14 Kinetic energy budgets

Examination of the energy budgets of small-scale flows can provide useful insights into the factors driving instability and turbulence. Energy budgets are especially useful to consider in stratified flows such as the atmosphere and oceans, where there is exchange between potential and kinetic energy, and hence changes in the flow can be related to changes in the buoyancy field.

Questions to be asked when examining the energy budgets include:

1. What is the energy source for the instability or turbulence i.e. what aspects of the large-scale flow/buoyancy field lead to generation of instability or turbulence?

2. Where does the small-scale energy go? Does it feedback on the large-scale, or is it "lost" to molecular processes?

3. Once generated, how is small-scale energy redistributed both spatially, and between different components of the small-scale flow?

To examine the small-scale energy budgets, we make a separation of fields into a large scale component and a small-scale component:

$$u = \overline{U} + u' \tag{344}$$

where

$$\int_V u dV = \overline{U} \; ; \; \int_V u' dV = 0 \tag{345}$$

where V is the volume over which the spatial averaging takes place. The small-scale flow is arbitrarily defined to be any flow below the averaging scale. Of course this might include waves as well as instabilities and turbulence, and here we will not be able to distinguish between them. Hereafter we're going to use "turbulence" to mean any small-scale flow (including waves and instabilities)

The turbulent flow could also be defined as the perturbation from a steady flow, as in the Reynolds averaged equations. However, we shall see that the energy equations imply a net change in the energy of the "steady" flow, so the definition of turbulence as the small-scale component rather than the time-fluctuating component is more consistent.

14.1 Kinetic Energy Budget definitions

$$KE = \frac{1}{2}\mathbf{u}.\mathbf{u} \tag{346}$$

$$\text{KE of mean or large-scale flow} = KE_{mean} = \frac{1}{2}\overline{\mathbf{U}}.\overline{\mathbf{U}} \quad (347)$$

$$\text{KE of turbulent or fluctuating flow} = KE_{turb} = \frac{1}{2}\overline{\mathbf{u}'.\mathbf{u}'} \quad (348)$$

14.2 KE of large-scale flow

We begin with the Boussinesq equations, to derive equations for the evolution of KE_{mean}. Consider the portion due to each velocity component separately. In the x-direction, multiply the evolution equation for $\overline{U}$ by $\overline{U}$:

$$\frac{\partial}{\partial t}\left(\frac{\overline{U}^2}{2}\right) + \overline{U}\overline{\mathbf{U}}.\nabla\overline{U} + \overline{U}\overline{\mathbf{u}'.\nabla u'} = -\frac{\overline{U}}{\rho_0}\frac{\partial \overline{P}}{\partial x} + \nu\overline{U}\nabla^2\overline{U} + f\overline{U}\overline{V} \quad (349)$$

Now $\overline{U}(\overline{\mathbf{U}}.\nabla\overline{U}) = \overline{\mathbf{U}}.\nabla(\overline{U}^2/2)$;

$\overline{U}(\overline{\mathbf{u}'.\nabla u'}) = \nabla.(\overline{\mathbf{u}'u'}\overline{U}) - \overline{u'\mathbf{u}'}.\nabla\overline{U}$

and $\overline{U}\nabla^2\overline{U} = \nabla^2(\overline{U}^2/2) - \nabla\overline{U}.\nabla\overline{U}$. Then we can rearrange the U-component of the mean KE equation to give:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \overline{\mathbf{U}}.\nabla\right)\frac{\overline{U}^2}{2} = &-\frac{1}{\rho_0}\frac{\partial}{\partial x}(\overline{P}\overline{U}) + \nu\nabla^2(\frac{\overline{U}^2}{2}) - \nabla.(\overline{\mathbf{u}'u'}\overline{U}) \\ &-\nu\nabla\overline{U}.\nabla\overline{U} + \overline{\mathbf{u}'u'}.\nabla U \\ &+f\overline{U}\overline{V} + \frac{\overline{P}}{\rho_0}\frac{\partial\overline{U}}{\partial x}\end{aligned} \quad (350)$$

The first three terms on the right hand side describe redistribution of mean KE within the volume:

$-\frac{1}{\rho_0}\frac{\partial}{\partial x}(\overline{P}\overline{U})$: pressure work

$\nu\nabla^2(\frac{\overline{U}^2}{2})$: transport by viscous stresses

$-\nabla.(\overline{\mathbf{u}'u'}\overline{U})$: transport by Reynolds stresses.

