# Convergence Framework for Trapezoidal-Like Semi-Implicit Time-Stepping Schemes for Multi Body Dynamics with Contacts and Joints


Mihai Anitescu

Argonne National Laboratory

(joint work with Bogdan Gavrea (Penn-U Cluj), Jeff Trinkle (RPI), and Florian Potra (UMBC))

# Problem Description

- Rigid bodies. Generalized positions and velocities: $(q, v)$.
- Joint constraints:

$$\Theta^{(i)}(q) = 0, \quad i = 1, 2, ..., m.$$

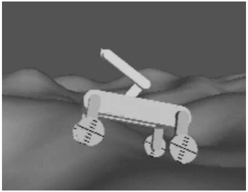
- No inter-penetration (Hard-Constraints):

$$\Phi^{(j)}(q) \geq 0, \quad j = 1, 2, ..., p.$$

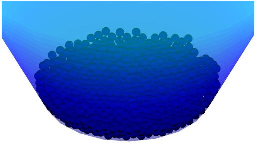
- Compressive normal forces at contact.
- Impact resolution (Poisson hypothesis).
- Coulomb friction.

# Applications

- Robot simulation and design.



- Granular and rock dynamics.



- Other areas: interactive virtual reality, masonry stability analysis, pedestrian and evacuation dynamics, e.t.c.

# Contact and Frictional Constraints

- Contact Constraints:

$$0 \le c_n^{(j)} \perp \Phi^{(j)}(q) \ge 0, \;\; j = 1, ..., p.$$

- Coulomb Friction. Maximal Dissipation Principle.

$$\min_{\bar{\beta}^{(j)} \in \mathbb{R}^d} v^T \bar{D}^{(j)}(q) \bar{\beta}^{(j)} \quad \text{subject to} \;\; ||\beta^{(j)}||_2 \le \mu c_n^{(j)}$$

- Optimality Conditions with Polyhedral Approximation.

$$\begin{array}{lllll} 0 \le D^{(j)}(q)^T v + \lambda^{(j)} e^{(j)} & \perp & \beta^{(j)} \ge 0\,, & j = 1, 2, \ldots, p \\ 0 \;\; \le \mu^{(j)} c_n^{(j)} - {e^{(j)}}^T \beta^{(j)} & \perp & \lambda^{(j)} \ge 0\,, & j = 1, 2, \ldots, p\,, \end{array}$$

where $e^{(j)} = (1, 1, ..., 1)^T \in \mathbb{R}^{m_C^j}$ and $m_C^j$ is the number of facets used in the approximation.

# The Classical Mechanics Framework

$$M(q)\frac{d^2q}{dt^2} - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} - \sum_{j=1}^{p} \left( n^{(j)}(q) c_n^{(j)} + D^{(j)} \beta^{(j)} \right) = k\left(t, q, \frac{dq}{dt}\right)$$

$$\begin{aligned}
\Theta^{(i)}(q) &= 0, \quad i = 1, 2, \cdots, m \\
\Phi^{(j)}(q) &\geq 0, \quad \perp \ c_n^{(j)} \geq 0, \quad j = 1, \ldots, p \\
D^{(j)}(q)^T v + \lambda^{(j)} e^{(j)} &\geq 0, \quad \perp \ \beta^{(j)} \geq 0 \\
\mu^{(j)} c_n^{(j)} - {e^{(j)}}^T \beta^{(j)} &\geq 0, \quad \perp \ \lambda^{(j)} \geq 0, \quad j = 1, \ldots, p
\end{aligned}$$

$M(q)$: generalized mass matrix (symmetric positive definite)
$k(t, q, dq/dt)$: external and inertial forces

- The above DCP doesn't always have a solution in the classical sense (which cases we call Painleve paradoxes).
- We discretize it as if it does, multiply through with time step (move to velocity-impulse) and investigate weak convergence.

