PERFORMANCE OF AMPLIFY-AND-FORWARD AND DECODE-AND-FORWARD RELAYING IN RAYLEIGH FADING WITH TURBO CODES

Michael R. Souryal *

National Institute of Standards and Technology
Wireless Communication Technologies Group
Gaithersburg, Maryland

Branimir R. Vojcic †

The George Washington University
Dept. of Electrical and Computer Engineering
Washington, DC

ABSTRACT

Cooperative transmission, in which a source and relay cooperate to send a message to a destination, can provide spatial diversity against fading in wireless networks. We derive analytical expressions for the error probability of amplify-and-forward (AF), decode-and-forward (DF), and a new hybrid AF/DF relaying protocol, for systems using strong forward error correction in quasi-static Rayleigh fading channels, and these expressions are shown to compare favorably with simulation results using turbo codes. Analytical results include an exact expression for the distribution of the SNR in AF transmission. For the protocols that achieve diversity (AF, adaptive DF, and hybrid AF/DF), the optimum position of the relay is midway between the source and destination, implying that mutual relaying (or partnering) to a common destination is suboptimal.

1. INTRODUCTION

In cooperative wireless communication, a source transmits a message to a destination with the assistance of a relay. The relay listens to the source's transmission and may retransmit the message to the destination. By combining the source and relay transmissions, and depending on the relaying protocol used, the destination can achieve diversity against fading without the use of an antenna array at any terminal.

An information-theoretic analysis of outage behavior [1] has shown that a fixed decode-and-forward (DF) relaying protocol—in which the relay always decodes, re-encodes and transmits the message—does not achieve diversity. However, adaptive DF—in which the source uses either source-relay channel state information (CSI) or feedback from the relay to decide between retransmitting the message or permitting the relay to forward the message—does achieve second-order diversity in the high signal-to-noise ratio (SNR) region. The same analysis showed that diversity is also achieved when the relay simply amplifies and forwards the message at all times, known as fixed amplify-and-forward (AF).

Previous studies have also considered the performance of these protocols in practical systems. For example, an analysis of uncoded systems concluded that AF transmission does not achieve full diversity when no source-relay CSI is available at the destination [2]. However, it is not clear that the concatenated source-relay-destination channel needed for optimum combining cannot be estimated in practice. An analysis of a system using distributed turbo codes found that AF and adaptive DF perform comparably, both achieving diversity, and concluded that a hybrid of the two would therefore not be beneficial [3].

In this paper, we derive expressions for the error rate performance of relaying protocols in quasi-static Rayleigh fading when the success/failure of a transmission on a link can be modeled using an SNR-threshold model. Such a model approximates scenarios in which the fading process is slow and where strong channel codes are used, such as turbo codes with iterative decoding, at each receiver. Among the new results is an exact expression for the outage probability of AF transmission, which had only been approximated in previous work. We also analyze a new hybrid AF/DF protocol that achieves diversity against fading without the need for adaptivity at the source (and the associated overhead), and that avoids noise amplification when the source-relay transmission succeeds. We observe that the optimal position of the relay is midway between the source and destination, suggesting that mutual relaying, where two sources relay for one another to a common destination, is suboptimal. Finally, analytical results are compared with simulation results using a practical turbo code.

2. SYSTEM MODEL

A source and one relay cooperate in time-division manner to transmit a message to a destination. The source encodes the message and transmits it in the first time slot. In the second time slot, either the source or relay retransmits the message to the destination. When the relay transmits, it either fully decodes and re-encodes the message, or it amplifies and forwards its received signal. We consider decode-and-forward protocols which use the same code for both transmissions.

The channel propagation model includes path loss with distance and Rayleigh fading that is constant during the two-slot transmission and independent from one transmission to the next. Furthermore, the fading is mutually independent among the three links in the system (see Fig. 1). The channel also includes additive white Gaussian noise with two-sided power spectral density $N_0/2$. The sampled output of the demodulator of a receiver is thus modeled as

$$y_i = \alpha_i s_i + z_i \tag{1}$$

where $\alpha_i s_i$ is the attenuated signal contribution, z_i is the noise contribution, all terms are complex representing in-phase and quadrature components, and the subscript $i \in \{0, 1, 2\}$ denotes the source-destination, source-relay, and relay-destination links, respectively. Under the stated channel assumptions, the channel gain

*This research was performed while the author held a National Research Council Research Associateship Award at NIST.

