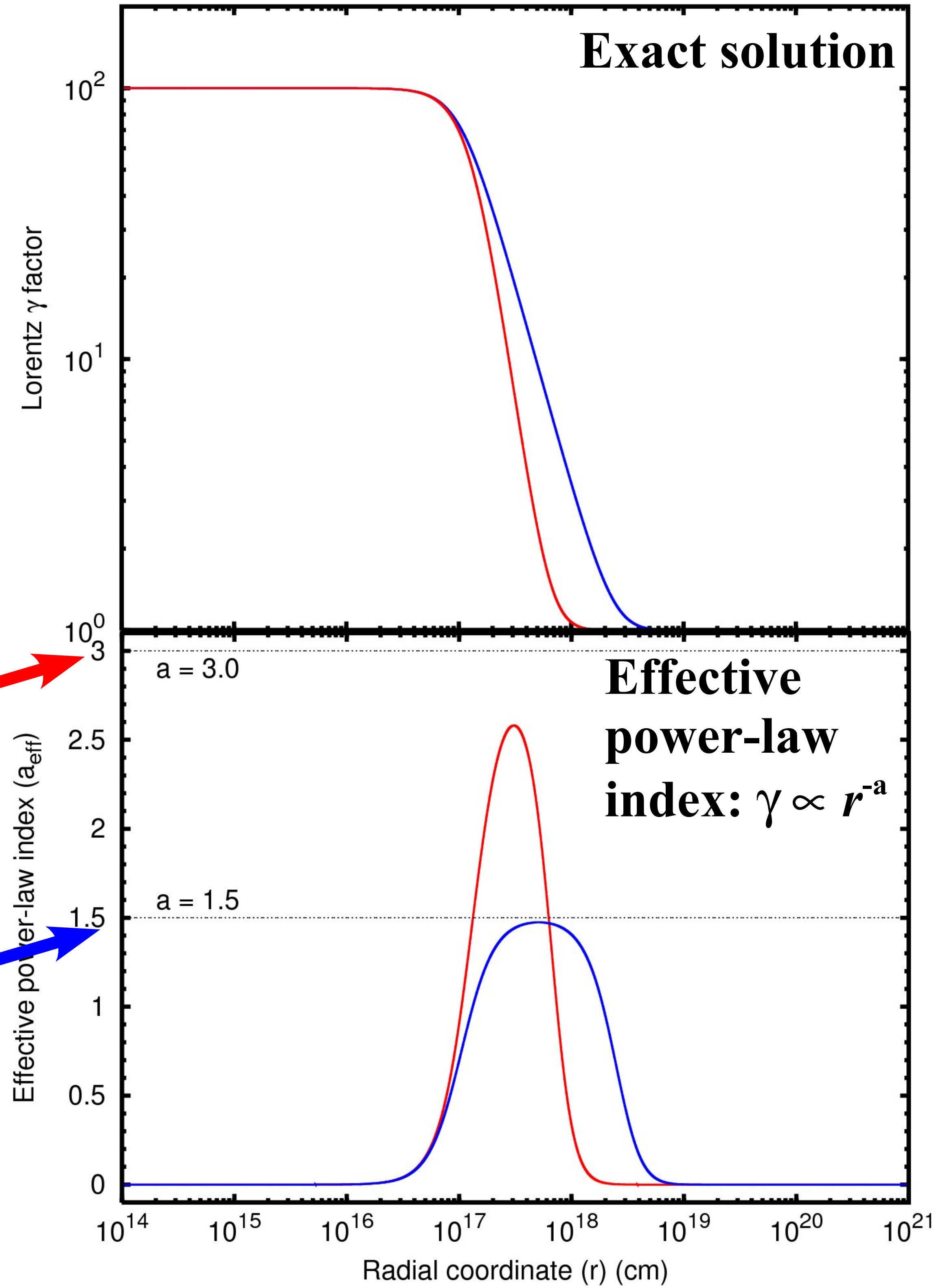
Comparison between exact and approximate afterglow equations of motion (radiative and adiabatic)

$a = 3.0$ ($\gamma_0 >> \gamma >> 1$)

$a = 1.5$ ($\gamma_0^2 >> \gamma^2 >> 1$)

$\gamma_0 = 10^2$



Bianco, Ruffini, in preparation (2005)

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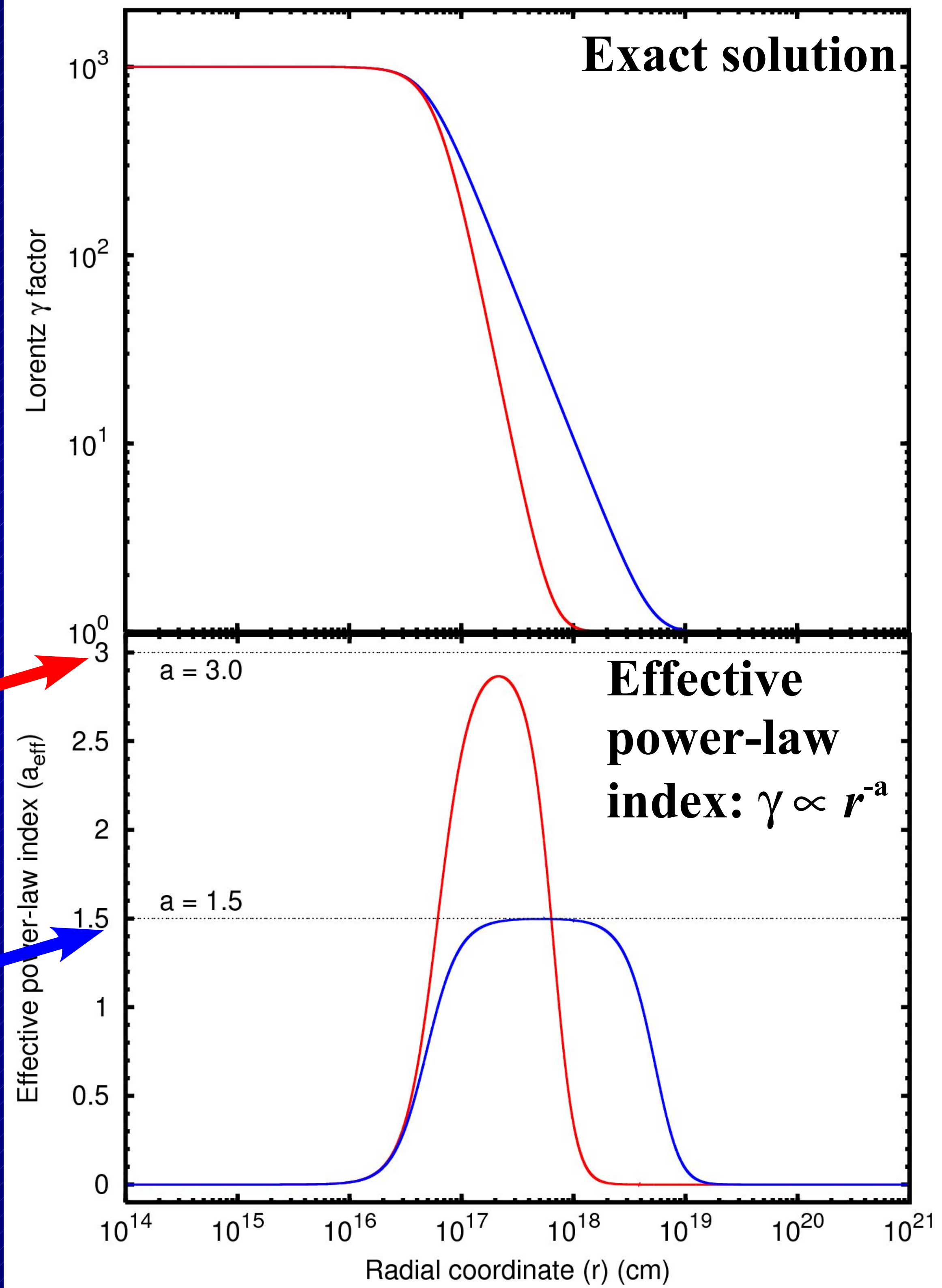