When integrated over a volume with no flux of KE in or out, these terms are zero.

The 4th and 5th terms represent net sources/sinks of mean KE:

$-\nu\nabla\overline{U}.\nabla\overline{U}$: loss of KE to dissipation;

$\overline{\mathbf{u}'u'}.\nabla U$: transfer of mean KE to the fluctuating/turbulent part of the flow.

The 6th and 7th terms represent transfer of kinetic energy from the $\overline{U}$-component of the flow to the $\overline{V}$- and $\overline{W}$- components.

We can write down similar equations for the time-evolution of $\overline{V}^2$ and $\overline{W}^2$:

$$\left(\frac{\partial}{\partial t} + \overline{\mathbf{U}}.\nabla\right)\frac{\overline{V}^2}{2} = -\frac{1}{\rho_0}\frac{\partial}{\partial y}(\overline{PV}) + \nu\nabla^2(\frac{\overline{V}^2}{2}) - \nabla.(\overline{\mathbf{u}'v'}\overline{V})$$
$$-\nu\nabla\overline{V}.\nabla\overline{V} + \overline{\mathbf{u}'v'}.\nabla V$$
$$-f\overline{UV} + \frac{\overline{P}}{\rho_0}\frac{\partial\overline{V}}{\partial y} \tag{351}$$

$$\left(\frac{\partial}{\partial t} + \overline{\mathbf{U}}.\nabla\right)\frac{\overline{W}^2}{2} = -\frac{1}{\rho_0}\frac{\partial}{\partial z}(\overline{PW}) + \nu\nabla^2(\frac{\overline{W}^2}{2}) - \nabla.(\overline{\mathbf{u}'w'}\overline{W})$$
$$-\nu\nabla\overline{W}.\nabla\overline{W} + \overline{\mathbf{u}'w'}.\nabla W$$
$$+\overline{Wb} + \frac{\overline{P}}{\rho_0}\frac{\partial\overline{W}}{\partial z} \tag{352}$$

The $\overline{V}^2$ equation contains terms analogous to the $\overline{U}^2$ equation, while the $\overline{W}^2$ equation lacks the coriolis term (since we have assumed Coriolis is aligned with the vertical), but includes a buoyancy term, through which large scale potential energy is converted to kinetic energy.

If we sum these three equations, to obtain the evolution equation for $1/2\overline{\mathbf{U}}.\overline{\mathbf{U}}$, and rewrite it in Einstein notation, we have

$$\left(\frac{\partial}{\partial t} + \overline{U}_j\frac{\partial}{\partial x_j}\right)\frac{\overline{U}_i^2}{2} = \frac{\partial}{\partial x_j}\left(-\frac{\overline{P}}{\rho_0}\overline{U}_j\delta_{i,j} + \nu\frac{\partial}{\partial x_j}\frac{\overline{U}_i^2}{2} - \overline{u_j'u_i'}\overline{U}_i\right)$$
$$-\nu\left(\frac{\partial\overline{U}_i}{\partial x_j}\right)^2 + \overline{u_j'u_i'}\frac{\partial}{\partial x_j}\overline{U}_i + \overline{Wb} \tag{353}$$

where the first three terms on the right hand side are once again the transport terms: pressure work, transport by viscous stresses and transport by Reynolds stresses. The 4th term is again the dissipation, and the 5th term represents the transfer of kinetic energy between the mean flow and the turbulent fluctuating flow. This term is known as the **Shear production** term, since the shear in the mean flow (finite gradients in $\overline{U}_i$) leads to production of turbulent kinetic energy (see figure 1). The final term on the right hand side is the large-scale buoyancy production term (see figure 2).

Note that the terms $\overline{P}/\rho_0\partial\overline{U}_i/\partial x_i$, which transfer kinetic energy between the different components of the flow $\overline{U}, \overline{V}, \overline{W}$ vanish from the equation for the total, due to the divergence relation $\nabla.\mathbf{U} = 0$. The Coriolis term similarly does not influence the total kinetic energy, but only its transfer between $\overline{U}$ and $\overline{V}$ components.

