Serial and Hybrid Concatenated Codes with Applications

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Abstract— **Analytical bounds on the performance of concatenated codes on a tree structure are obtained. Analytical results are applied to examples of parallel concatenation of two codes (turbo codes), serial concatenation of two codes, hybrid concatenation of three codes, and self concatenated codes, over AWGN and fading channels. Based on the analysis, design criteria for the selection of component codes are presented. Asymptotic results for large interleavers are extended to MPSK modulations over AWGN and Rayleigh fading channels. Simulation results are only given for examples of coded modulation and fading channels.**

1. Introduction

Turbo codes proposed by Berrou et al. represent a recent breakthrough in coding theory [1], which has stimulated a large amount of new research. These codes are *parallel concatenated convolutional codes* (PCCC) whose encoder is formed by two (or more) *constituent* systematic encoders joined through one (or more) interleavers. Analytical performance bounds for PCCC with uniform interleaver and maximum likelihood receiver were obtained in [2], and [3] for AWGN channel, and in [4] for Rayleigh fading channel with binary modulation.

Parallel concatenated convolutional codes yield very large coding gains (10-11 dB) at the expense of bandwidth expansion. Trellis coded modulation (TCM) proposed by Ungerboeck in 1982 [5] is now a well-established technique in digital communications. In essence, it is a technique to obtain significant coding gains (3-6 dB) sacrificing neither data rate nor bandwidth. In [6] and references there for prior work, TCM was merged with PCCC in order to obtain large coding gains and high bandwidth efficiency. It is called parallel concatenated trellis coded modulation (PCTCM), also addressed as "turbo TCM". Later we considered merging TCM with the recently discovered *serial concatenated convolutional codes* (SCCC) [7]. We refer to the concatenation of an outer convolutional code with an inner TCM as serial concatenated TCM (SCTCM).

In this paper we propose a design method for turbo trellis coded modulation and serial trellis coded modulation, over Rayleigh fading channels for mobile communications. New concatenations of three codes, called *hybrid concatenated convolutional codes* (HCCC), and their special case, *self concatenated codes*[1] are introduced, analyzed, and design rules for these codes are presented[2].

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2. Analytical Bounds on the Performance of Codes over AWGN and Fading Channels

Consider a linear (n,k) block code C with code rate $R_c = k/n$ and minimum distance h_m. An upper bound on the bit-error probability of the block code C over memoryless binary-input channels, with coherent detection, and, using maximum likelihood decoding, can be obtained as

$$P_b \leq \sum_{h=d_{min}}^{n} \sum_{w=1}^{k} \frac{w}{k} A_{w,h}^C D(R_c E_b/N_0, h) \qquad (1)$$

where E_b/N_0 is the signal-to-noise ratio per bit, and $A_{w,h}^C$ for the block code C represents the number of codewords of the block code with output weight h associated with an input sequence of weight w. $A_{w,h}^C$ is the input–output weight coefficient (IOWC). The function $D(\cdot)$ represents the pairwise error probability which is a monotonic decreasing function of the signal to noise ratio and the output weight h. For AWGN channels we have $D(R_c E_b/N_0, h) = Q(\sqrt{2R_c\, h\, E_b/N_0})$. For fading channels, assuming coherent detection, and perfect Channel State Information (CSI) the conditional pairwise error probability is given by

$$D\left(R_c \frac{E_b}{N_0}, h \,\middle|\, \rho\right) = Q\left(\sqrt{2R_c \frac{E_b}{N_0} \sum_{i=1}^{h} \rho_i^2}\right) \qquad (2)$$

The Q function can be represented as [10]

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} d\theta \leq \frac{1}{2} e^{-\frac{x^2}{2}} \qquad (3)$$

To obtain the unconditional pairwise error probability, we have to average over the joint density function

[1]A special case of self concatenated codes where the interleaver is split into two parts corresponding to the original data and its duplicate, was proposed by Berrou [8]

[2]A strictly related paper [9] presents the iterative decoding of various forms of code concatenation discussed in this paper

of fading samples. For simplicity, assume independent Rayleigh fading samples. This assumption is valid if we use an interleaver after the encoder and a deinterleaver before the decoder. Thus the fading samples ρ_i are i.i.d. random variables with Rayleigh density of the form $f(\rho) = 2\rho e^{-\rho^2}$. Using (1) ,(2), (3) and results in [11], by averaging the conditional bit error rate over fading we obtain

$$P_b \leq \int_0^{\frac{\pi}{2}} \sum_{h=d_{min}}^{n} \sum_{w=1}^{k} \frac{w}{k\pi} A^C_{w,h} \left[\frac{\sin^2\theta}{\sin^2\theta + R_c \frac{E_b}{N_o}} \right]^h d\theta \tag{4}$$

We can further upper bound the above result and obtain [11]

$$P_b \leq \frac{1}{2} \sum_{h=h_m}^{n} \sum_{w=1}^{k} \frac{w}{k} A^C_{w,h} \left[\frac{1}{1 + R_c E_b/N_o} \right]^h \tag{5}$$

All these results apply to convolutional codes as well, if we construct an equivalent block code from the convolutional code. Obviously results apply also to concatenated codes including parallel and serial concatenations and other types of code concatenations discussed in this paper. As soon as we obtain the input–output weight coefficients $A^C_{w,h}$ for a particular concatenated code, we can compute its performance.