†This work was supported in part by the National Science Foundation under grant CCF-042299228.

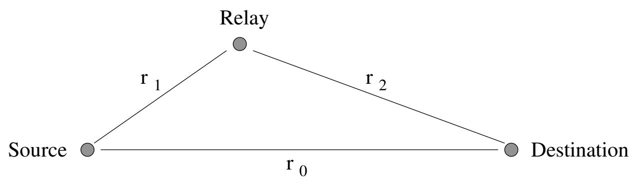

Fig. 1. Sample topology of source, relay and destination

$\Gamma_i = |\alpha_i|^2$ has an exponential distribution with mean $1/r_i^n$, where r_i is the link distance and n is the path loss exponent. The noise term, z_i, is complex Gaussian with variance $\sigma^2 = N_0/E_s$, where E_s is the average received signal energy, and s_i has unit energy.

To obtain analytical results, an ideal SNR-threshold model is assumed for the success/failure of a transmission on a link. When the received SNR exceeds a threshold, the message is correctly decoded; otherwise, it has uncorrectable errors. Under this model, the probability of successful transmission is given by $\Pr[\mu > \mu_c]$ where μ denotes the received SNR, random due to channel effects, and μ_c is the threshold, or cut-off, SNR. The threshold model approximates the sharp transition in error rate with SNR that is characteristic of systems with strong forward error correction, where the threshold value is a function of the specific modulation-coding scheme used. For example, our simulation results for the performance of a rate-1/3 and a punctured rate-2/3 binary turbo code in AWGN show that the frame error rate (FER) decreases from 0.5 to 10^{-2} with an increase in SNR of less than one decibel. These results were obtained for 1024-bit frames, encoded with generator polynomial $(1, 13/15)_8$, mapped to quadrature phase shift keying (QPSK) channel symbols, after eight iterations of soft-input/soft-output (SISO) MAP decoding at the receiver. If we use the 0.5 FER point to define the threshold SNR, the rate-1/3 and rate-2/3 codes in this example are characterized by thresholds of -1.3 dB and 3.1 dB, respectively.

3. ANALYSIS

In this section, we derive the performance of various relaying protocols in terms of probabilities that the overall SNR exceeds the cut-off value for successful reception. In the following section, these results are compared with simulation results for a specific turbo-coded system.

3.1. Non-Cooperative Transmission

In direct, non-cooperative transmission, the source transmits the message to the destination without the assistance of a relay. From (1) with $i = 0$, the received SNR is $\mu_0 = \Gamma_0/\sigma^2$. Under the threshold model for successful reception, the probability of success with direct transmission is

$$P_{s,\mathrm{dir}} = \Pr[\mu_0 > \mu_{c1}] = e^{-\sigma^2 r_0^n \mu_{c1}}$$

where μ_{c1} is the threshold SNR of the modulation-coding scheme.

For comparison purposes, we also consider a non-cooperative, fixed two-hop transmission scheme in which a relay decodes and forwards the source's transmission to the destination. This scheme is non-cooperative in the sense that the destination does not attempt to combine the transmissions of the source and relay, but only listens to the relay's forwarded transmission.

In the two-hop case, the message is received only if both the source-relay and relay-destination transmissions are successful. Since the fading is independent on the two links, the probability of success is

$$\begin{aligned} P_{s,2-\mathrm{hop}} &= \Pr[\mu_1 > \mu_{c2},\, \mu_2 > \mu_{c2}] \\ &= e^{-\sigma^2 \left(r_1^n + r_2^n\right)\mu_{c2}} \end{aligned}$$

where, to achieve the same throughput as the direct transmission, μ_{c2} is the threshold SNR for a modulation-coding scheme with twice the spectral efficiency of the direct transmission.

3.2. Fixed Decode-and-Forward

In fixed decode-and-forward transmission, the message is received at the destination if the combined received signal from the source and relay is successfully decoded. If the relay's transmission contains errors, it is unlikely that the destination will recover the message from the combined signal. We, therefore, approximate the overall probability of success with the following joint probability, where $\mu_{0,2}$ represents the SNR of the combined signal:

$$P_{s,\mathrm{DF}} \simeq \Pr[\mu_{0,2} > \mu_{c2},\, \mu_1 > \mu_{c2}]\,. \tag{2}$$