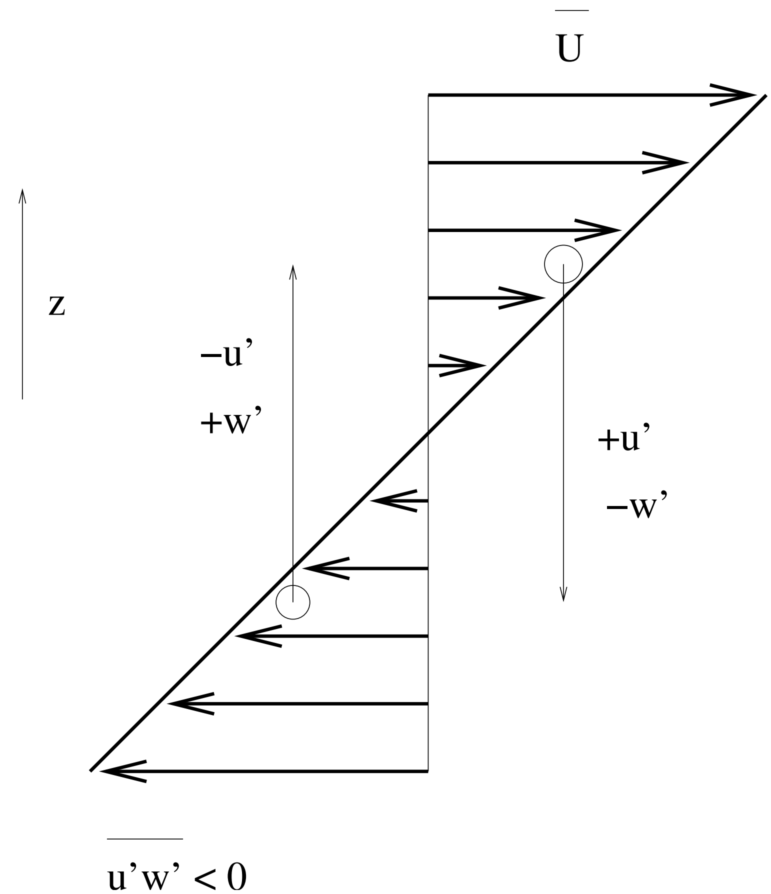

Figure 1: The sign of energy transfer from large-scale to small-scale through shear production: A vertical shear in large scale zonal velocity $\partial\overline{U}/\partial z > 0$ gives rise to a negative shear production term $\overline{w'u'}\partial\overline{U}/\partial z < 0$ if parcels preserve their initial momentum upon exchange. Hence the large scale flow loses energy to the turbulent flow.