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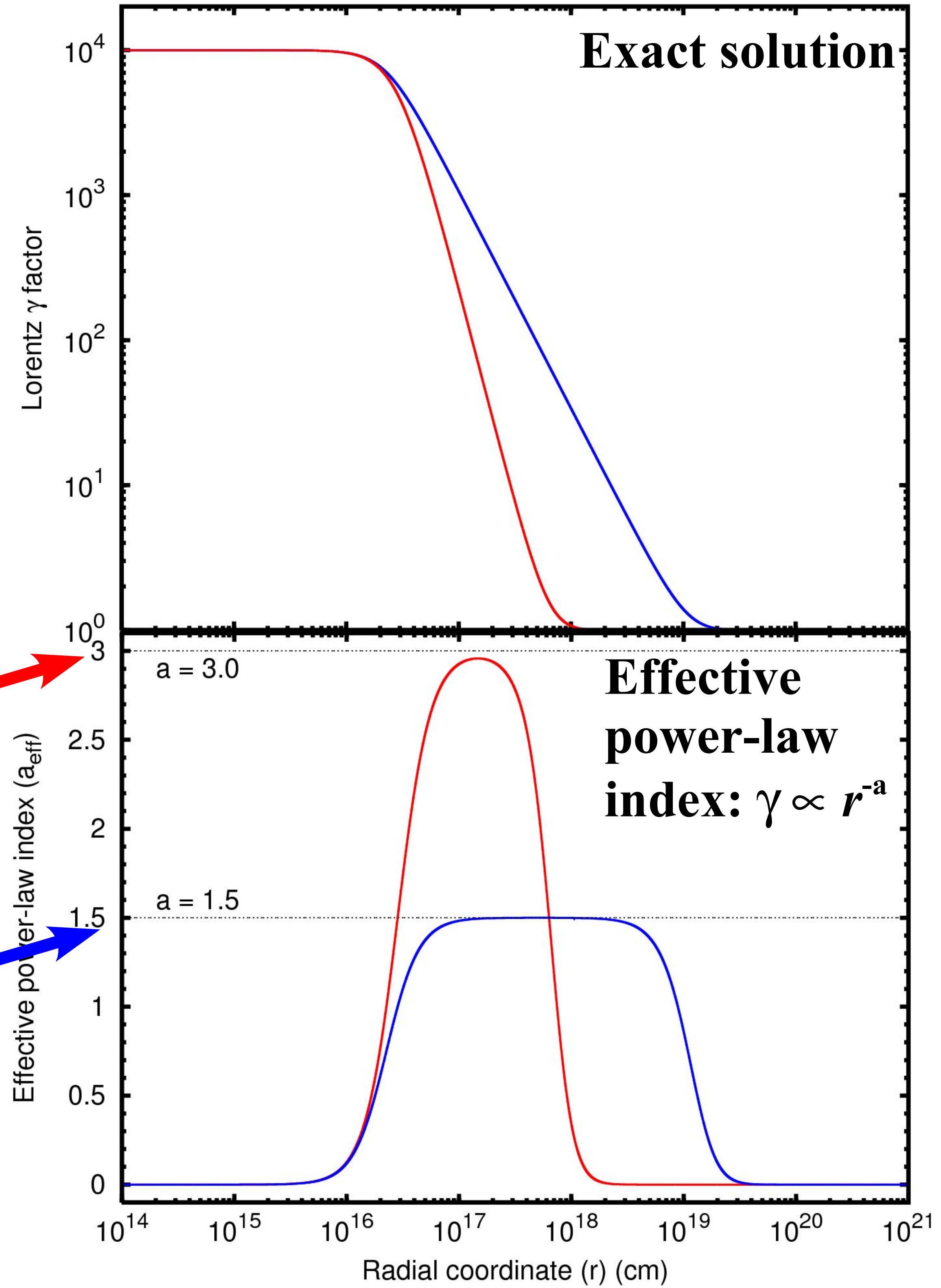
$\gamma_0 = 10^4$



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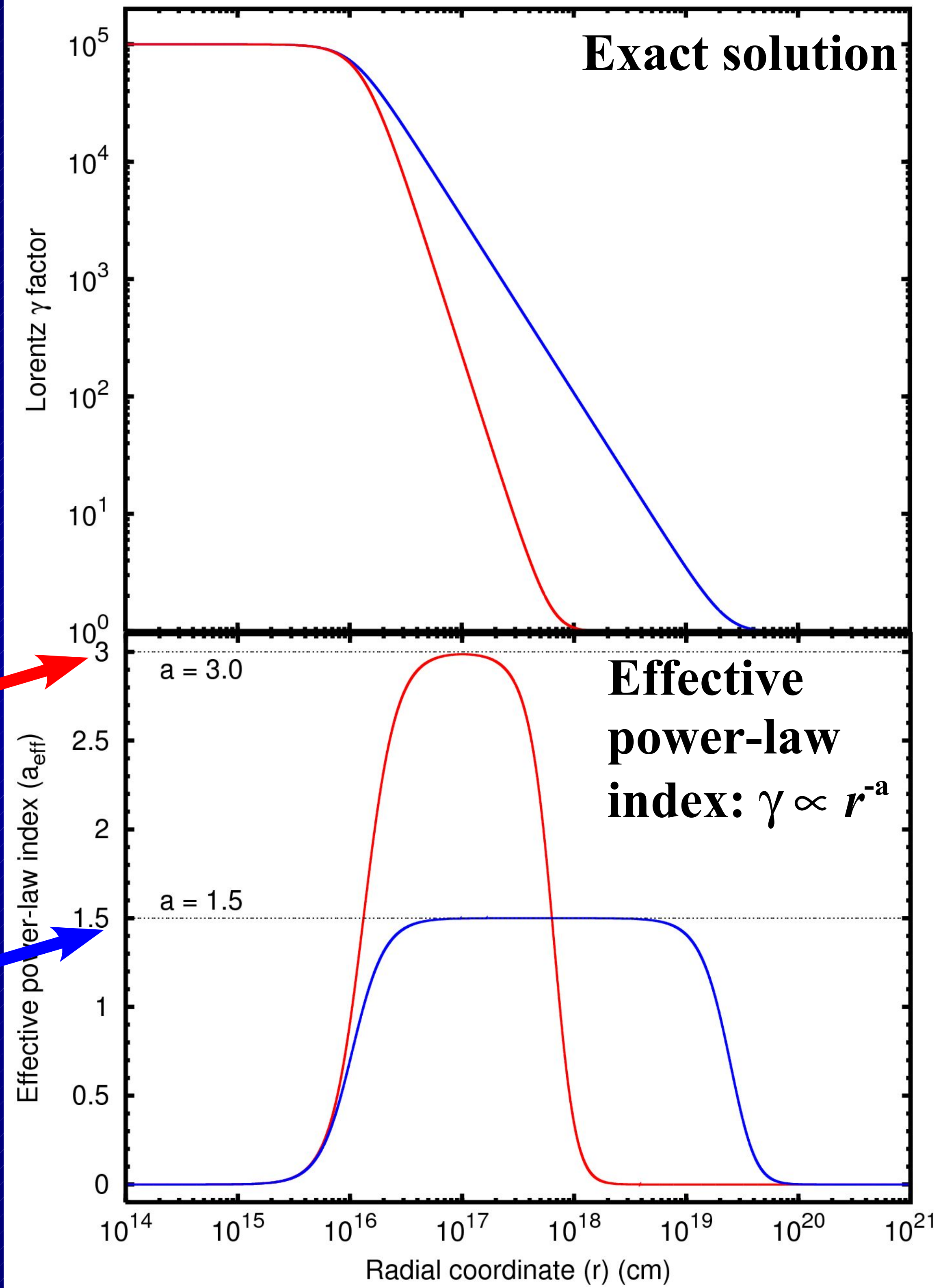
$\gamma_0 = 10^5$



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Comparison between exact and approximate afterglow equations of motion (radiative and adiabatic)

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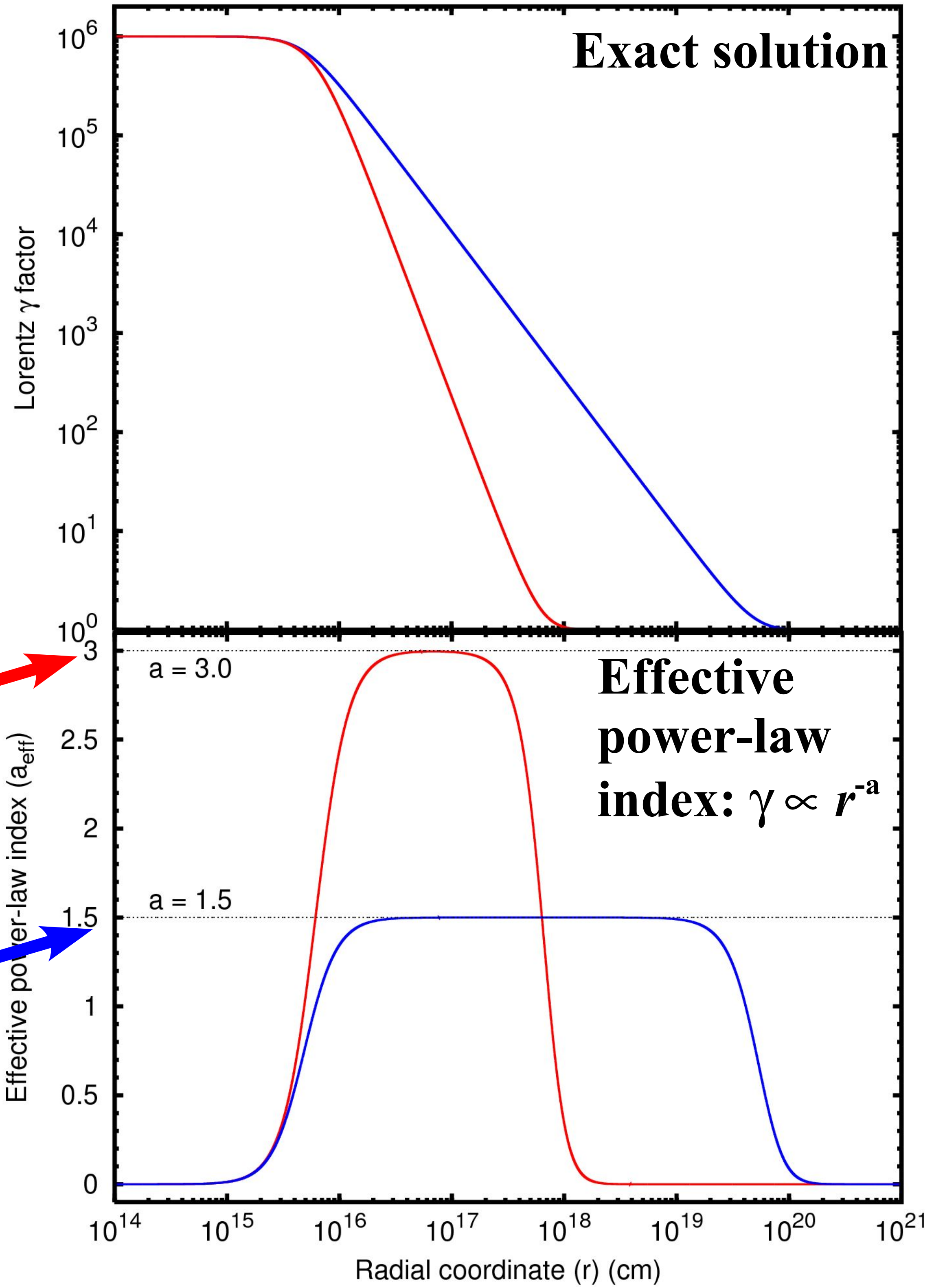
$a = 1.5\ (\gamma_0^2 >> \gamma^2 >> 1)$