2.1. Computation of $A^C_{w,h}$ for Concatenated Codes with Random Interleavers

The average input-output weight coefficients $A^C_{w,h}$ for q concatenated codes with $q-1$ interleavers can be obtained by averaging (1) over all possible interleavers. This average is obtained by replacing the actual ith interleaver ($i = 1, 2. \ldots, q-1$), that performs a permutation of the N_i input bits, with an abstract interleaver called **uniform interleaver** [2], defined as a probabilistic device that maps a given input word of weight w into all distinct $\binom{N_i}{w}$ permutations of it with equal probability $p = 1/\binom{N_i}{w}$.

For a concatenated code with q codes and $q-1$ uniform interleavers, each constituent code C_i; $i \in s_q = \{1, 2, \ldots, q\}$ is preceded by a uniform interleaver of size N_i except say C_1 which is not preceded by an interleaver, but it is connected to the input. Define the subsets of the set s_q by $s_i = \{i \in s_q : C_i \text{ connected to input}\}$, $s_o = \{i \in s_q : C_i \text{ connected to Channel}\}$, and its complement $\bar{s}_o$. With the knowledge of the $A^{C_i}_{w_i,h_i}$ for the constituent codes using the concept of uniform interleaver, the $A^C_{w,h}$ for a concatenated code, with tree structure (see Fig. 1), can be obtained as

$$A^C_{w,h} = \sum_{\substack{h_i: i \in s_o, \\ \sum h_i = h}} \sum_{h_i: i \in \bar{s}_o} A_{w,h_1} \prod_{i \in sq: i \neq 1} \frac{A^{C_i}_{w_i,h_i}}{\binom{N_i}{w_i}} \tag{6}$$

Note that $w_i = w$; $i \in s_i$, and $w_j = h_i$ if C_i is connected to C_j by interleaver N_i.

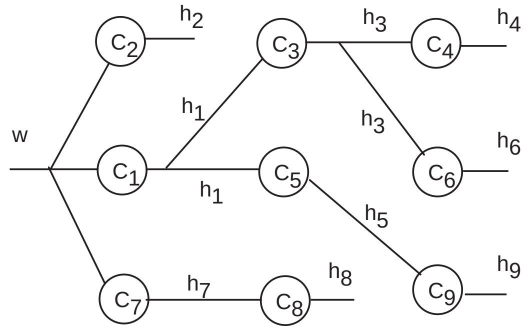

Fig. 1. Example of concatenated codes with tree structure $s_i = \{1, 2, 7\}$, $s_o = \{2, 4, 6, 8, 9\}$

2.2. Design of Concatenated Codes

Design of concatenated codes is based on the asymptotic behavior of the upper bound in (1) after replacing $A^C_{w,h}$ in (1) with the result obtained in (6) for large interleavers. The reason for the good performance of concatenated codes with input block size of N symbols is that the normalized coefficients $A^C_{w,h}/N$ decrease with interleaver size. For a given signal to noise ratio and large interleavers the maximum component of $A^C_{w,h}/N$ over all input weights w and output weights h, is proportional to N^{α_M}, with corresponding minimum output weight $h(\alpha_M)$. If $\alpha_M < 0$ then for a given SNR the performance of concatenated code improves as the input block size is increased. If the input block size increases then the size of interleavers used in the concatenated code should also increase. When $\alpha_M < 0$ we say we have "interleaving gain". The more negative is α_M the more interleaving gain we can obtain. In order to compute α_M we proceed as follows. Consider a rate $R = p/n$ convolutional code C with memory ν, and its equivalent $(N/R, N - p\nu)$ block code whose codewords are all sequences of length N/R bits of the convolutional code starting from and ending at the zero state. By definition, the codewords of the equivalent block code are concatenations of error events of the convolutional codes. Let $A^{C_c}_{w,h,j}$ be the input–output weight coefficients given that the convolutional code generates j error events with total input weight w, and output weight h (see Fig. 2). The $A_{w,h,j}$ actually represents the number of sequences of weight h, input weight w, and the number of concatenated error events j without any gap between them, starting at the beginning of the block. For N much larger than the memory of the convolutional code, the coefficient $A^{C_c}_{w,h}$ of the equivalent block code can be approximated by

$$A^{C_c}_{w,h} \sim \sum_{j=1}^{n_M} \binom{N/p}{j} A^{C_c}_{w,h,j} \tag{7}$$

where n_M, the largest number of error events concatenated in a codeword of weight h and generated by a weight w input sequence, is a function of h and w that depends on the encoder.

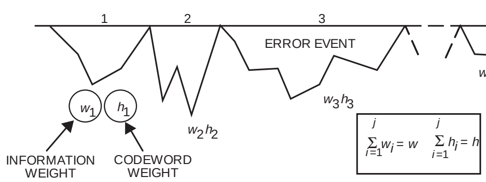

Fig. 2. A code sequence in $A^C_{w,h,j}$

The large N assumption permits neglecting the length of error events compared to N, which also implies that the number of ways j input sequences producing j error events can be arranged in a register of length N is $\binom{N/p}{j}$. The ratio N/p derives from the fact that the code has rate p/n, and thus N bits corresponds to N/p input symbols or, equivalently, trellis steps. We are interested in large interleaver lengths and thus use for the binomial coefficient the asymptotic approximation

$$\binom{N}{j} \sim \frac{N^j}{j!} \tag{8}$$

Substitution of this approximation in the previous equation yields

$$A^{C_c}_{w,h} \sim \sum_{j=1}^{n_M} \left(\frac{N^j}{j!p^j} \right) A^{C_c}_{w,h,j} \tag{9}$$

Finally, substituting (9) in (6) for each constituent code, and then the result in (1) gives the bit-error probability bound in a desired form for design of concatenated codes, from which we obtain

$$\alpha_M = \max_{w,h} \left\{ \sum_{i \in s_q} j_i - \sum_{i \in s_q : i \neq 1} w_i - 1 \right\} \tag{10}$$

where j_i denotes the number of concatenated error events for code C_i. Computation of α_M depends on the concatenated code structure and constituent codes. Next we obtain $A^C_{w,h}$ using (6), and α_M, using (10), to compute upper bounds, and design rules for the following concatenated codes.