Rewriting the right-hand side of (2) in terms of conditional and *a priori* probabilities, we have

$$\begin{aligned} &\Pr[\mu_{0,2} > \mu_{c2},\, \mu_1 > \mu_{c2}] \\ &\quad= \Pr[\mu_1 > \mu_{c2}]\Pr[\mu_{0,2} > \mu_{c2} \,|\, \mu_1 > \mu_{c2}] \\ &\quad= e^{-\sigma^2 r_1^n \mu_{c2}} \Pr[\mu_{0,2} > \mu_{c2} \,|\, \mu_1 > \mu_{c2}]\,. \end{aligned} \tag{3}$$

For maximal ratio combining, which is optimum when CSI is known [4], the combined SNR of the source and relay transmissions, given that the relay's transmission is the same as that of the source, is easily found to be $\mu_{0,2} = (\Gamma_0 + \Gamma_2)/\sigma^2$, which is the scaled sum of two independent, exponentially distributed variates with different means. By convolving their density functions and integrating, the complementary distribution function of $\mu_{0,2}$ can be shown to be

$$\begin{aligned} &\Pr[\mu_{0,2} > \mu_{c2} \,|\, \mu_1 > \mu_{c2}] \\ &\quad= \begin{cases} \frac{r_2^n e^{-r_0^n \sigma^2 \mu_{c2}} - r_0^n e^{-r_2^n \sigma^2 \mu_{c2}}}{r_2^n - r_0^n} & ; \quad r_0 \neq r_2 \\ \left(r_0^n \sigma^2 \mu_{c2} + 1\right) e^{-r_0^n \sigma^2 \mu_{c2}} & ; \quad r_0 = r_2. \end{cases} \end{aligned} \tag{4}$$

The probability of success with fixed DF, then, is approximated by substituting (4) in (3).

3.3. Adaptive Decode-and-Forward

Adaptive DF (ADF) is different from fixed DF only in that if the source-relay transmission is unsuccessful, the source retransmits to the destination. The success probability with adaptive DF can be written as

$$\begin{aligned} P_{s,\mathrm{ADF}} &= \Pr[\mu_{0,2} > \mu_{c2},\, \mu_1 > \mu_{c2}] \\ &\quad + \Pr[\mu_{0,0} > \mu_{c2},\, \mu_1 \leq \mu_{c2}] \end{aligned}$$

where $\mu_{0,0}$ denotes the combined SNR of the source's two transmissions. The first term is identical to (3). The second term can be written as

$$\begin{aligned} &\Pr[\mu_1 \leq \mu_{c2}]\Pr[\mu_{0,0} > \mu_{c2} \,|\, \mu_1 \leq \mu_{c2}] \\ &\quad= \Pr[\mu_1 \leq \mu_{c2}]\Pr[\mu_{0,0} > \mu_{c2}] \\ &\quad= \left(1 - e^{-\sigma^2 r_1^n \mu_{c2}}\right) e^{-\sigma^2 r_0^n \mu_{c2}/2} \end{aligned}$$

where we have used the fact that the combined SNR, $\mu_{0,0} = 2\Gamma_0/\sigma^2$, is independent of μ_1.

3.4. Amplify-and-Forward

In fixed amplify-and-forward transmission, the received signals of the source-destination, source-relay, and relay-destination links, respectively, are

$$\begin{aligned} y_0 &= \alpha_0 s + z_0 \\ y_1 &= \alpha_1 s + z_1 \\ y_2 &= \alpha_2 \beta y_1 + z_2 \\ &= \alpha_1 \alpha_2 \beta s + \alpha_2 \beta z_1 + z_2 \end{aligned}$$

where β is the amplification factor. The output of the maximal ratio combiner at the destination is

$$y_{0,2} = \frac{\alpha_0^*}{\sigma^2} y_0 + \frac{\alpha_1^* \alpha_2^* \beta}{\sigma^2 \left(|\alpha_2|^2 \beta^2 + 1\right)} y_2$$

and it is straightforward to show that its SNR is

$$\begin{aligned} \mu_{\mathrm{AF}} &= \frac{1}{\sigma^2}\left(\Gamma_0 + \frac{\Gamma_1 \Gamma_2 \beta^2}{1 + \Gamma_2 \beta^2}\right) \\ &= \frac{1}{\sigma^2}\left(\Gamma_0 + \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2 + \sigma^2}\right) \end{aligned} \quad (5)$$

where, in the second line, we have set $\beta = \sqrt{1/\left(\Gamma_1 + \sigma^2\right)}$ to meet a constant power constraint.