$\gamma_0 = 10^6$



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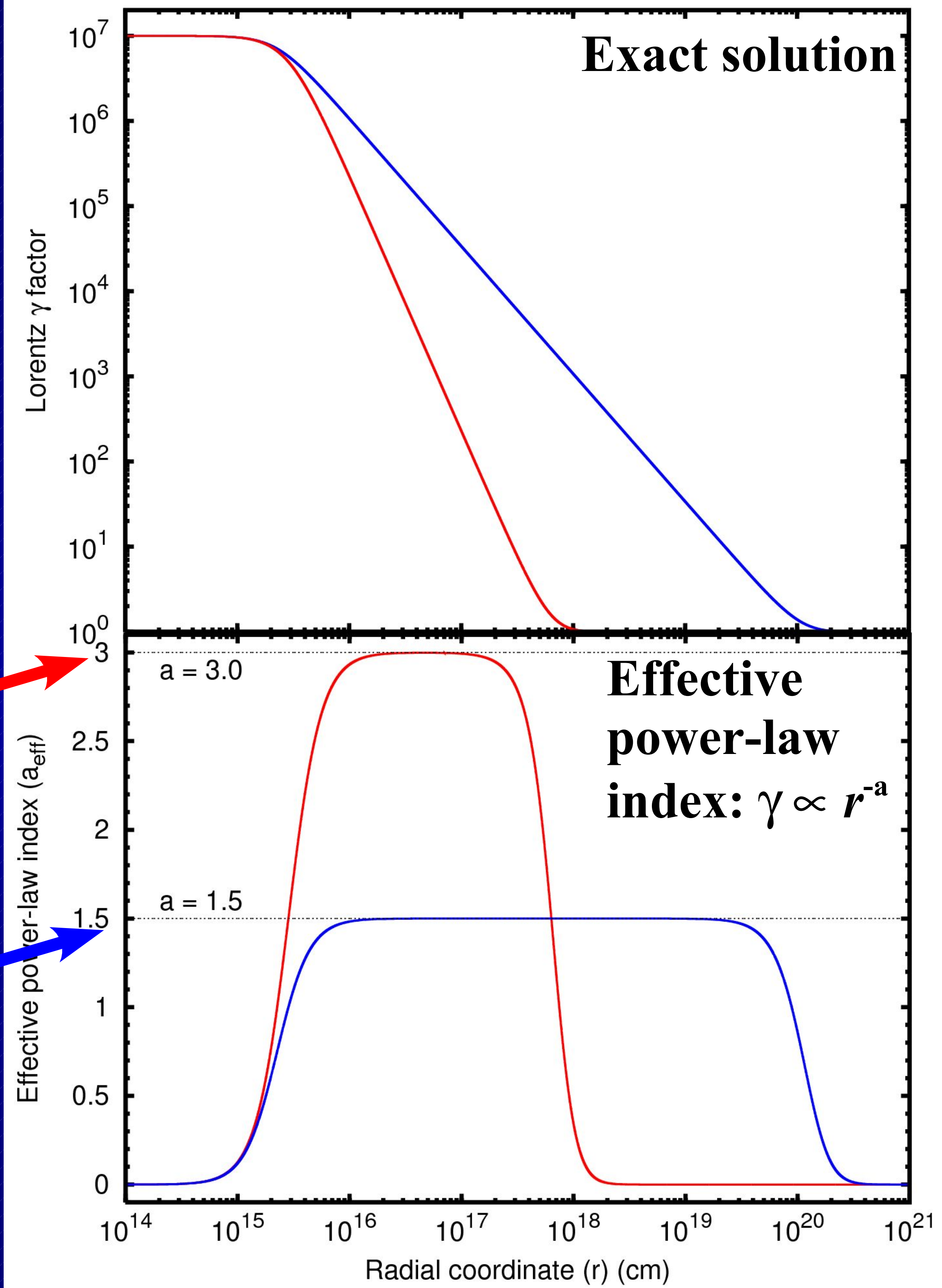

$a = 3.0$ ($\gamma_0 >> \gamma >> 1$)

$a = 1.5$ ($\gamma_0^2 >> \gamma^2 >> 1$)

$\gamma_0 = 10^7$

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Comparison between exact and approximate afterglow equations of motion (radiative and adiabatic)

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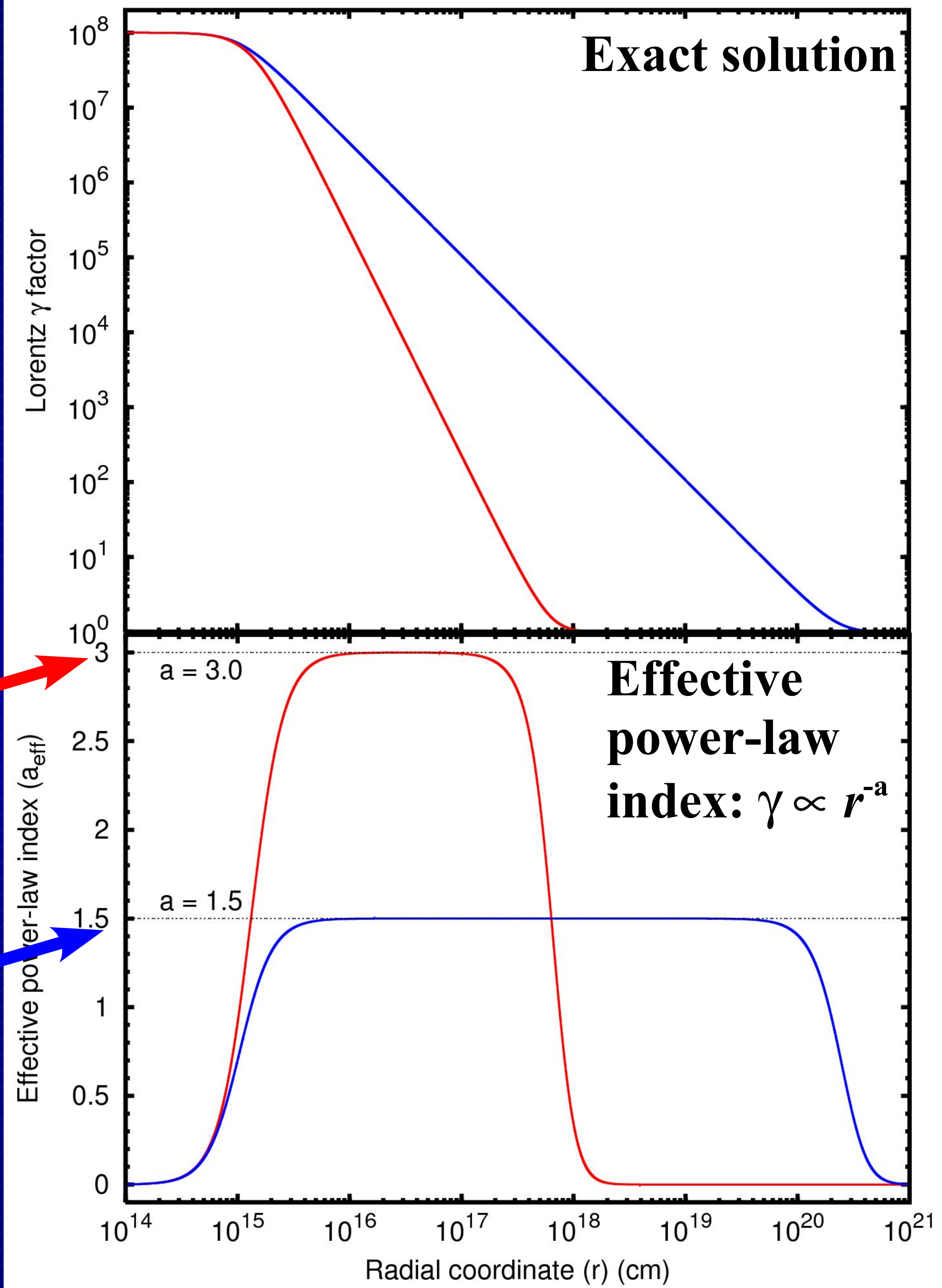
$a = 1.5\ (\gamma_0^2 >> \gamma^2 >> 1)$