3. Parallel Concatenated Convolutional Codes

The structure of a parallel concatenated convolutional code (PCCC) or "turbo code" is shown in Fig. 3. Figure 3 refers to the case of two convolutional codes, code C_1 with rate $R_c^1 = p/q_1$, and code C_2 with rate $R_c^2 = p/q_2$, where the constituent code inputs are joined by an interleaver of length N, generating a PCCC, C_P, with rate $R_c = \frac{R_c^1 R_c^2}{R_c^1 + R_c^2}$. Note that N is an integer multiple of p. The input block length $k = N$, and the output codeword length $n = n_1 + n_2$ as shown in Fig. 3.

3.1. *Computation of input–output weight coefficient (IOWC) $A^{C_P}_{w,h}$ for PCCC (turbo codes)*

With the knowledge of the $A^{C_1}_{w,h_1}$ for code C_1, and $A^{C_2}_{w,h_2}$ for code C_2, using (6), IOWC $A^{C_P}_{w,h}$ for PCCC

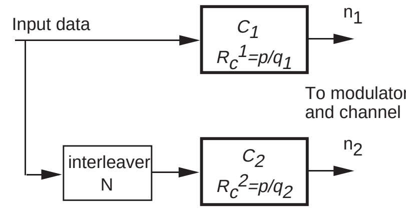

Fig. 3. Parallel Concatenated Convolutional Codes (PCCC).

can be obtained as follows.

$$A^{C_P}_{w,h} = \sum_{\substack{h_1,h_2: \\ h_1+h_2=h}} A^{C_P}_{w,h_1,h_2} = \sum_{\substack{h_1,h_2: \\ h_1+h_2=h}} \frac{A^{C_1}_{w,h_1} \times A^{C_2}_{w,h_2}}{\binom{N}{w}} \tag{11}$$

where $A^{C_P}_{w,h_1,h_2}$ is the number of codewords of the PCCC with output weights h_1,and h_2 associated with an input sequence of weight w.

Example 1. Consider a rate 1/2 PCCC formed by two identical 4-state convolutional codes: Code C_1 with rate 2/3 and code C_2 with rate 1/1 (this is obtained by not sending the systematic bits of the rate 2/3 C_2 convolutional code). The inputs of encoders are joined by a uniform interleaver of lengths $N = 50$, 100 and 256. Both codes are systematic and recursive, and are shown in Fig. 4. Using the previously outlined analysis for PCCC, we have obtained the bit-error probability bounds shown in Fig. 4. The performance is shown both for AWGN and Rayleigh fading channels.

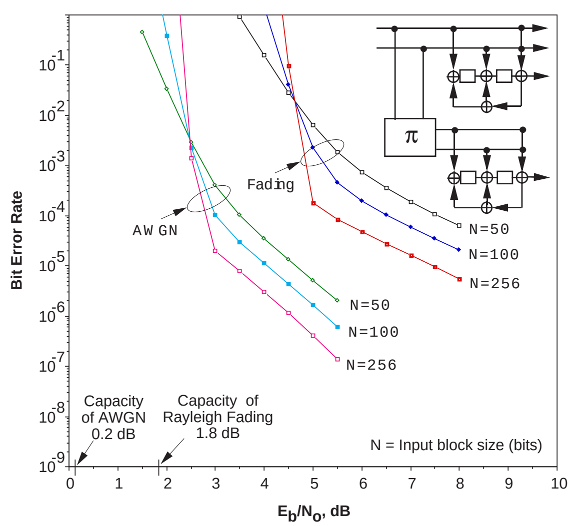

Fig. 4. Performance of rate 1/2 PCCC over AWGN and Rayleigh Fading Channels

Using (10) we obtain the following results. If both convolutional codes are recursive then $\alpha_M \leq -1$. Any other choice of encoders results in $\alpha_M \geq 0$. Thus, for all $h = h_1 + h_2$, the coefficients of the exponents in h decrease with N, and we always have an *interleaving gain* [2].

Define $d_{i,f,eff}$ as the minimum weight of codewords of a recursive code C_i, $i = 1, 2$ generated by weight-2 input sequences. We call it the effective free Hamming distance of a recursive convolutional code. To maximize the *interleaving gain*, i.e., minimize N^{α_M} corresponding to output weight $h_1(\alpha_M)$, and $h_2(\alpha_M)$ we should maximize the $d_{i,f,eff}$, $i = 1, 2$. The sum $d_{1,f,eff} + d_{2,f,eff}$ represents the effective free distance of the turbo code. Thus, substituting the exponent α_M into the expression for bit error rate (5) approximated by keeping only the term of the summation in h_1, and h_2 corresponding to $h_1 = h_1(\alpha_M)$, and $h_2 = h_2(\alpha_M)$, yields

$$\lim_{N\to\infty} P_b(e) \simeq BN^{-1}\left[\frac{1}{1+R_c\frac{E_b}{N_0}}\right]^{d_{1,f,eff}+d_{2,f,eff}} \tag{12}$$

where B is a constant independent of N.