# Trapezoidal-like LCP Time-Stepping -(Potra et al.),

- Position update, $\gamma \in [0,1]$:

$$q^{l+1} = q^{(l)} + h\left((1-\gamma)v^l + \gamma v^{l+1}\right)$$

- Velocity $v^{l+1}$ is found from the solution of the MLCP ($\alpha \in [0,1]$):

$$\begin{aligned}
\widetilde{M}v^{l+1} - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} - \sum_{j\in\mathcal{A}} (n^{(j)} c_n^{(j)} + D^{(j)}\beta^{(j)}) &= \widetilde{M}v^l + \widetilde{k} \\
\nu^{(i)T}\left(\alpha v^{(l+1)} + (1-\alpha)v^{(l)}\right) &= 0\,,\ i = 1,2,\ldots m \\
0 \le \rho^{(j)} := n^{(j)T}\left(\alpha v^{(l+1)} + (1-\alpha)v^{(l)}\right) &\perp c_n^{(j)} \ge 0, \quad j \in \mathcal{A} \\
0 \le \sigma^{(j)} := \lambda^{(j)} e^{(j)} + D^{(j)T}\left(\alpha v^{(l+1)} + (1-\alpha)v^{(l)}\right) &\perp \beta^{(j)} \ge 0, \quad j \in \mathcal{A} \\
0 \le \zeta^{(j)} := \mu^{(j)} c_n^{(j)} - {e^{(j)}}^T \beta^{(j)} &\perp \lambda^{(j)} \ge 0, \quad j \in \mathcal{A}
\end{aligned}$$

where $\widetilde{M} = \left(M - \alpha h \widetilde{k}_v^l - \alpha\gamma h^2\, \widetilde{k}_q^l\right)$ and

$$\widetilde{k} = h\left((1-\alpha)k\left(t_l, q^l, v^l\right) + \alpha k\left(t_{l+1}, q^l, v^l\right)\right) + \alpha h^2 \widetilde{k}_q^l v^l$$

# Historical and Computational Considerations for Time-Stepping

- First described by Stewart and Trinkle, 1996, Anitescu and Potra (1997-LCP only), with some similarities with past work of Moreau.
- Subsequently dealt with stiffness (Anitescu and Potra, 2002), constraint stabilization (Anitescu and Hart 2006), nonsmooth shapes – polyhedral bodies(Hart, Ph.D thesis, 2007).
- Is not affected by Painleve paradoxes.
- Does not suffer from the stiffness of the compliant contact or spring and dashpot - or its parameter tuning.
- The solution set of the LCP may be nonconvex (though it is always solvable) but optimization-based relaxation may provide efficient approximations especially for slow granular flow simulations (Alessandro Tasora's talk this morning), which are also quite accurate if the flow is slow.

# Solvability of the Integration Step

- Matrix Form of the Integration Step.

$$\begin{bmatrix} \widetilde{M} & -\widetilde{\nu} & -\widetilde{n} & -\widetilde{D} & 0 \\ \widetilde{\nu}^T & 0 & 0 & 0 & 0 \\ \widetilde{n}^T & 0 & 0 & 0 & 0 \\ \widetilde{D}^T & 0 & 0 & 0 & \widetilde{E} \\ 0 & 0 & \widetilde{\mu} & -\widetilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{l+1,\alpha} \\ \widetilde{c}_\nu \\ \widetilde{c}_n \\ \widetilde{\beta} \\ \widetilde{\lambda} \end{bmatrix} - \begin{bmatrix} \widetilde{M}v^l + \alpha\widetilde{k} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \widetilde{\rho} \\ \widetilde{\sigma} \\ \widetilde{\zeta} \end{bmatrix}$$

$$0 \leq \left[\widetilde{c}_n, \widetilde{\beta}, \widetilde{\lambda}\right], \perp \left[\widetilde{\rho}, \widetilde{\sigma}, \widetilde{\zeta}\right] \geq 0$$

where $v^{l+1,\alpha} = \alpha v^{l+1} + (1-\alpha)v^l$.

- Solvability.