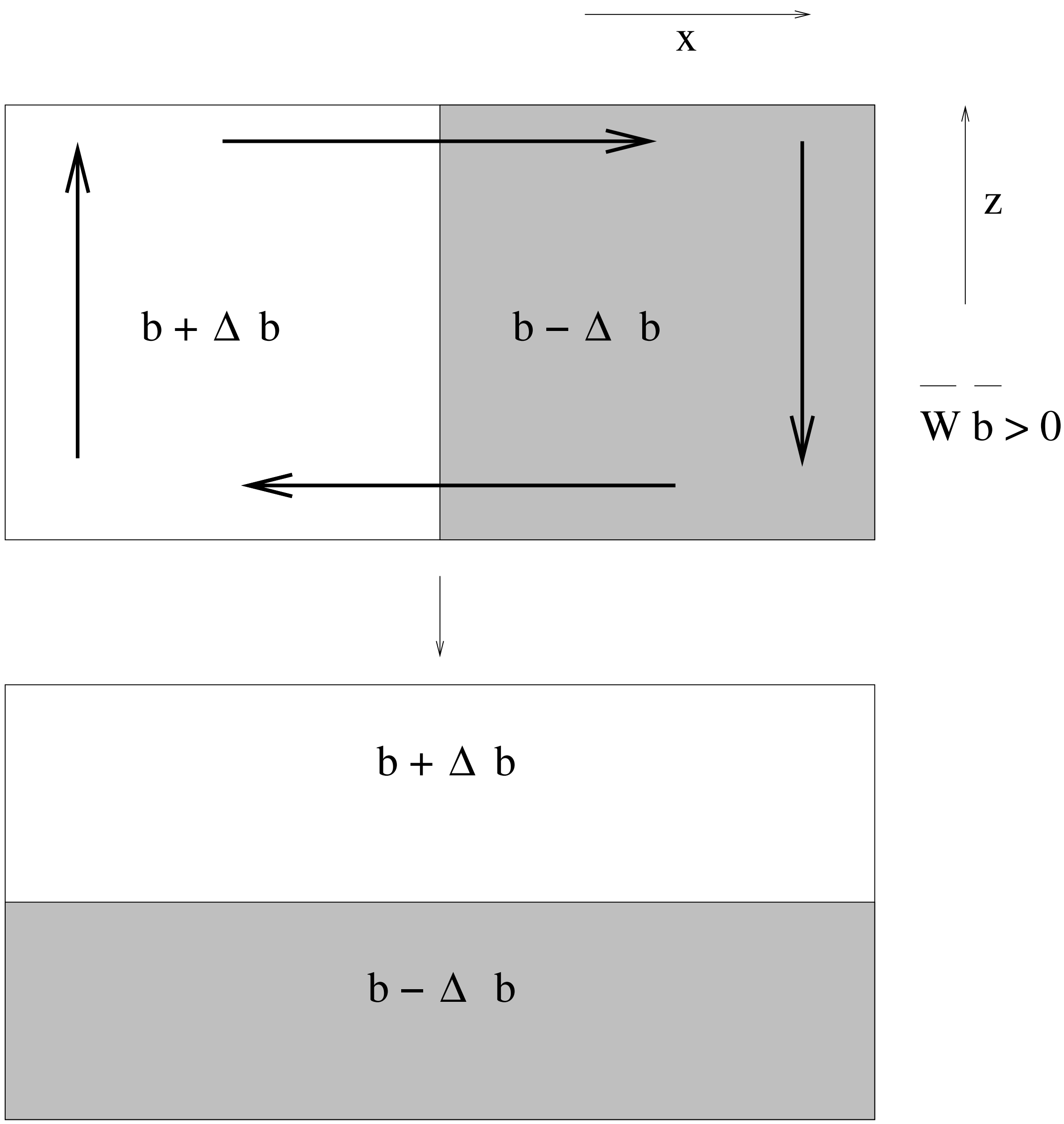

Figure 2: The large-scale reordering of the buoyancy field to give a lower potential energy state implies $\overline{Wb} > 0$, generating kinetic energy.

14.3 KE of turbulent (small-scale) flow

To find the evolution equation for the x-component of the turbulent kinetic energy (TKE) multiply $\partial u'/\partial t = \partial U/\partial t - \partial \overline{U}/\partial t$ by u' and take the spatial average:

$$\left(\frac{\partial}{\partial t} + \overline{\mathbf{U}}.\nabla\right)\frac{\overline{u'^2}}{2} = -\frac{1}{\rho_0}\frac{\partial}{\partial x}\overline{u'p'} + \nu\nabla^2\frac{\overline{u'^2}}{2} - \nabla.\left(\frac{\overline{\mathbf{u}'u'^2}}{2}\right)$$
$$-\nu\overline{\nabla u'.\nabla u'} - \overline{u'\mathbf{u}'}.\nabla\overline{U}$$
$$+f\overline{u'v'} + \frac{1}{\rho_0}\overline{p'\frac{\partial u'}{\partial x}} \tag{354}$$

Comparing with the equation for $\overline{U}^2/2$ we see that once again, there are three transport terms: pressure work, transport by viscous stresses and transport by Reynolds stresses, and loss of TKE to dissipation. The shear production terms appears once again, but with the opposite sign to that in eqn 350 - hence this term represents no net loss of KE but a transfer between mean and turbulent components.

The analogous equations for $\overline{v'^2}$ and $\overline{w'^2}$ are:

$$\left(\frac{\partial}{\partial t} + \overline{\mathbf{U}}.\nabla\right)\frac{\overline{v'^2}}{2} = -\frac{1}{\rho_0}\frac{\partial}{\partial y}\overline{v'p'} + \nu\nabla^2\frac{\overline{v'^2}}{2} - \nabla.\left(\frac{\overline{\mathbf{u}'v'^2}}{2}\right)$$
$$-\nu\overline{\nabla v'.\nabla v'} - \overline{v'\mathbf{u}'}.\nabla\overline{V}$$
$$-f\overline{u'v'} + \frac{1}{\rho_0}\overline{p'\frac{\partial v'}{\partial y}} \tag{355}$$

$$\left(\frac{\partial}{\partial t} + \overline{\mathbf{U}}.\nabla\right)\frac{\overline{w'^2}}{2} = -\frac{1}{\rho_0}\frac{\partial}{\partial z}\overline{w'p'} + \nu\nabla^2\frac{\overline{w'^2}}{2} - \nabla.\left(\frac{\overline{\mathbf{u}'w'^2}}{2}\right)$$
$$-\nu\overline{\nabla w'.\nabla w'} - \overline{w'\mathbf{u}'}.\nabla\overline{W}$$
$$+\overline{w'b'} + \frac{1}{\rho_0}\overline{p'\frac{\partial w'}{\partial z}} \tag{356}$$