4. Parallel Concatenated Trellis Coded Modulation

The basic structure of parallel concatenated trellis coded modulation is shown in Fig. 5.

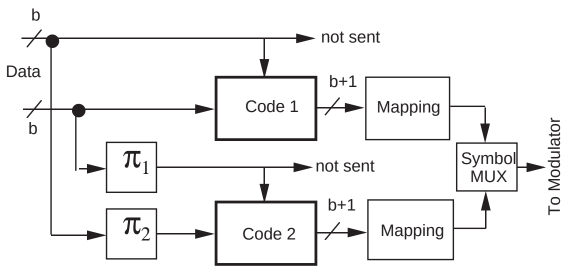

Fig. 5. Block Diagram of the Encoder for Parallel Concatenated Trellis Coded Modulation.

This structure uses two rate $\frac{2b}{2b+1}$ constituent convolutional codes. The first b most significant output bits of each convolutional code are only connected to the shift register of the TCM encoder and are not mapped to the modulation signals. The last $b+1$ least significant output bits however are mapped to the modulation signals. This method requires at least two interleavers. The first interleaver permutes the b least significant input bits. This interleaver is connected to the b most significant bits of the second TCM encoder. The second interleaver permutes the b most significant input bits. This interleaver is then connected to the b least significant input bits of the second TCM encoder.

4.1. *Design Criteria for PCTCM over Rayleigh Fading Channels*

To extend the asymptotic results we obtained for binary modulation to M-ary Modulation (e.g. MPSK), let $\mathbf{x}_i$ represent the sequence of M-ary output (complex) symbols $\{x_{i,j}\}$ of trellis code i ($i = 1, 2$). Complex symbols have unit average power. Let $\mathbf{x}'_i$ represent another sequence of the output symbols $\{x'_{i,j}\}$ for $i = 1, 2$. Then the asymptotic result in (12) should be modified to

$$P_b(e) \simeq BN^{-1}\prod_{n_1\in\eta_1}\left[\frac{1}{1+|x_{1,n_1}-x'_{1,n_1}|^2R_c\frac{E_b}{4N_0}}\right]\times$$

$$\prod_{n_2\in\eta_2}\left[\frac{1}{1+|x_{2,n_2}-x'_{2,n_2}|^2R_c\frac{E_b}{4N_0}}\right]$$

where, for $i = 1, 2$, η_i is the set of all n_i with the smallest cardinality $d_{i,f,eff}$ such that $x_{i,n_i} \neq x'_{i,n_i}$. Then $d_{i,f,eff}$ represents the minimum (M-ary symbol) Hamming distance of trellis code i ($i = 1, 2$) corresponding to input Hamming distance 2 between binary input sequences that produce $d_{i,f,eff}$. The $d_{i,f,eff}$, $i = 1, 2$ is also called the minimum diversity of trellis code i. We note that the asymptotic result on the bit error rate is inversely proportional to the product of the squared Euclidean distances along the error event paths which result in $d_{i,f,eff}$ i=1,2. Therefore the criterion for optimization of the component trellis codes is to maximize the minimum diversity of the code and then maximize the product of the squared Euclidean distances which result in minimum diversity.

4.2. *2 bits/sec/Hz PCTCM with 8PSK for AWGN and Fading Channels*

The code we propose has $b = 2$, and employs 8PSK modulation in connection with two 8-state, rate 4/5 constituent codes. The selected code uses reordered mapping: If b_2, b_1, b_0 represents a binary label for natural mapping for 8PSK, where b_2 is the MSB and b_0 is the LSB, then the reordered mapping is given by b_2, (b_2+b_1), b_0. The effective Euclidean distance of this code is $\delta^2_{f,eff} = 5.17$ (unit-norm constellation is assumed), using two interleavers.

The structure of this code is shown in Fig. 6, and its BER for AWGN and Rayleigh fading channels in Fig. 7. The size of each interleaver is 8192 bits.

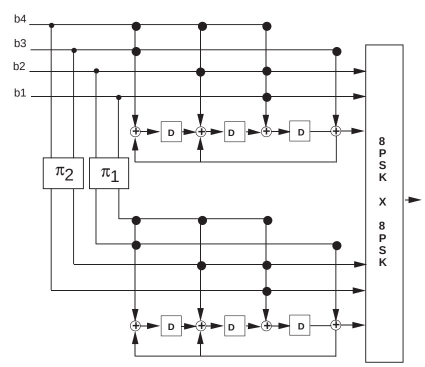

Fig. 6. Parallel Concatenated Trellis Coded Modulation, 8PSK, 2 bits/sec/Hz.