*Theorem (Anitescu, Potra 97, applied for $\alpha = 0$ but also here) If the matrix $\tilde{M}$ is positive definite, then the solution set $\mathcal{L}(q^l, v^l, h, \tilde{k}, \alpha)$ of the above mixed LCP is nonempty. A solution can be found by Lemke's algorithm.*

# Simulations-UMBRA

- The trapezoidal method was implemented in UMBRA, an industrial strength package for multibody dynamics simulations.
- Umbra software package built at Sandia National Laboratories by Eric Gottlieb, Fred Oppel, and Patrick Xavier. The software is now distributed and maintained by Orion, Inc, in Albuquerque, NM.
- Simulations at webpage of Jeff Trinkle.

# What is the big deal with convergence for joint constraints?

- The original proof by Stewart was predicated heavily on a friction cone being properly pointed, and its extension to the joint case is non-trivial.
- Of course one could reduce the joints by changing parameterization. This is an incredibly costly process at times, since it involves a nonlinear solve at every step. It is the reason we study DAEs as opposed to pure ODEs.
- We prove convergence of time-stepping schemes, as actually implemented.

# Strong Formulation in Terms of Friction Cone

$$
\begin{aligned}
\frac{dq}{dt} &= v \\
M\frac{dv}{dt} &= k(q,v) + \rho \\
\Theta^{(i)}(q) = 0, \quad i &= 1,2,\ldots,m \\
\Phi^{(j)}(q) \geq 0, \quad j &= 1,\ldots,p \\
\rho(t) = \bar{\rho}(t) + \sum_{j=1}^{p} \rho^{(j)}(t) &\in \mathcal{FC}(q) \\
\bar{\rho}(t) \in \mathrm{span}\{\nu^{(i)}(q(t)) \; : \; i &= 1,\ldots,m\} \\
||\rho^{(j)}||\Phi^{(j)}(q) = 0, \quad j &= 1,2,\ldots,p
\end{aligned}
$$

We cannot expect that the velocity is even continuous. Therefore, we must consider a weak form of the differential inclusion above.

# Measure Differential Inclusions

We must assign a meaning to inclusions of the form

$$M\frac{dv}{dt} - k(q,v) \in \mathcal{FC}(q).$$

**Definition (Stewart,1998).** If $v$ is a function of bounded variation and $K(\cdot)$ is a convex–set valued mapping we say that $v$ satisfies the differential inclusion

$$\frac{dv}{dt} \in K(t),$$

if, for all continuous $\Phi \geq 0$ with compact support, not identically 0, we have that

$$\frac{\int \Phi(t)dv(t)}{\int \Phi(t)dt} \in \bigcup_{\tau:\Phi(\tau)\neq 0} K(\tau).$$

# Pointed Friction Cone

- closed cone $K \in IR^n$ is pointed if it does not contain any proper linear subspace (has truly a point at 0).
- Main convergence tool in all MDI convergence proofs (Stewart 98, 00, Anitescu 06) Theorem 4 (Stewart 1998b) Suppose that $q_{\hat{n}}(\cdot)$ are continuous, $v_{\hat{n}}(\cdot)$ have uniformly bounded variation and $k_{\hat{n}}(\cdot)$ are uniformly bounded, all on $[0, T]$, and $q_{\hat{n}}(\cdot) \rightarrow q(\cdot)$ uniformly, $v_{\hat{n}}(\cdot) \rightarrow v(\cdot)$ pointwise a.e. and $k_{\hat{n}}(\cdot) \rightarrow k(\cdot)$ pointwise a.e. Suppose also that $K : IR^n \Rightarrow \mathcal{C}(IR^n)$ has closed graph, $\min\{||z|| \ |z \in K(w)\}$ is uniformly bounded and $K(w)$ is pointed for all $w \in IR^n$. Then if

$$\frac{dv_{\hat{n}}}{dt}(t) \in K(q_{\hat{n}}(t)) - k_{\hat{n}}(t)$$

for all $\hat{n}$, the limit satisfies

$$\frac{dv}{dt}(t) \in K(q(t)) - k(t).$$

Here $\mathcal{C}(IR^n)$ denotes all the closed and convex subsets of $IR^n$.