The SNR, μ_{AF}, is a non-trivial function of three exponentially distributed variates, and obtaining its distribution function is somewhat involved. As shown below, it can be expressed in terms of a single-dimension integration.

The distribution of μ_{AF} is obtained in two steps. First, let U equal the second term in the parentheses in (5). Conditioned on $\Gamma_2 = \gamma$, the conditional distribution of U is

$$F_U(u|\gamma) = \begin{cases} 1 - \exp\left[-r_1^n \frac{u(\gamma+\sigma^2)}{\gamma - u}\right] & ; \quad 0 < u < \gamma \\ 1 & ; \quad u \geq \gamma. \end{cases}$$

Unconditioning with $f_{\Gamma_2}(\gamma) = r_2^n e^{-r_2^n \gamma}$ for $\gamma > 0$, we have for $u > 0$

$$\begin{aligned} F_U(u) &= \int_0^\infty F_U(u|\gamma) f_{\Gamma_2}(\gamma) d\gamma \\ &= 1 - r_2^n e^{-\left(r_1^n + r_2^n\right)u} \\ &\quad \cdot \int_0^\infty \exp\left[-r_1^n \frac{u\left(u + \sigma^2\right)}{t} - r_2^n t\right] dt \\ &= 1 - e^{-\left(r_1^n + r_2^n\right)u} g(u) K_1(g(u)) \end{aligned} \quad (6)$$

where $g(u) = 2\sqrt{r_1^n r_2^n u\left(u + \sigma^2\right)}$, $K_1(\cdot)$ is the first-order modified Bessel function of the second kind, the second line was obtained by the change of variable $t = \gamma - u$, and the third line was obtained using (3.324.1) in [5, p. 334]. A result analogous to (6) was also shown in [6].

Since U is independent of Γ_0, the distribution of $\mu_{\mathrm{AF}} = (\Gamma_0 + U)/\sigma^2$ is obtained by convolution:

$$F_{\mathrm{AF}}(\mu) = \int_0^{\sigma^2 \mu} F_U(u) r_0^n e^{-r_0^n\left(\sigma^2 \mu - u\right)} du. \quad (7)$$

Finally, substituting (6) in (7), gives

$$\begin{aligned} F_{\mathrm{AF}}(\mu) &= 1 - e^{-\sigma^2 r_0^n \mu} \\ &\quad - r_0^n e^{-\sigma^2 r_0^n \mu} \int_0^{\sigma^2 \mu} e^{-\left(r_1^n + r_2^n - r_0^n\right)u} g(u) K_1(g(u)) du. \end{aligned}$$

The success probability of AF, then, is just $P_{s,\mathrm{AF}} = 1 - F_{\mathrm{AF}}\left(\mu_{c2}\right)$.

3.5. Hybrid Amplify-and-Forward/Decode-and-Forward

In the hybrid AF/DF protocol, the relay detects whether it successfully receives the source's transmission (e.g., using a CRC). If successful, it re-encodes and transmits the message to the destination as in the DF protocol. If the source's transmission is not correctly decoded, the relay amplifies and forwards the message as in the AF protocol.

The success probability with hybrid AF/DF can be written as

$$\begin{aligned} P_{s,\mathrm{hyb}} &= \Pr\left[\mu_{0,2} > \mu_{c2},\, \mu_1 > \mu_{c2}\right] \\ &\quad + \Pr\left[\mu_{\mathrm{AF}} > \mu_{c2},\, \mu_1 \leq \mu_{c2}\right]. \end{aligned}$$

The first term is identical to (3). The second term can be written in terms of a conditional probability of μ_{AF}:

$$\begin{aligned} &\Pr\left[\mu_{\mathrm{AF}} > \mu_{c2},\, \mu_1 \leq \mu_{c2}\right] \\ &\quad = \Pr\left[\mu_1 \leq \mu_{c2}\right] \Pr\left[\mu_{\mathrm{AF}} > \mu_{c2} \,|\, \mu_1 \leq \mu_{c2}\right] \\ &\quad = \left(1 - e^{-\sigma^2 r_1^n \mu_{c2}}\right) \Pr\left[\mu_{\mathrm{AF}} > \mu_{c2} \,|\, \Gamma_1 \leq \sigma^2 \mu_{c2}\right]. \end{aligned} \quad (8)$$