$\gamma_0 = 10^8$



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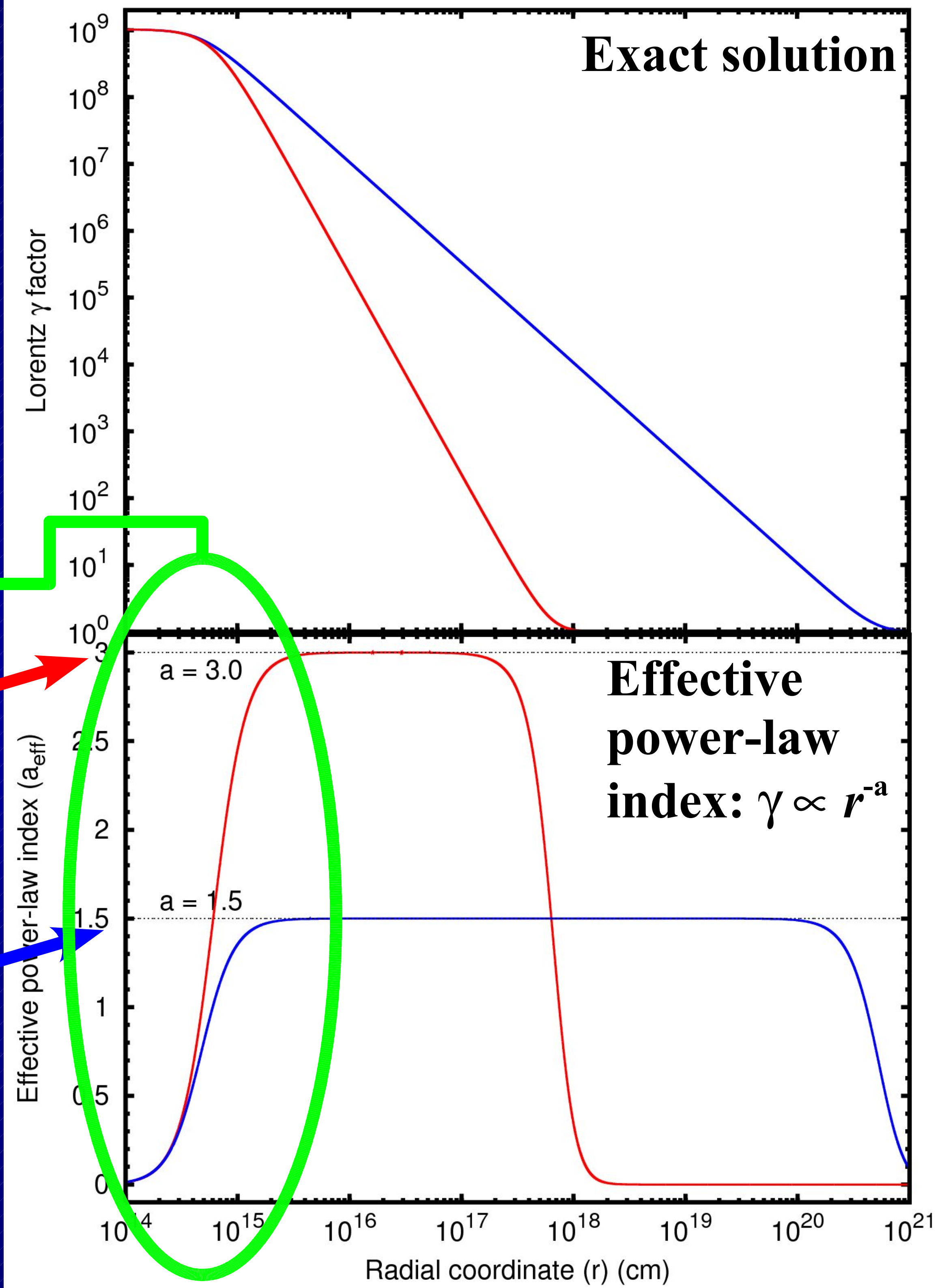
Comparison between exact and approximate afterglow equations of motion (radiative and adiabatic)

The power-law expansion ***never*** applies in GRBs' first three hours

$a = 3.0$ ($\gamma_0 >> \gamma >> 1$)

$a = 1.5$ ($\gamma_0^2 >> \gamma^2 >> 1$)

$\gamma_0 = 10^9$



Bianco, Ruffini, in preparation (2005)

Comparison between exact and approximate afterglow equations of motion (radiative and adiabatic)

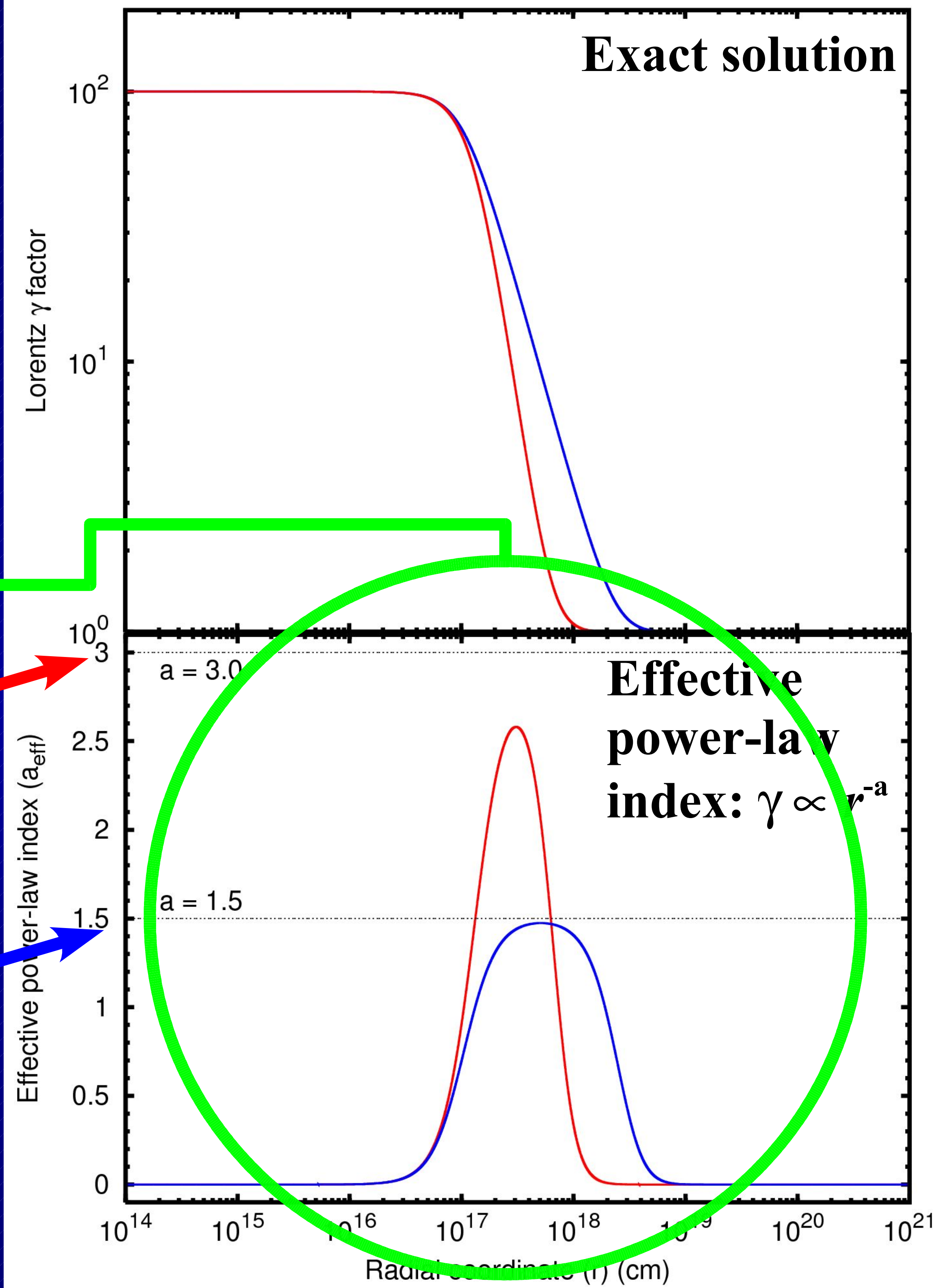
The power-law expansion ***never*** applies to actual GRBs at all

$a = 3.0$ ($\gamma_0 >> \gamma >> 1$)

The relevant case for GRBs!