5. Serially Concatenated Convolutional Codes

The structure of a serially concatenated convolutional code (SCCC) is shown in Fig. 8. Figure 8 refers to the case of two convolutional codes, the outer code C_o with rate $R_c^o = q/p$, and the inner code C_i with rate $R_c^i = p/m$, joined by an interleaver of length N bits, generating an SCCC C_S

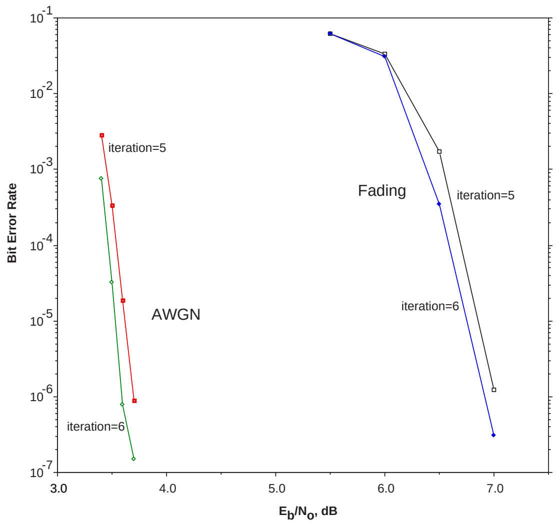

Fig. 7. BER Performance of Parallel Concatenated Trellis Coded 8PSK, 2 bits/sec/Hz.

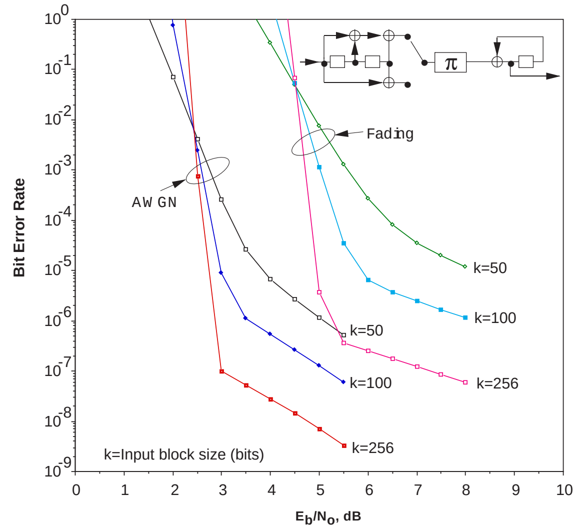

Fig. 9. Performance of rate 1/2 SCCC over AWGN and Rayleigh Fading Channels

with rate $R_c = k/n$. Note that N must be an integer multiple of p. The input block size is $k = Nq/p$ and the output block size of SCCC is $n = Nm/p$.

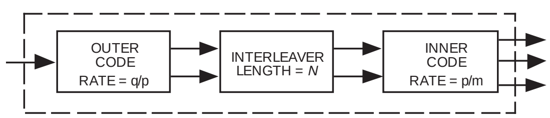

Fig. 8. Serial Concatenated Convolutional Codes (SCCC).

5.1. *Computation of input–output weight coefficient (IOWC) $A^{C_S}_{w,h}$ for SCCC*

With the knowledge of the $A^{C_o}_{w,l}$ for the outer code, $A^{C_i}_{l,h}$ for the inner code, and using (6), the IOWC $A^{C_S}_{w,h}$ for SCCC can be obtained as

$$A^{C_S}_{w,h} = \sum_{l=0}^{N} \frac{A^{C_o}_{w,l} \times A^{C_i}_{l,h}}{\binom{N}{l}} \tag{13}$$

Example 2. Consider a rate 1/2 SCCC formed by a 4-state convolutional code C_o with rate 1/2 and an inner 2-state convolutional code C_i with rate 1/1 (this is obtained by not sending the systematic bits of the rate 1/2 C_i convolutional code). The two codes are joined by a uniform interleaver. Input blocks of length $N = 50, 100$ and 256 were considered. The outer code is a nonrecursive code, the inner code is systematic and recursive, and the generators are shown in Fig. 9. Using the previously outlined analysis for SCCC, we have obtained the bit-error probability bounds shown in Fig. 9. The performance was obtained both for AWGN and Rayleigh fading channels. Comparing to Fig. 4, the performance of SCCC is better than PCCC both over AWGN and fading channels.

Using (6) we obtain α_M and the corresponding output weight $h(\alpha_M)$. If the inner convolutional code is recursive then $\alpha_M = -\left\lfloor \frac{d_f^o+1}{2} \right\rfloor$ where d_f^o is the free (minimum) distance of the outer convolutional code.

The value of α_M shows that the exponents of N are always negative integers. Thus, for all h, the coefficients of the exponents in h decrease with N, and we always have an "interleaving gain".

Define $d^i_{f,eff}$ as the minimum weight of codewords of the inner code generated by weight-2 input sequences. We obtain a different weight $h(\alpha_M)$ for even and odd values of d_f^o. For even d_f^o, the weight $h(\alpha_M)$ associated to the highest exponent of N is given by

$$h(\alpha_M) = \frac{d_f^o d^i_{f,eff}}{2} \tag{14}$$

For d_f^o odd, the value of $h(\alpha_M)$ is given by

$$h(\alpha_M) = \frac{(d_f^o - 3) d^i_{f,eff}}{2} + h_m^{(3)} \tag{15}$$

where $h_m^{(3)}$ is the minimum weight of sequences of the inner code generated by a weight-3 input sequence.

Thus, substituting the exponent α_M into the expression for bit error rate in (5) approximated by keeping only the term of the summation in h corresponding to $h = h(\alpha_M)$ yields

$$\lim_{N\to\infty} P_b \simeq B N^{-\left\lfloor \frac{d_f^o+1}{2} \right\rfloor} \left[\frac{1}{1 + R_c \frac{E_b}{N_0}} \right]^{h(\alpha_M)} \tag{16}$$

where B is a constant independent of N.

6. Serial Concatenated Trellis Coded Modulation

The basic structure of serially concatenated trellis coded modulation is shown in Fig. 10.