Following a similar approach as in Section 3.4, one can derive the distribution of μ_{AF} conditioned on the event that $\Gamma_1 \leq \sigma^2 \mu_{c2}$. The conditional distribution of the intermediate variate U can be shown to be

$$\begin{aligned} &F_U\left(u \,|\, \Gamma_1 \leq \sigma^2 \mu_{c2}\right) \\ &\quad = 1 - e^{-r_2^n \gamma^*} + \frac{1}{1 - e^{-r_1^n \sigma^2 \mu_{c2}}} \\ &\qquad \cdot \int_{\gamma^*}^\infty \left\{1 - \exp\left[-r_1^n \frac{u\left(\gamma + \sigma^2\right)}{\gamma - u}\right]\right\} r_2^n e^{-r_2^n \gamma} d\gamma \end{aligned}$$

where $\gamma^* = u\sigma^2(1 + \mu_{c2})/(\sigma^2 \mu_{c2} - u)$. Convolving with the density of Γ_0 and scaling as before (7) gives the the conditional distribution of μ_{AF} as a double-integral, which can be used in (8) to evaluate the success probability for hybrid AF/DF.

4. NUMERICAL RESULTS

We compare the error probability of the protocols analyzed above with each other and with the FER obtained by simulation using the turbo-coded system described in Section 2. Analytical results are evaluated using SNR threshold values pertaining to this code of $\mu_{c1} = -1.3$ dB and $\mu_{c2} = 3.1$ dB, and a path loss exponent of $n = 4$. The soft channel measurements used by the SISO decoder are the log-likelihood ratios (LLRs) of each coded bit, defined for the jth coded bit, c_j, as

$$L_j(\mathbf{y}, \boldsymbol{\alpha}) \triangleq \log \frac{\Pr\left[c_j = 1 \,|\, \mathbf{y}, \boldsymbol{\alpha}\right]}{\Pr\left[c_j = 0 \,|\, \mathbf{y}, \boldsymbol{\alpha}\right]} = \log \frac{\sum_{s: c_j = 1} f\left(\mathbf{y}|s, \boldsymbol{\alpha}\right)}{\sum_{s: c_j = 0} f\left(\mathbf{y}|s, \boldsymbol{\alpha}\right)}$$

where $\mathbf{y}$ and $\boldsymbol{\alpha}$ are the observed channel outputs and known CSI, respectively, the summations are over all (equally likely) points in

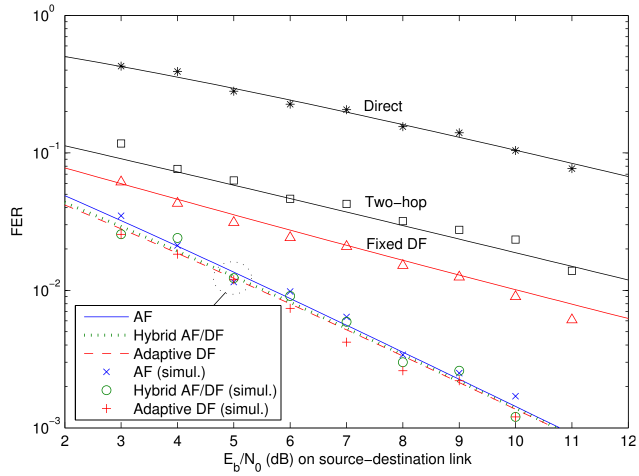

Fig. 2. FER vs. SNR, Rayleigh fading, relay at midpoint

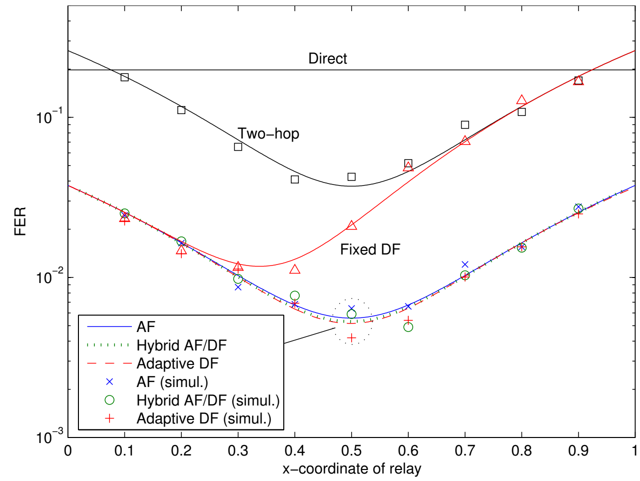

Fig. 3. FER vs. position, Rayleigh fading, $E_b/N_0 = 7$ dB

the signal constellation for which c_j is one or zero, and $f(\mathbf{y}|s, \boldsymbol{\alpha})$ is the conditional density of $\mathbf{y}$. At the destination, the LLR is a function of y_0 and y_2, and their joint conditional density for DF transmission is