$a = 1.5$ ($\gamma_0^2 >> \gamma^2 >> 1$)

$\gamma_0 = 10^2$



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EQTS analytic expression: radiative condition

$$t_a^d = (1+z)\left[t\left(r\right) - \frac{r}{c}\cos\vartheta + \frac{r^\star}{c}\right]$$

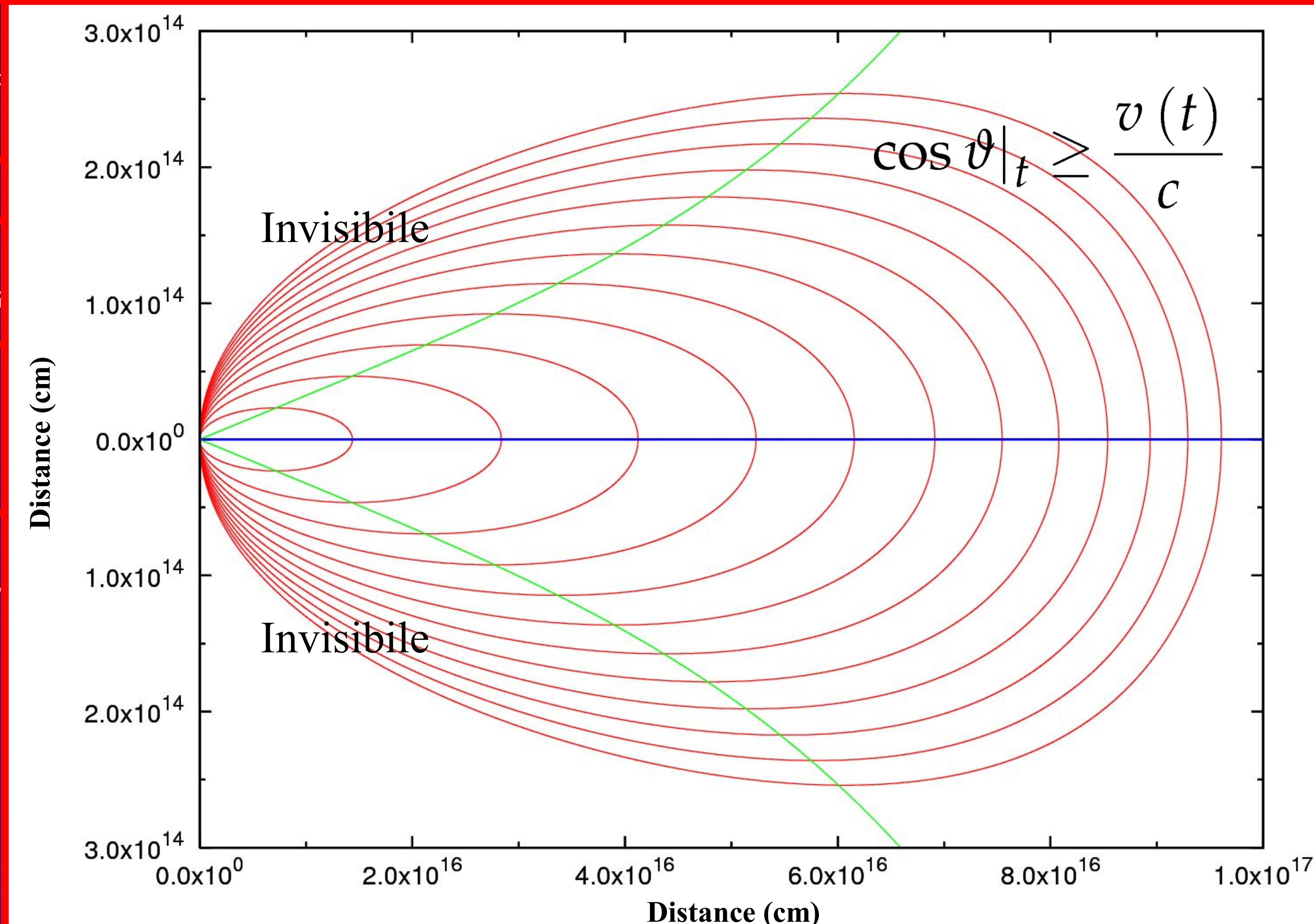
Using the approximate solution for $t = t(r)$ (Panaitescu & Mészáros, 1998)

$$\vartheta = 2\arcsin\left[\frac{1}{2\gamma_\circ}\sqrt{\frac{2\gamma_\circ^2 c t_a}{r} - \frac{1}{7}\left(\frac{r}{r_\circ}\right)^6}\right]$$

Using analytic solution for $t = t(r)$

$$\cos\vartheta = \frac{M_B - m_i^\circ}{2r\sqrt{C}}\left(r - r_\circ\right) + \frac{m_i^\circ r_\circ}{8r\sqrt{C}}\left[\left(\frac{r}{r_\circ}\right)^4 - 1\right]$$
$$+ \frac{r_\circ\sqrt{C}}{12 r m_i^\circ A^2}\ln\left\{\frac{\left[A + \left(r/r_\circ\right)\right]^3\left(A^3+1\right)}{\left[A^3 + \left(r/r_\circ\right)^3\right]\left(A+1\right)^3}\right\} + \frac{c t_\circ}{r} - \frac{c t_a^d}{r\left(1+z\right)}$$
$$+ \frac{r^\star}{r} + \frac{r_\circ\sqrt{3C}}{6 r m_i^\circ A^2}\left[\arctan\frac{2\left(r/r_\circ\right) - A}{A\sqrt{3}} - \arctan\frac{2-A}{A\sqrt{3}}\right]$$

EQTS analytic expression: radiative condition



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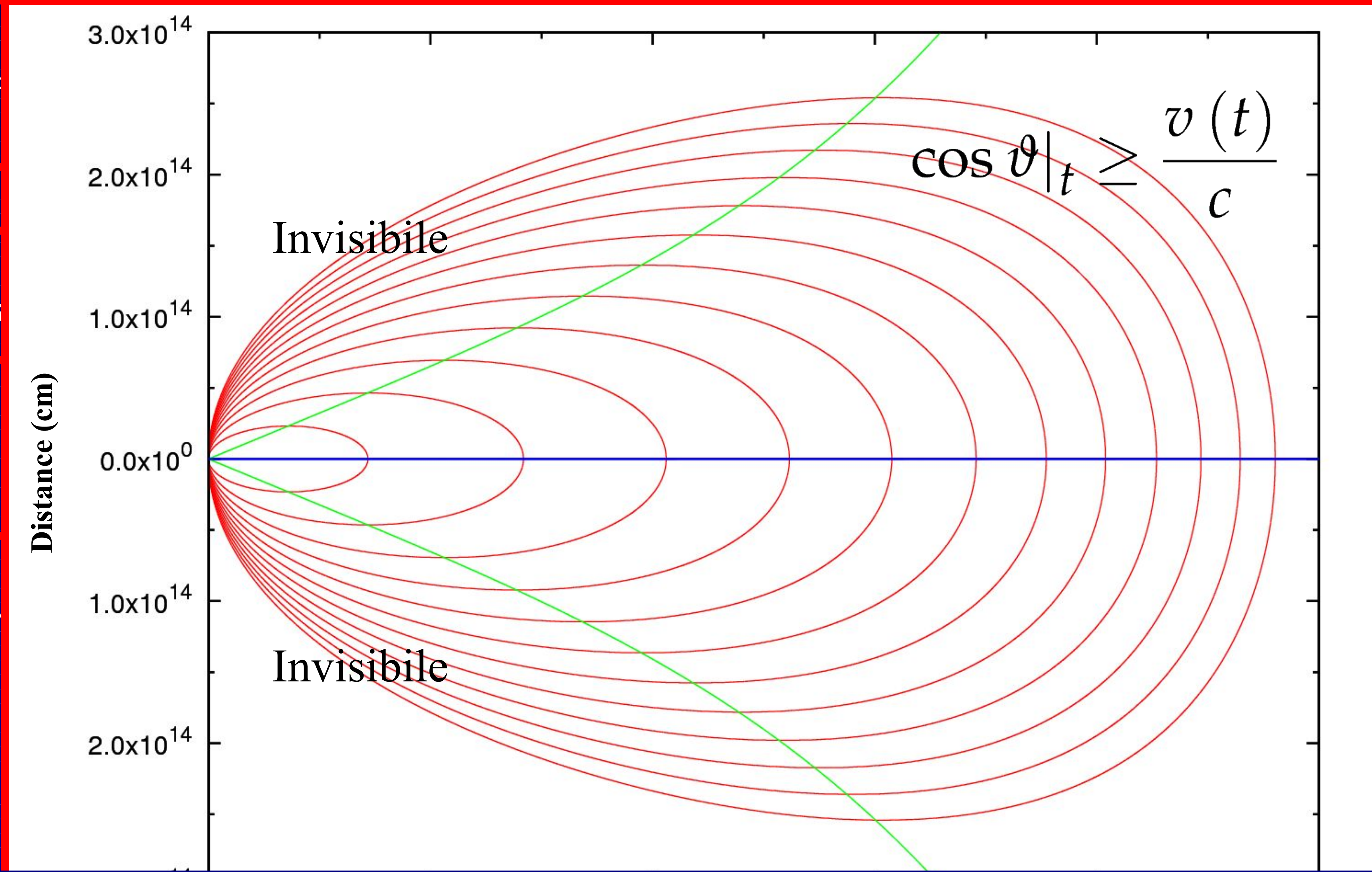