We propose a novel method to design serial concatenated TCM for Rayleigh fading channels, which

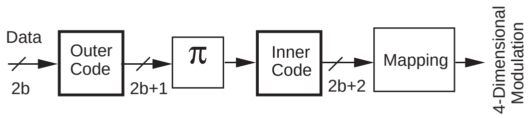

Fig. 10. Block Diagram of the Encoder for Serial Concatenated Trellis Coded Modulation.

achieves b bits/sec/Hz, using a rate $2b/(2b+1)$ non-recursive binary convolutional encoder with maximum free Hamming distance as outer code. We interleave the output of the outer code with a random permutation. The interleaved data enters a rate $(2b+1)/(2b+2)$ recursive convolutional inner encoder. The $2b+2$ output bits are mapped to two symbols belonging to a 2^{b+1} level modulation (four dimensional modulation). In this way, we are using $2b$ information bits for every two modulation symbol intervals, resulting in b bit/sec/Hz transmission (when ideal Nyquist pulse shaping is used) or, in other words, b bits per modulation symbol. For the AWGN channel the inner code and the mapping are jointly optimized based on maximizing the effective Euclidean distance of the inner TCM. The optimum 2-state inner trellis code is shown in Fig. 11. The effective Euclidean distance of this code is 1.76 (for unit norm constellation) and its minimum M-ary Hamming distance is 1.

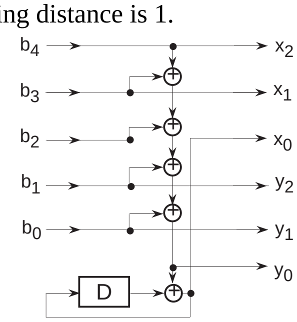

Fig. 11. Optimum 2-state inner trellis encoder for SCTCM with 2× 8PSK Modulation.

6.1. *Design Criteria for SCTCM over Rayleigh Fading Channels*

To extend the asymptotic results obtained for binary modulation to to M-ary modulation (e.g., MPSK), criteria similar to those discussed for parallel concatenated trellis coded modulation (PCTCM) are now applied to serial concatenated trellis coded modulation (SCTCM). The interleaving gain is still $N^{-\lfloor (d_f^o+1)/2 \rfloor}$, however now the minimum diversity is $\frac{d_f^o d_{f,eff}^i}{2}$ for even d_f^o, and $\frac{(d_f^o-3)d_{f,eff}^i}{2} + h_m^{(3)}$ for odd d_f^o, where $d_{f,eff}^i$ represents the minimum (M-ary symbol) Hamming distance of the inner trellis code corresponding to input Hamming distance 2 between binary input sequences to the trellis code that produce $d_{f,eff}^i$. Therefore the criterion for optimizing the inner trellis code in SCTCM is to maximize the minimum diversity of the code and then maximize the product of the squared Euclidean distances which result in minimum diversity. For odd d_f^o, first we maximize $d_{f,eff}^i$, then among the codes with maximum $d_{f,eff}^i$, we maximize $h_m^{(3)}$, the minimum (M-ary symbol) Hamming distance of the inner trellis code corresponding to input Hamming distance 3 between binary input sequences to the trellis code that produce $h_m^{(3)}$. As is seen from the previous results, large d_f^o produces large interleaving gain and diversity.

6.2. *Design Method for Inner TCM*

To illustrate the design methodology we developed the following example. Let the eight phases of 8PSK $\pi i/4$, $i = 0, 1, \ldots, 7$ be denoted by $\{0, 1, 2, 3, 4, 5, 6, 7\}$. Consider the 2×8PSK signal set $A_0 = [(0,0), (1,3), (2,6), (3,1), (4,4), (5,7), (6,2), (7,5)]$. Each element in the set has two components. The second component is 3 times the first one modulo 8. Also consider the 2×8PSK signal set $B_0 = [(0,0), (1,5), (2,2), (3,7), (4,4), (5,1), (6,6), (7,3)]$. Each element in the set has two components. The second component is 5 times the first one modulo 8. For these sets, the Hamming distance between elements in each set is 2, and the minimum of the product of square Euclidean distances is the largest possible.

The following sets are constructed from A_0 and B_0 as: $A_2 = A_0 + (0,2)$, $A_4 = A_0 + (0,4)$, $A_6 = A_0 + (0,6)$, $A_1 = B_0 + (0,1)$, $A_3 = B_0 + (0,3)$, $A_5 = B_0 + (0,5)$, $A_7 = B_0 + (0,7)$, where addition is component-wise modulo 8. Map the first and last 2 bits of input labels to the 8PSK signals as $\{00, 00, 01, 01, 11, 11, 10, 10\} \Rightarrow \{0, 1, 2, 3, 4, 5, 6, 7\}$.

The fifth bit for the input label is the parity check bit. Use an even parity check bit for signal sets A_0, A_4, A_1, A_5 and an odd parity check bit for signal sets A_2, A_6, A_3, A_7. This completes the input label assignments to signal sets.

Now the Hamming distance between input labels for each set A_i i=0,1,2,..., 7, is at least 2 and the corresponding M-ary Hamming distance between signal elements in each set is 2. Consider a 4-state trellis code with full transition. Assign A_0, A_2, A_4, A_6, to the first state, and A_1, A_3, A_5, A_7 to the second state, and permutations of these sets to the third and fourth states. This completes the input label and 2×8PSK signal set assignments to the edges of the 4-state trellis. Therefore the minimum Hamming distance of the 4-state trellis code is 2. At this point to obtain a circuit that generates this trellis we need to use an output label. We used reordered mapping as it was discussed before to obtain the circuit for the encoder.