The $\overline{w'^2}$ equation contains an additional term: the buoyant production of kinetic energy, representing conversion from potential to kinetic energy.

Adding the contributions due to the 3 velocity components and rewriting in Einstein notation we have

$$\left(\frac{\partial}{\partial t}+\overline{U}_j\frac{\partial}{\partial x_j}\right)\frac{\overline{u_i'^2}}{2}=\frac{\partial}{\partial x_j}\left(-\frac{1}{\rho_0}\overline{u_i'p'}\delta_{i,j}+\nu\frac{\partial}{\partial x_j}\frac{\overline{u_i'^2}}{2}-\overline{u_j'u_i'u_i'}\right)$$
$$-\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}-\overline{u_j'u_i'}\frac{\partial}{\partial x_j}\overline{U}_i+\overline{b'w'} \tag{357}$$

Hence TKE is generated by (a) shear production,

$$P=-\overline{u_j'u_i'}\frac{\partial}{\partial x_j}\overline{U}_i \tag{358}$$

and (b) buoyant production

$$B=\overline{b'w'} \tag{359}$$

and lost through dissipation

$$\epsilon=\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \tag{360}$$

The buoyant production term may be either positive (generation of kinetic energy, loss of potential energy) or negative (loss of KE, increase in PE) (see figure 3).

Now we see the importance of turbulence to the total energy of the system. The viscous terms in the KE of the mean flow can be quite small, so that most KE loss from the mean flow might be due to transfer to the turbulence via the shear production term. Then once in the turbulent regime the KE may either be dissipated, or converted to potential energy via the buoyancy term.

Note that the TKE equations are far from isotropic. Shear production reflects any isotropy in the mean flow, while buoyant production appears only in the $\overline{w'^2}$ equation. The pressure interaction terms (and coriolis terms) transfer the TKE between different velocity components.

If (a) the Turbulence is stationary ($D/Dt(KE)=0$), and (b) we integrate over a volume bounded by surfaces through which there are no energy fluxes, then there is a balance between production and dissipation of TKE:

$$P+B=\epsilon \tag{361}$$

Of course if our small scale motion consists of inviscid instabilities, this steady state balance cannot be reached, and for growing instability we must have $P+B>0$.