Ruffini, Bianco, Chardonnet, Fraschetti, Xue, *ApJ*, 581, L19, (2002)
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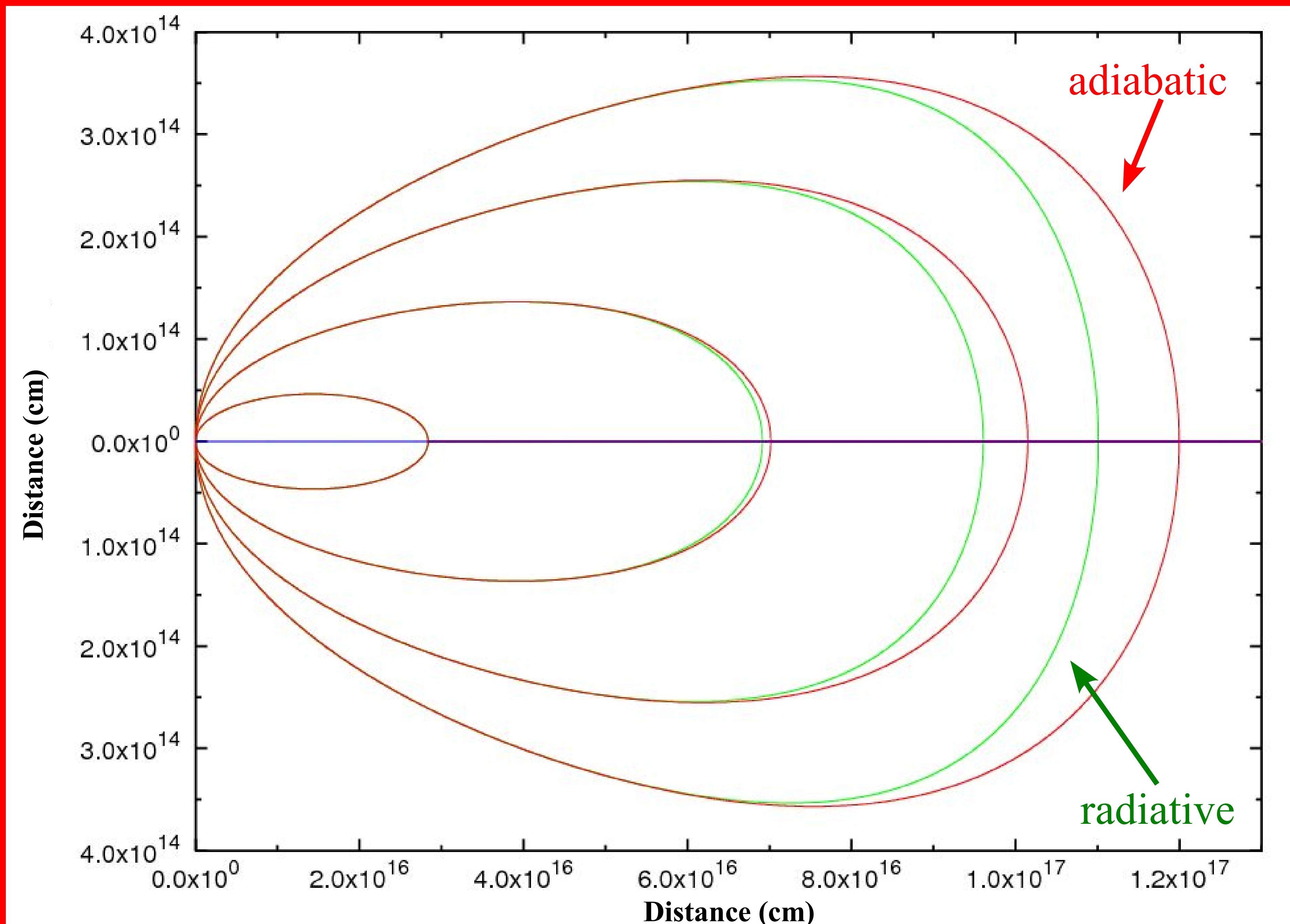
$$\vartheta = 2\arcsin\left[\frac{1}{2\gamma_\circ}\sqrt{\frac{2\gamma_\circ^2 c t_a}{r} - \frac{1}{4}\left(\frac{r}{r_\circ}\right)^3}\right]$$

Using the analytic solution for $t = t(r)$

$$\cos\vartheta = \frac{m_i^\circ}{4M_B\sqrt{\gamma_\circ^2-1}}\left[\left(\frac{r}{r_\circ}\right)^3 - \frac{r_\circ}{r}\right] + \frac{ct_\circ}{r} - \frac{ct_a}{r} + \frac{r^\star}{r} - \frac{\gamma_\circ - (m_i^\circ/M_B)}{\sqrt{\gamma_\circ^2-1}}\left[\frac{r_\circ}{r} - 1\right]$$

EQTS analytic expression: adiabatic condition



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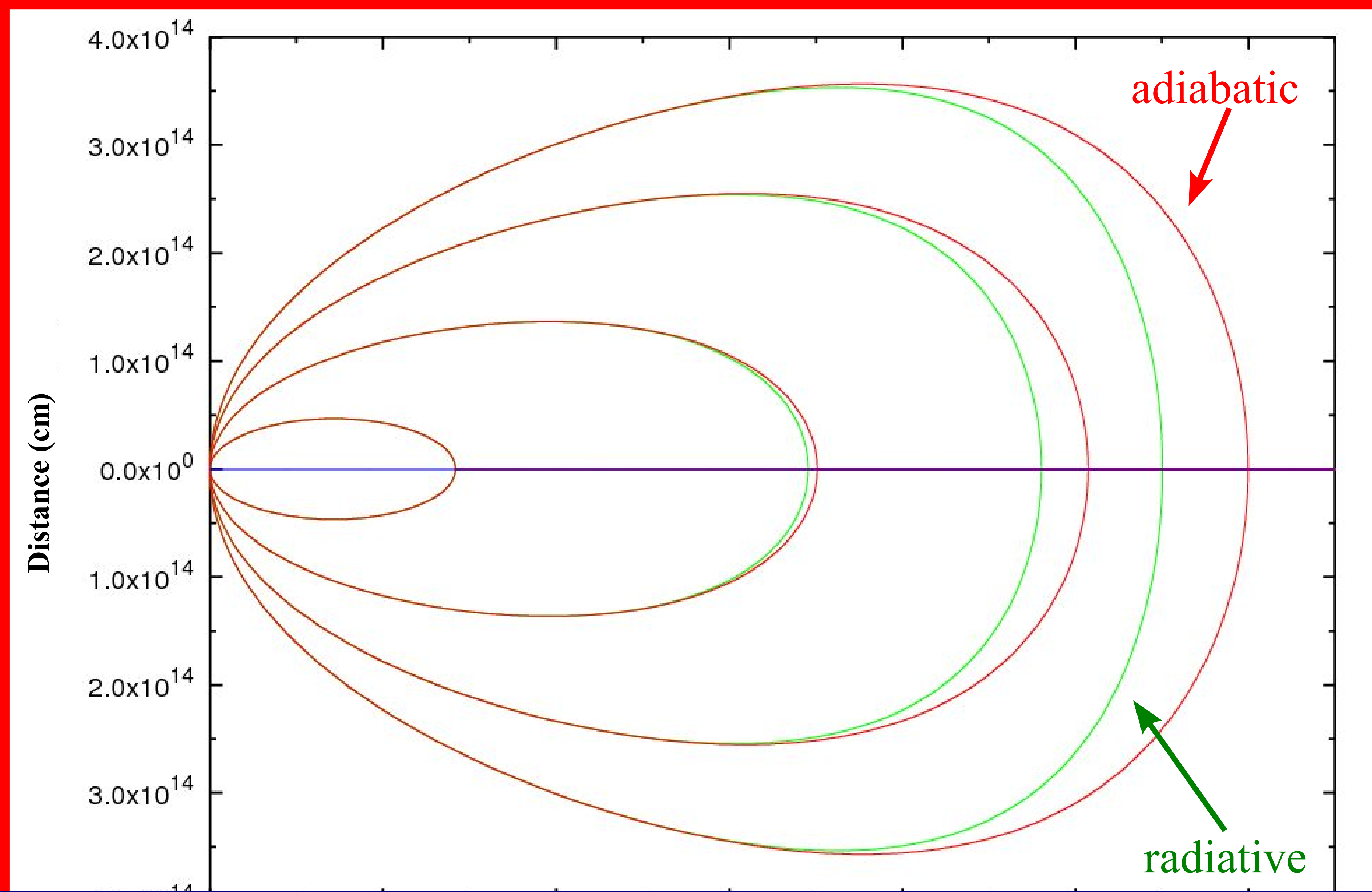