$$f_{\mathrm{DF}}(\mathbf{y}|s, \boldsymbol{\alpha}) \propto \exp\left[-\frac{1}{\sigma^2}\left(|y_0 - \alpha_0 s|^2 + |y_2 - \alpha_2 s|^2\right)\right].$$

For AF transmission, the joint density can be shown to be

$$f_{\mathrm{AF}}(\mathbf{y}|s, \boldsymbol{\alpha}) \propto \exp\left[-\frac{1}{\sigma^2}\left(|y_0 - \alpha_0 s|^2 + \frac{|y_2 - \alpha_1\alpha_2\beta s|^2}{|\alpha_2|^2\beta^2 + 1}\right)\right].$$

Fig. 2 plots the error rate as a function of the SNR per bit (E_b/N_0) on the source-destination link, when the relay is located midway between the source and destination. Lines represent analytical results, and symbol markers are simulation results. One observes that direct, two-hop, and fixed DF transmission do not achieve diversity, as expected, whereas AF, ADF, and hybrid AF/DF do exhibit 2nd-order diversity. AF and hybrid transmission are only 0.6 and 0.1 dB worse than ADF, respectively, at a FER of 0.1, and this gap narrows even further at higher SNR. Simulation results show that the SNR-threshold model is useful for predicting performance with turbo codes, although the slight differences in the diversity schemes are only perceptible in the analytical results.

Next, the relay position is varied along the line between the source (at $x = 0$) and destination ($x = 1$). Fig. 3 illustrates the error rate as a function of relay position for a source-destination E_b/N_0 of 7 dB. The diversity-achieving protocols perform very closely across the range, and are optimum when the relay is at the midpoint. Fixed DF converges with these schemes as the relay approaches the source and converges with two-hop transmission as it approaches the destination. These results suggest that in large networks, mutual relaying, where two users relay for each other, is less advantageous than allowing each node to choose its most helpful relay.

Though the diversity protocols perform similarly in Rayleigh fading, they differ in other respects. AF does not require the relay to decode the source's transmission, which can be a major processing savings in the case of turbo codes. Relative to ADF, the hybrid scheme has the advantage of not requiring any feedback from the relay to the source. Furthermore, relative to AF, it avoids noise accumulation when the relay successfully decodes the source's transmission. Preliminary work, not shown here, indicates a more pronounced performance advantage over AF in channels with specular components such as Ricean fading, approaching a 3 dB gain in AWGN channels at the optimum relay position.

We note that optimum combining of the relay and source's transmissions in AF mode requires knowledge by the destination of the cumulative signal attenuation along the relay path, $\alpha_1\alpha_2\beta$. This factor can presumably be estimated from appropriately placed pilots in the source's transmission. The accuracy of this estimate, impacted by noise at both the relay and destination, is beyond the scope of this paper but should be investigated in future work.

5. REFERENCES

[1] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior," *IEEE Trans. Inform. Theory*, vol. 50, no. 12, pp. 3062–3080, Dec. 2004.

[2] E. Zimmermann, P. Herhold, and G. Fettweis, "On the performance of cooperative diversity protocols in practical wireless systems," in *Proc. Vehicular Technology Conf. (VTC)*, vol. 4, Oct. 2003, pp. 2212–2216.

[3] M. Yu and J. Li, "Is amplify-and-forward practically better than decode-and-forward or vice versa?" in *Proc. IEEE Int. Conf. Acoustics, Speech, and Signal Processing (ICASSP)*, vol. 3, March 2005, pp. 365–368.

[4] J. G. Proakis, *Digital Communications*. 4th ed., New York: McGraw-Hill, 2001.

[5] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*. 6th ed., New York: Academic Press, 2000.

[6] V. Emamian, P. Anghel, and M. Kaveh, "Multi-user spatial diversity in a shadow-fading environment," in *Proc. Vehicular Technology Conf. (VTC)*, vol. 1, Sept. 2002, pp. 573–576.