The implementation of the 4-state inner trellis code is shown in Fig. 12. The ROM maps 32 addresses in the range of 0 to 31 to a single output. The 32 binary outputs can be summarized in hex as 3A53ACC5.

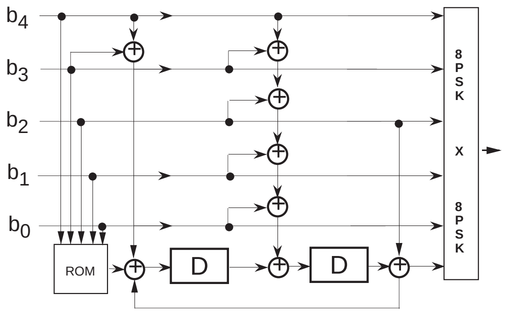

Fig. 12. 4-state inner trellis encoder for SCTCM with 2×8PSK modulation for Rayleigh fading.

7. Simulation of Serial Concatenated Trellis Coded Modulation with Iterative Decoding

In this section the simulation results for serial concatenated TCM, with 2×8PSK over the Rayleigh fading channel are presented. For SCTCM with 2×8PSK, the outer code is a rate 4/5, 8-state nonrecursive convolutional encoder with $d_f^o = 3$, and the inner code is the 4-state TCM designed for $2\times 8PSK$ in subsection 6.2. The bit error probability vs. bit signal-to-noise ratio E_b/N_o for various numbers of iterations is shown in Fig. 13. The performance of the inner 2-state code in Fig. 11 is also shown in Fig. 13for the input block of 16384 bits. This example demonstrates the power and bandwidth efficiency of SCTCM, over a Rayleigh fading channel at low BERs.

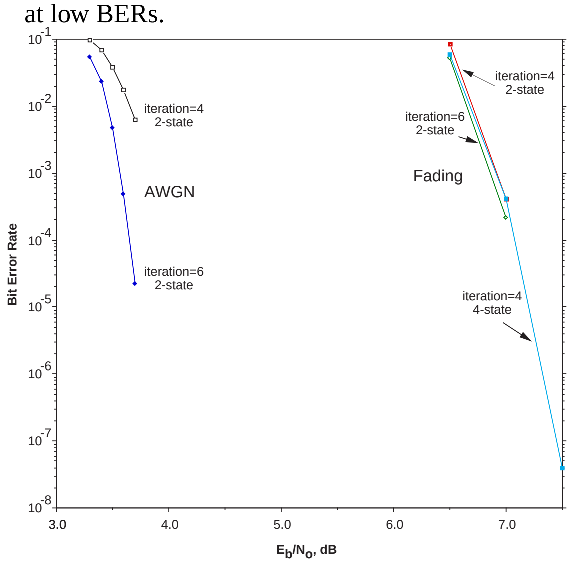

Fig. 13. Performance of Serial Concatenated Trellis Coded Modulation, 8-state outer, 2-state or 4-state inner, with 2×8PSK, 2 bits/sec/Hz

8. Hybrid concatenated convolutional codes

The hybrid structure shown in Fig. 14 includes a *parallel* convolutional code C_p with rate $R_c^p = k/n_1$ and equivalent block code representation $(N_1/R_c^p, N_1)$, an *outer* $(N_1/R_c^o, N_1)$ code C_o with rate $R_c^o = k/p$, (this code can be a repetition code), an *inner* $(N_2/R_c^i, N_2)$ code C_i with rate $R_c^i = p/n_2$, plus a N_1-bit and a N_2-bit interleaver. This gives an HCCC with overall rate $R_c = k/(n_1 + n_2)$. In special case the outer code can be a repetition code. Further if the parallel code is rate 1, 1-state code (no code) we obtain self concatenated code which is discussed in the next section.

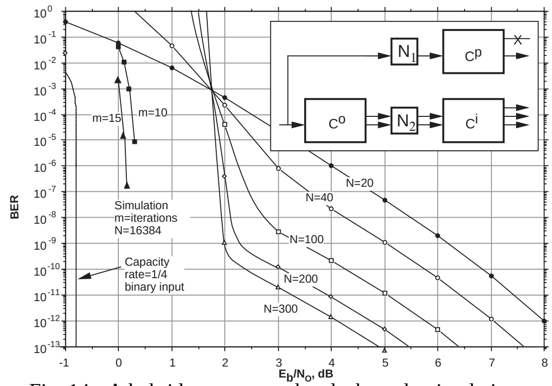

Fig. 14. A hybrid concatenated code, bounds, simulations.

Using (6) we obtain

$$A_{w,h_1,h_2}^{C_H} = \sum_{l=0}^{N_2} \frac{A_{w,h_1}^{C_p} \times A_{w,l}^{C_o} \times A_{l,h_2}^{C_i}}{\binom{N_1}{w}\binom{N_2}{l}} \qquad (17)$$