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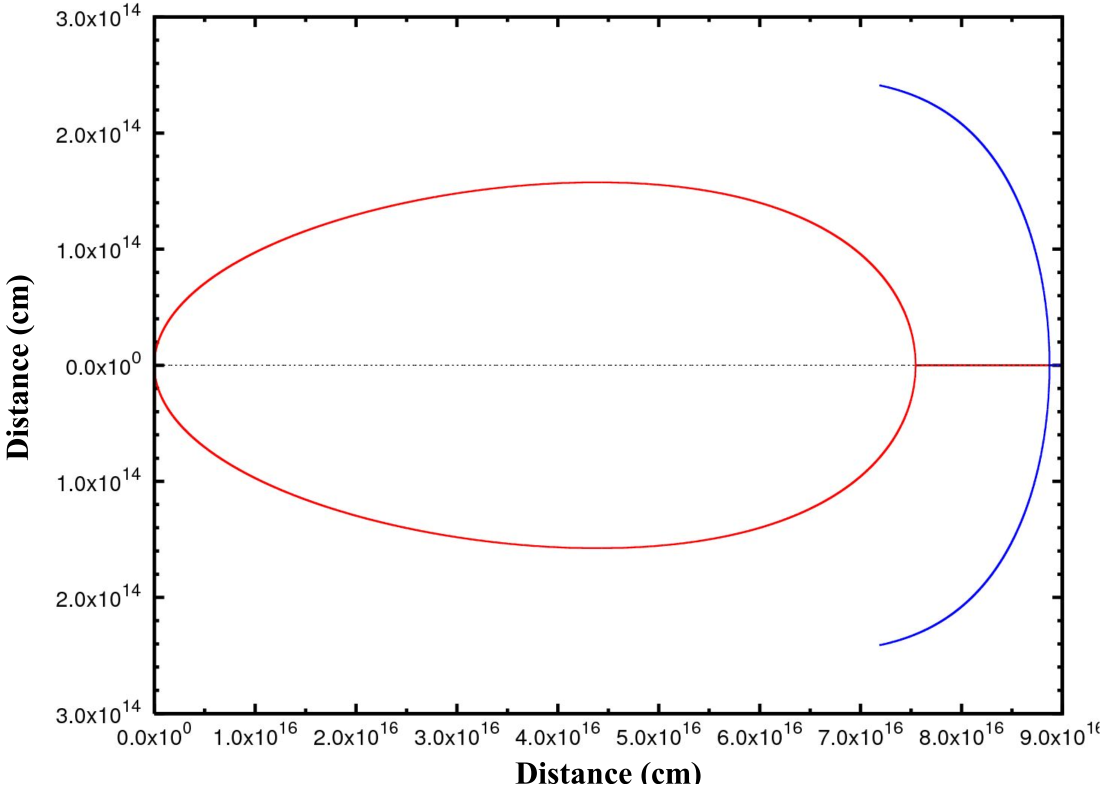
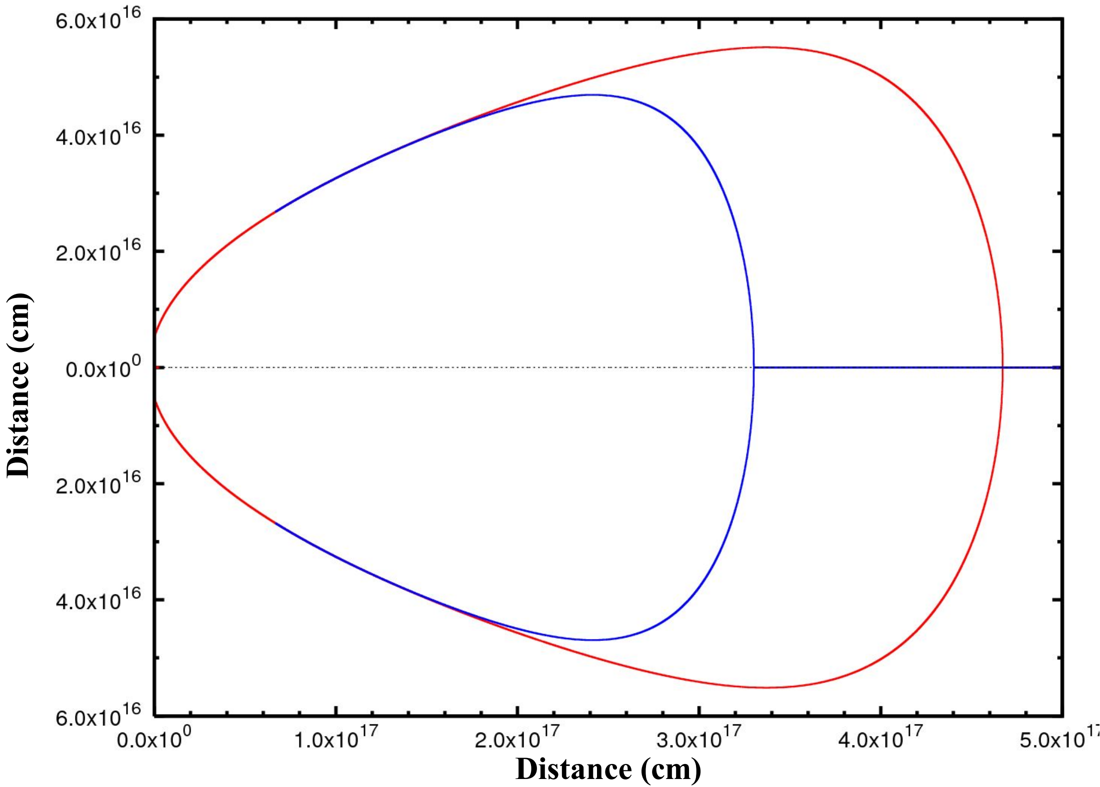
EQTS exact and approximate (radiative cd.)

t_a = 35 seconds:



t_a = 4 days:



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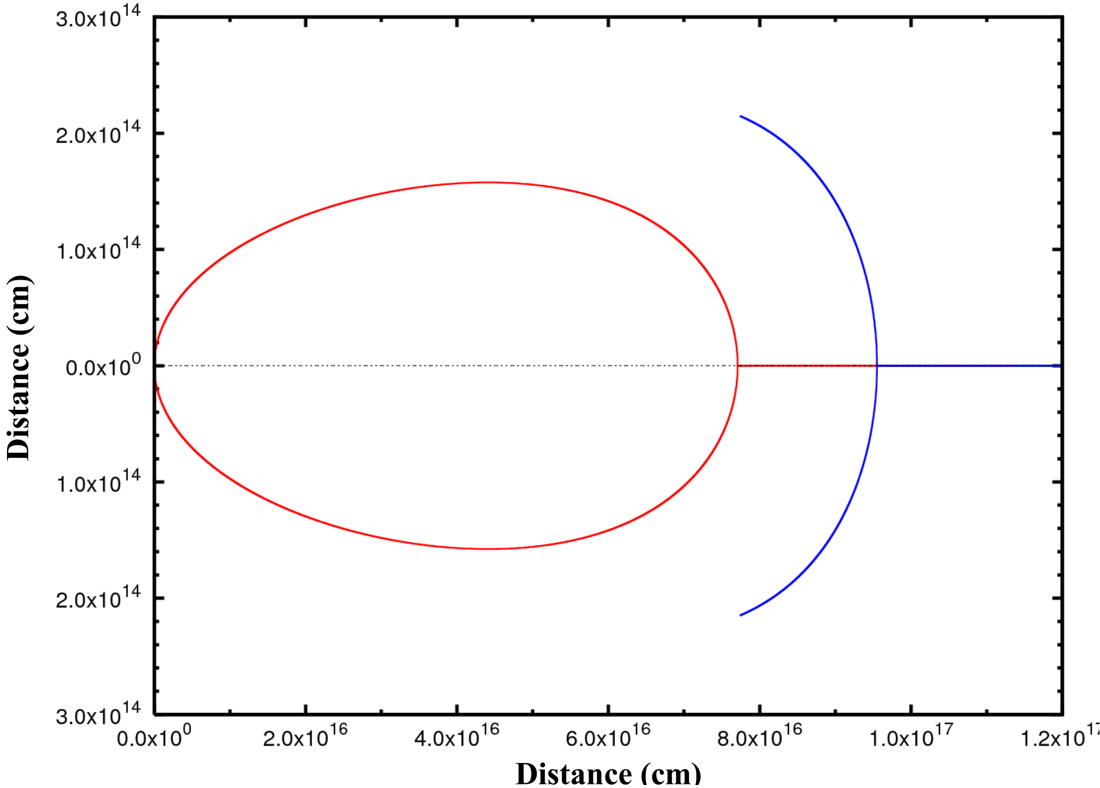

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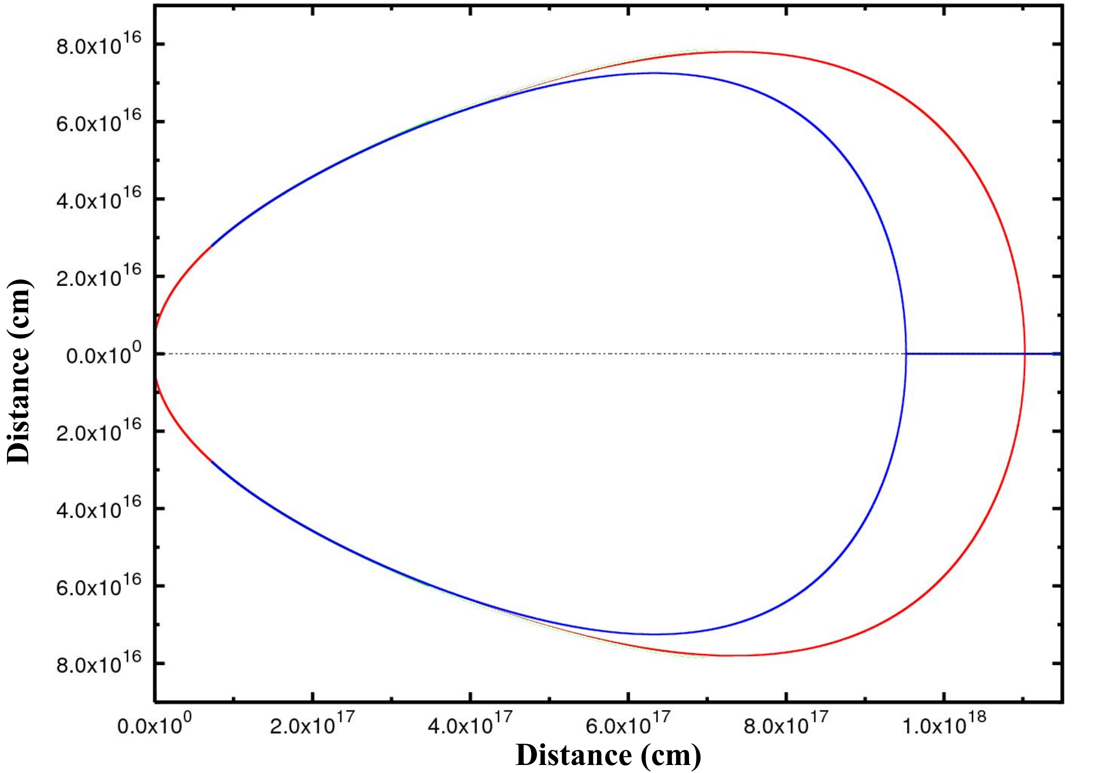