# The Joint Constrained Friction Cone

- Corresponding to each contact define

$$\mathcal{FC}^{(j)}(q) = \left\{ z = \tilde{\nu}\tilde{c}_\nu + n^{(j)} {c_n}^{(j)} + \bar{D}^{(j)}\beta^{(j)} \,\middle|\, {c_n}^{(j)} \geq 0,\ ||\beta^{(j)}||_2 \leq \mu^{(j)} c_n^{(j)} \right\}$$

and the *total friction cone*

$$\mathcal{FC}(q) = \sum_{\Phi^{(j)}(q)=0} \mathcal{FC}^{(j)}(q).$$

- In a similar fashion define the polyhedral approximation

$$\widehat{\mathcal{FC}}(q) = \sum_{\Phi^{(j)}(q)=0} \widehat{\mathcal{FC}}^{(j)}(q).$$

where $\widehat{\mathcal{FC}}^{(j)}(q)$

$$\widehat{\mathcal{FC}}^{(j)}(q) = \left\{ z = \tilde{\nu}\tilde{c}_\nu + n^{(j)} c_n^{(j)} + D^{(j)}\beta^{(j)} \,\middle|\, c_n^{(j)},\ \beta^{(j)} \geq 0,\ ||\beta^{(j)}||_1 \leq \mu^{(j)} c_n^{(j)} \right\}.$$

- None can be pointed – they contain the range of $\tilde{\nu}$

# How does a mathematician solve a problem?

- A physicist and a mathematician sit in a faculty lounge. Suddenly, the coffee machine catches on fire.
- The physicist grabs a bucket and leaps towards the sink, fills the bucket with water and puts out the fire.
- Second day, the same two sit in the same lounge. Again, the coffee machine catches on fire.
- This time, the mathematician grabs the bucket and hand it to the physicist, and reduces the problem to a previously solved one.
- This is our approach: We hand the bucket to Theorem 4.

# Reduced Friction Cone. Regularity Assumption.

- The Reduced Friction Cone (Gavrea, Ph.D. thesis). where $\tilde{\nu}_\perp(q)$ is the orthogonal complement of $\tilde{\nu}(q)$.

$$\mathcal{FC}_r(q) := \left(\tilde{\nu}_\perp(q)\right)^T \mathcal{FC}(q) = \left\{ \left(\tilde{\nu}_\perp(q)\right)^T z \quad : \quad z \in \mathcal{FC}(q) \right\}$$

- We assume that the friction cone is *uniformly pointed*: $\tilde{\nu}(q)$ has a lowest singular value bounded away from 0, and the angle between any two vectors in $\mathcal{FC}_r(q)$ is bounded away from $\pi$.
- Equivalent to

$$\exists c_{\mathcal{FC}} > 0, \quad \text{such that} \quad \tilde{c}_n, \tilde{\beta} \geq 0, \quad \text{and} \quad z = \tilde{\nu}\tilde{c}_\nu + \tilde{n}\tilde{c}_n + \tilde{D}\tilde{\beta} \in \mathcal{FC}(q).$$
$$\Rightarrow ||(\tilde{c}_\nu, \tilde{c}_n, \tilde{\beta})|| \leq c_{\mathcal{FC}} ||z||$$

with $c_{\mathcal{FC}}$ uniformly bounded.

# Solution of MDI reduced cone problem

We say that $q(t)$, $w(t)$ is a reduced weak solution on $[0, T]$ if

1. $w(\cdot)$ is a function of bounded variation on $[0, T]$.
2. $q(\cdot)$ is an absolutely continuous function that satisfies

$$q(t) = q(0) + \int_0^t \widetilde{\nu}_\perp(q(\tau))w(\tau)d\tau, \ \text{ for } \ t \in [0, T].$$