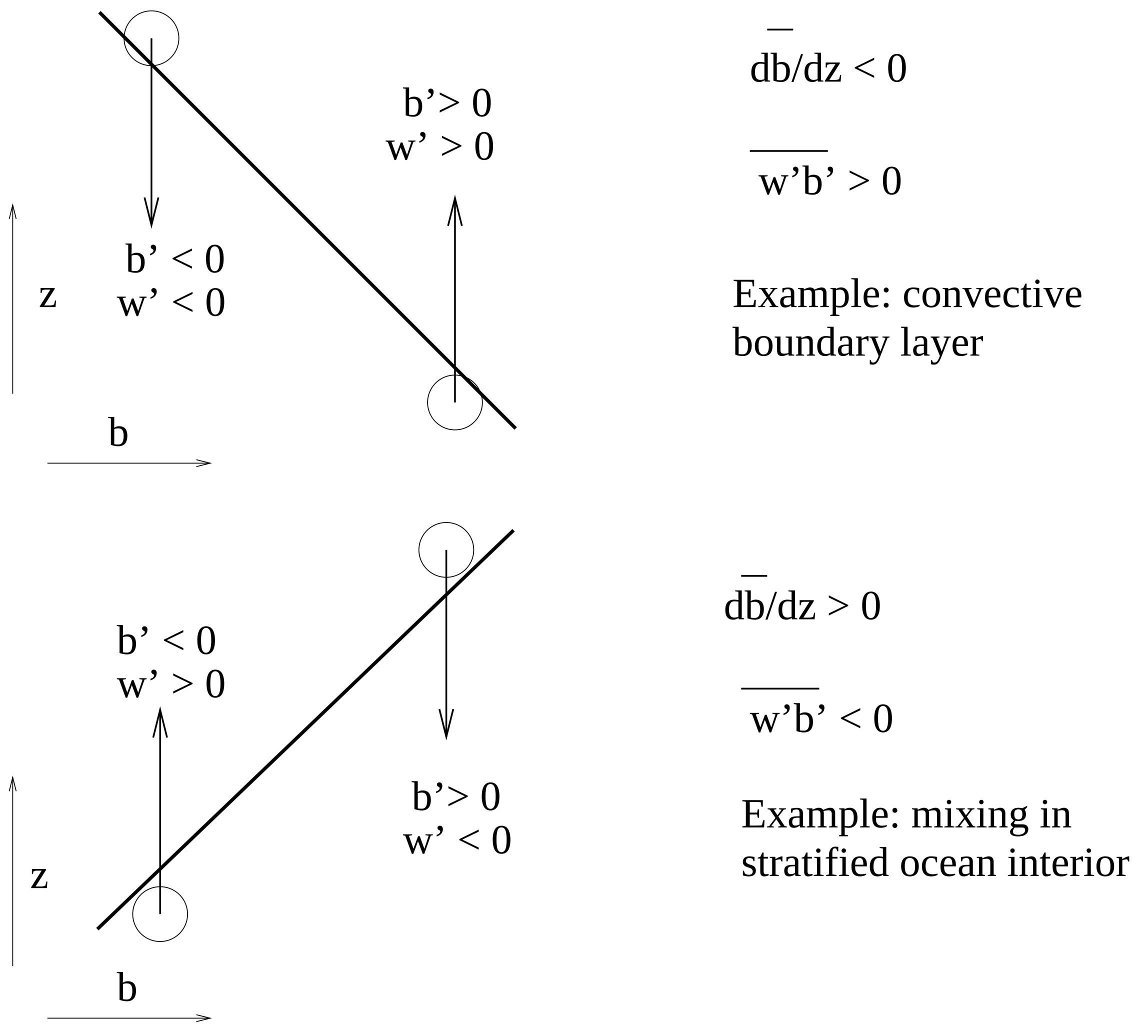

Figure 3: The sign of the turbulent buoyancy production term: $\overline{w'b'} > 0$ (i.e. transfer of PE to KE) occurs if parcels in a statically unstable mean buoyancy gradient $\partial\bar{b}/\partial z < 0$ are exchanged - convection. $\overline{w'b'} < 0$ (i.e. increase in PE by loss of KE) occurs if parcels in a statically stable mean buoyancy gradient $\partial\bar{b}/\partial z > 0$ are exchanged - diapycnal mixing.

14.4 Pure Shear flow

If the large-scale flow consists of a pure shear flow of the form $(U, V, W) = (U(z), 0, 0)$ with no buoyancy forcing, then the TKE shear production term becomes $\overline{u'w'}\partial\overline{U}/\partial z$, and it appears only in the $\overline{u'^2}$ equation. Hence the large-scale flow directly generates TKE only in the x-direction. $\overline{v'^2}$ and $\overline{w'^2}$ are then generated by transfer of TKE from the x-direction via the pressure interaction terms.

14.5 Pure convective flow

If there is no large-scale flow, and turbulence is generated entirely through buoyancy forcing, $P = 0$. The source of TKE is $\overline{w'b'}$, and TKE is directly generated only in the z-direction. Again, $\overline{u'^2}$ and $\overline{v'^2}$ are then generated by transfer of TKE via the pressure interaction terms.

14.6 Flux Richardson number

Obviously an important parameter is the ratio between shear and buoyancy production of TKE, known as the flux Richardson number:

$$R_f = \frac{B}{P} = \frac{\overline{w'b'}}{\overline{u'w'}\partial\overline{U}/\partial z} \tag{362}$$

If $\partial U/\partial z > 0$, then $\overline{u'w'} < 0$ if the flux of momentum is downgradient (positive eddy viscosity). Hence we expect $\overline{u'w'}\partial\overline{U}/\partial z < 0$. $R_f < 0$ therefore if $\overline{w'b'} > 0$ (convective instability, buoyancy generating TKE), and $R_f > 0$ if $\overline{w'b'} < 0$ (stable stratification, loss of TKE to PE).