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Observed break time vs. Beaming angle

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Observed break time vs. Beaming angle

~ adiabatic condition in the current literature ~

Panaitescu & Mészáros (1999): $$T_j = 10^2\,(1+z)\left(\frac{\varepsilon_{\circ,54}}{n_0}\right)^{1/3}\Omega_\circ^{4/3}\,days$$

Sari, Piran & Halpern (1999): $$t_{jet} = 6.2\left(\frac{E_{52}}{n_1}\right)^{1/3}\left(\frac{\vartheta}{0.1}\right)^{8/3}\,hr$$

Observed break time vs. Beaming angle

~ comparison with the current literature in the adiabatic condition ~

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Observed break time vs. Beaming angle

~ comparison between adiabatic and fully radiative conditions ~

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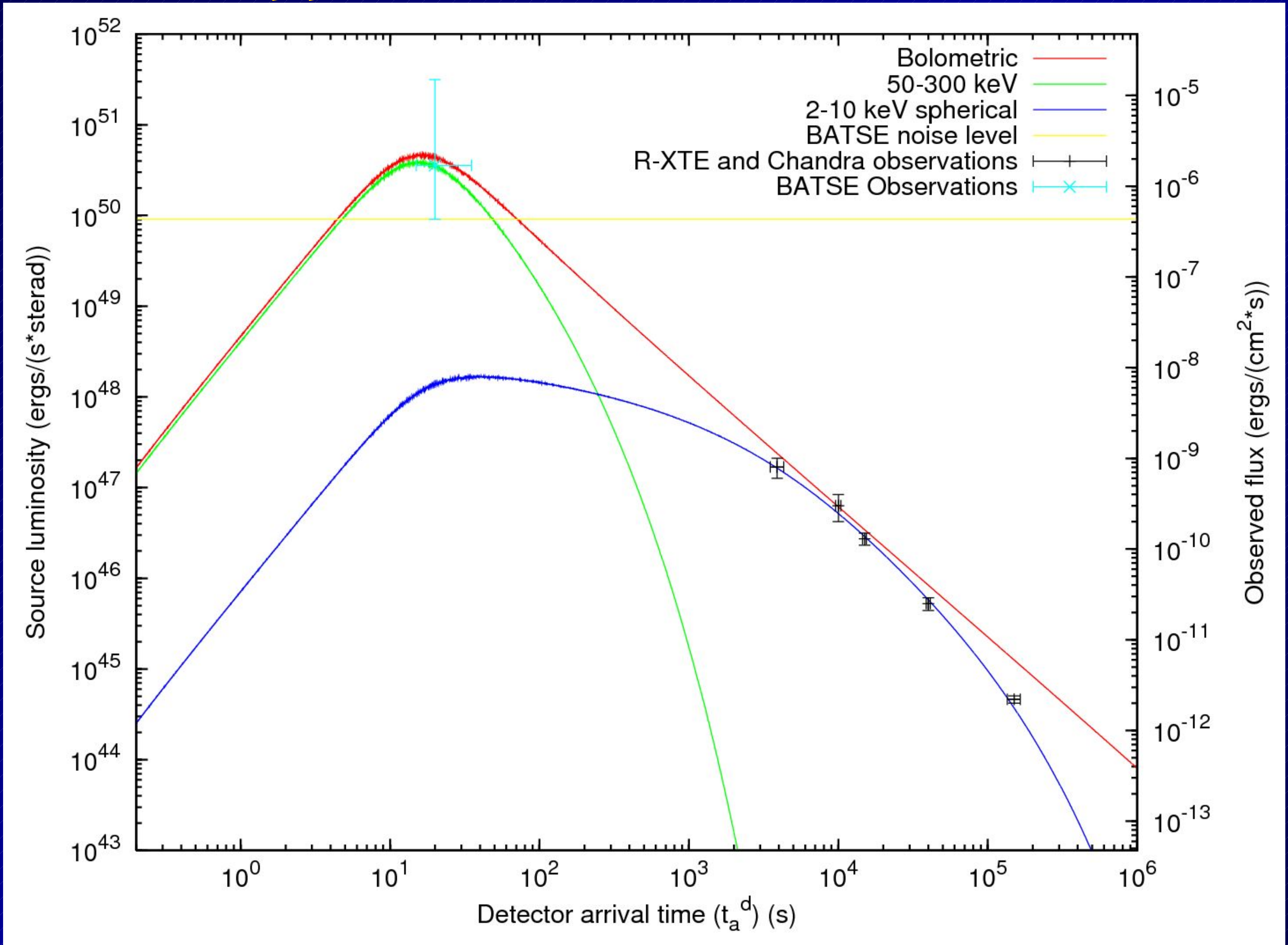
Observed break time vs. Beaming angle

~ application to GRB 991216 ~



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Observed break time vs. Beaming angle

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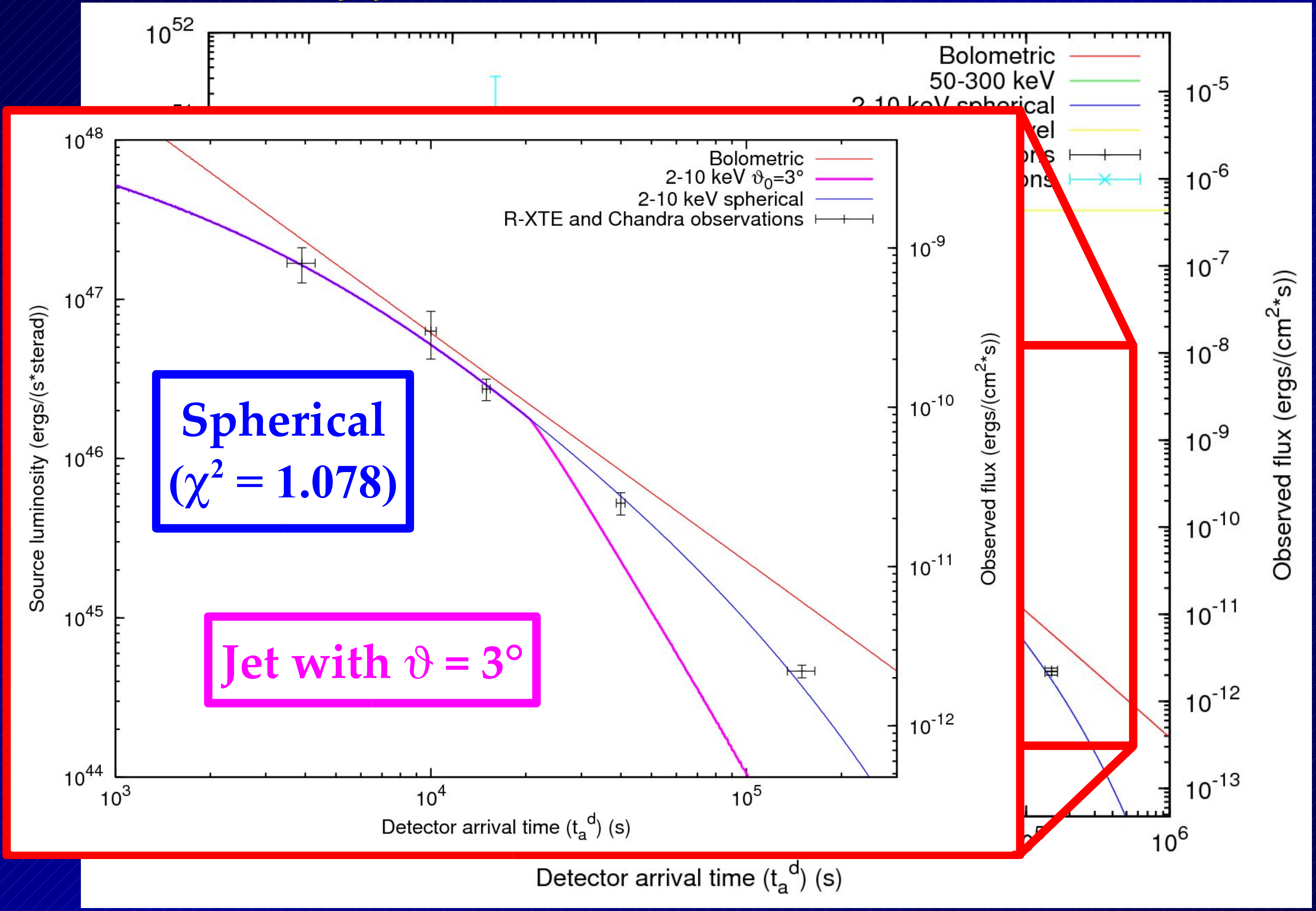

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Conclusions

- In the analysis of GRB afterglows we must use the exact solutions of the equations of motion, instead of the approximate power-law expansions, and the exact profiles of the EQTSes. This is especially fundamental when we try to infer a beaming angle from the observation of a "break" in the light curve.

- To do so, we must know the initial conditions (γ_0, r_0, t_0, etc.) at the beginning of the afterglow – i.e. we must have a *complete* theory of the GRB source.

- Anyway, the shapes of the afterglow light curves computed in fixed energy bands using our model show a curvature which can explain the observed "broken power-law" behavior of the observed data without introducing a beaming angle effect.