The coefficients $A_{w,h}^{C_H}$ can be obtained by summing $A_{w,h_1,h_2}^{C_H}$ over all h_1, and h_2 such that $h_1 + h_2 = h$. $A_{w,l}^{C_o}$ is the number of codewords of C_o of weight l given by the input sequences of weight w. Analogous definitions apply for $A_{w,h_1}^{C_p}$ and $A_{e,h_2}^{C_i}$. We have computed the bound in (1) over AWGN for a specific rate 1/4 HCCC formed by a 4-state C_p (recursive, systematic, $R_c^p = 1/2$), where, as in "turbo codes", the systematic bits are not transmitted, a 4-state C_o (nonrecursive, $R_c^0 = 1/2$), and a 4-state C_i (recursive, systematic, $R_c^i = 2/3$), joined by two interleavers of lengths $N_1 = N$ and $N_2 = 2N$. The respective generator matrices are $\left[\begin{array}{cc} 1, & \frac{1+D^2}{1+D+D^2} \end{array}\right]$, $\left[\begin{array}{cc} 1+D+D^2, & 1+D^2 \end{array}\right]$, and $\left[\begin{array}{ccc} 1, & 0, & (1+D^2)/(1+D+D^2) \\ 0, & 1, & (1+D)/(1+D+D^2) \end{array}\right]$. The BER performance bounds show a very significant interleaving gain , i.e., lower values of BER for higher values of N. At $E_b/N_o = 3$ dB, BER is 3×10^{-5}, 8×10^{-7}, 4×10^{-9}, 10^{-10}, and 2×10^{-11}, for $N = 20, 40, 100, 200, 300$, respectively. Simulation of the proposed iterative decoder produced BER=10^{-7} at $E_b/N_o = 0.2$ dB, with 15 iterations and $N = 16384$, as shown in Fig. 14.

8.1. Design of HCCCs

To evaluate α_M, again we use (10). If C_i is nonrecursive, and C_p or C_o are nonrecursive then we have $\alpha_M \geq 0$, and interleaving gain is not allowed. If C_i is nonrecursive, and both C_p and C_o are recursive then we have $\alpha_M = -1$, and interleaving gain is allowed, as for "turbo codes". If C_i is recursive, and

C_p is nonrecursive we have $\alpha_M \leq -\left\lfloor (d_f^o+1)/2 \right\rfloor$, and interleaving gain is allowed, as in serial concatenated codes. If C_i is recursive, and C_p is recursive we have $\alpha_M \leq -\left\lfloor (d_f^o+3)/2 \right\rfloor$, and interleaving gain is higher than for serial concatenated codes. Based on the above analysis, in order to achieve the highest interleaving gain in HCCCs, we should select the component codes as follows: a recursive C_i; a recursive C_p; C_o can be either nonrecursive or recursive but should have large d_f^o. Next we consider the weight $h(\alpha_M)$ which is the sum of output weights of C_i and C_p associated to the highest exponent of N. We have $h(\alpha_M) = d_f^o d^i_{\text{f,eff}}/2 + d^p_{\text{f,eff}}$, for d_f^o even, and $h(\alpha_M) = (d_f^o-3)d^i_{\text{f,eff}}/2 + h_m^{(3)} + d^p_{\text{f,eff}}$, for d_f^o odd, where $h_m^{(3)}$ is the minimum weight of codewords of C_i generated by a weight 3 input sequence, and $d^i_{\text{f,eff}}$, and $d^p_{\text{f,eff}}$ are the effective free distances of C_i and C_p.

9. Self-Concatenated Code

Consider a self concatenated code as shown in Fig. 15. This code can be considered as special case of hybrid concatenated code when the outer code is rate 1/p repetition code, and parallel code is a rate 1, 1-state code (no code). Since one nontrivial code is used we call it self concatenated code.

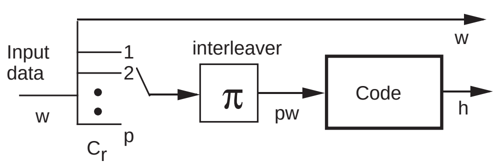

Fig. 15. A self concatenated code.

For a rate 1/p repetition code and its N concatenation we have $A^{C_r}_{w,l} = \binom{N}{w}; l = wp$, and zero otherwise. Using (1), and (6) we can obtain

$$P_b \leq \sum_{h=d_{min}}^{Np/R_c^i} \sum_{w=1}^{N} \frac{w}{N} \frac{\binom{N}{w} A^C_{pw,h}}{\binom{pN}{pw}} Q\left(\sqrt{\frac{2R_cE_b}{N_0}(h+w)}\right) \tag{18}$$

Using (10) we obtain $\alpha_M = \max_{w,h}\{w+j-pw-1\}$, where j is number of concatenated error events in code C. If the interleaver is split into p parts corresponding to the original data and its $p-1$ duplicates, the code will be equivalent to multiple turbo codes. The case of $p=2$, with structured interleaver which does not require trellis termination was proposed by Berrou [8] with good performance for short blocks.

9.1. *The Maximum Exponent of N*

For a nonrecursive convolutional encoder, we have $j \leq pw$. In this case $\alpha_M \geq 0$. Thus we have *no interleaving gain*. However, for recursive convolutional encoder C, the minimum weight of input sequences generating error events is 2. As a consequence, an input sequence of weight pw can generate at most $j = \lfloor \frac{pw}{2} \rfloor$ error events. In this case the exponent of N is negative. Thus, we have an *interleaving gain*. For $p=2$, the maximum exponent of N is -1, and the minimum output weight is $h+w = d_{f,eff}+1$. For $p=3$, the maximum exponent of N is -2, and the minimum output weight is $h+w = h_m^{(3)}+1$. However, if $h = h_m^{(3)} = \infty$ then the minimum output weight is $h+w = 3d_{f,eff}+2$.

10. Conclusions

General analytical upper bounds and design rules for concatenated codes with interleavers over AWGN, and Rayleigh fading channels were presented.

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