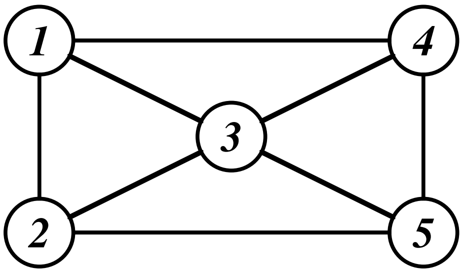
scsd1, scsd6, scsd8; truss: The application

A planar *structure* consists of a set of *joints* connected by *bars*. For example, in the following diagram, the joints are represented by circles, and the bars by lines between two circles:



Consider the problem of finding a minimum-weight structure to meet certain external forces. You are given a set $\mathcal{J}$ of joints, and a set $\mathcal{B} = (i, j)$ of *admissible* bars, where $i \in \mathcal{J}$, $j \in \mathcal{J}$, and $i \neq j$. In the diagram above, we could take $\mathcal{J} = \{1, 2, 3, 4, 5\}$, and

$$\mathcal{B} = \{(1,2), (1,3), (1,4), (2,3), (2,5), (3,4), (3,5), (4,5)\}.$$

The "origin" and "destination" of a bar are arbitrary. The bar between joints 1 and 2, for example, could be represented in $\mathcal{B}$ by either (1,2) or (2,1); but it need not be represented by both.

We can use two-dimensional Euclidean coordinates to specify the position of each joint in the plane, taking some arbitrary point as the origin:

a_i^x : horizontal position of joint i relative to the origin
a_i^y : vertical position of joint i relative to the origin

For the example, if the origin lies exactly at joint 2, we might have

$$\begin{array}{lll} (a_1^x, a_1^y) = (0, 2), & (a_3^x, a_3^y) = (2, 1), & (a_5^x, a_5^y) = (4, 0), \\ (a_2^x, a_2^y) = (0, 0), & (a_4^x, a_4^y) = (4, 2). & \end{array}$$

The remaining data consist of the external forces on the joints:

f_i^x : horizontal component of the external force on joint i
f_i^y : vertical component of the external force on joint i

To resist this force, subsets $\mathcal{J}_{\text{fix}}^x \subset \mathcal{J}$ and $\mathcal{J}_{\text{fix}}^y \subset \mathcal{J}$ of joints are fixed in the horizontal and vertical directions, respectively. (It can be proved that, under suitable conditions, fixing three directions is sufficient to guarantee a solution.)

The external forces induce stresses on the bars, which we can represent as

$$F_{ij} : \begin{cases} \text{if } > 0\text{, tension on bar } (i,j) \\ \text{if } < 0\text{, compression of bar } (i,j) \end{cases}$$

A set of stresses is in *equilibrium* if the external forces, tensions and compressions balance at all joints, in both the horizontal and vertical components—except at the fixed joints. That is, for each joint $k \notin \mathcal{J}_{\text{fix}}^x$,

$$\sum_{i \in \mathcal{J}:(i,k) \in \mathcal{B}} c_{ik}^x F_{ik} - \sum_{j \in \mathcal{J}:(k,j) \in \mathcal{B}} c_{kj}^x F_{kj} = f_k^x,$$

and for each joint $k \notin \mathcal{J}^y_{\text{fix}}$,

$$\sum_{i \in \mathcal{J}:(i,k) \in \mathcal{B}} c^y_{ik} F_{ik} - \sum_{j \in \mathcal{J}:(k,j) \in \mathcal{B}} c^y_{kj} F_{kj} = f^y_k,$$

where c^x_{st} and c^y_{st} are the cosines of the direction from joint s to joint t with the horizontal and vertical axes,

$$c^x_{st} = (a^x_t - a^x_s)/l_{st}, \quad c^y_{st} = (a^y_t - a^y_s)/l_{st},$$

and l_{st} is the length of the bar (s,t):

$$l_{st} = \sqrt{(a^x_t - a^x_s)^2 + (a^y_t - a^y_s)^2}.$$

In general, there are infinitely many different set of equilibrium stresses. However, it can be shown that a given system of stresses will be realized in the structure if and only if the cross-sectional areas of the bars are proportional to the absolute values of the stresses. Since the weight of a bar is proportional the cross section times length, we can take the (suitably scaled) weight of bar (i,j) to be

$$w_{ij} = l_{ij} \cdot |F_{ij}|.$$

The problem is then to find a system of stresses F_{ij} that meet the equilibrium conditions, and that minimize the sum of the weights w_{ij} over all bars $(i,j) \in \mathcal{B}$.

For a more detailed treatment of this problem, see [1].

References

[1] Gilbert Strang, *Introduction to Applied Mathematics. Wellesley-Cambridge Press, Wellesley, MA (1986).*