3. The measure $dw(t)$ must satisfy

$$\begin{aligned}\left(\left(\widetilde{\nu}_\perp(q)\right)^T M\widetilde{\nu}_\perp(q)\right)\frac{dw}{dt} \quad &- \quad k_{w,\perp}(t,q,w) \in \mathcal{FC}_r(q),\\ k_{w,\perp}(t,q,w) \quad &\triangleq \quad \left(\widetilde{\nu}_\perp(q)\right)^T k_w(t,q,w)\\ k_w(t,q,w) \triangleq k(t,q,\widetilde{\nu}_\perp(q)w) \quad &- \quad M\left(\left(\frac{\partial}{\partial q}\left(\widetilde{\nu}_\perp(q)w\right)\right)\widetilde{\nu}_\perp(q)w\right)\end{aligned}$$

4. $\Phi^{(j)}(q) \geq 0$, $j = 1, \ldots, p$.
5. Lemma: $q, w$ solve MDI reduced problem $\Leftrightarrow$ $q, w$ solve MDI original problem.

# Uniform Boundedness of the Numerical Velocities

- **Assumptions**
  - The generalized mass matrix $M$ is constant.
  - The number of collisions is uniformly upper bounded as $h \to 0$.
  - The friction cone $\mathcal{FC}(q)$ is uniformly pointed.
  - The applied forces are inertial + *at most linear growth*:

$$
\begin{aligned}
k(t,q,v) &= F(v)v + k_1(t,q,v), \text{ where } F(v) \text{ is skew-symmetric,} \\
k_1(t,q,v) &\leq c_1 + c_2||q|| + c_3||v||.
\end{aligned}
$$

  - We use $\widetilde{k}^l_q = \widetilde{k}^l_{1q}$, $\widetilde{k}^l_v = F(v^l) + \widetilde{k}^l_{1v}$ with $\widetilde{k}^l_{1q}$, $\widetilde{k}^l_{1v}$ uniformly bounded.
  - The parameter $\alpha$ satisfies

$$\frac{1}{2} \leq \alpha \leq 1.$$

- **Conclusion**

  The numerical velocities $v^{l,h}$ are uniformly bounded as $h \to 0$.

# Numerical Positions and Velocities

We define:

- Numerical velocity sequence, $v^{h,\alpha}(\cdot)$,

$$v^{h,\alpha}(t) = \begin{cases} (1-\alpha)v^{l,h} + \alpha v^{l+1,h} & \text{if no collision in } (t_l, t_{l+1}] \text{ and } t \in (t_l, t_{l+1}] \\ v^{l+1,h} & \text{if collision in } (t_l, t_{l+1}] \text{ and } t \in (t_l, t_{l+1}] \end{cases}$$

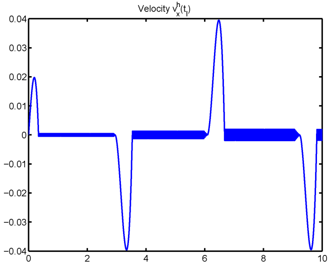
- Numerical position sequence $q^{h,\gamma}(\cdot)$,

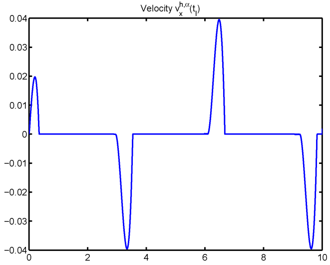
$$q^{h,\gamma}(t)\Big|_{t\in[t_l,t_{l+1}]} = \text{ linear interpolant of } (t_l, q^{l,h}), (t_{l+1}, q^{l+1,h})$$

*Obtaining convergent subsequences*

- Uniform bound for the numerical velocities $\Rightarrow$ existence of a uniform convergent position subsequence.
- Uniform pointedness of the reduced friction cone $\mathcal{FC}_r$ $\Rightarrow$ existence of a pointwise convergent velocity subsequence.

# What is the big deal with the definition of velocities?




- We have not been able to prove or disprove bounded variation of $v^h$. However, behavior of the numerical velocities suggests the latter (see-saw velocity profile). If amplified numerically by Coriolis ... But we could not find any such example at the moent.
- On the positive, we have been able to prove bounded variation of $v^{h,\alpha}$, and numerical results also suggest that its bounded variation is $O(1)$ smaller than the one of $v^{h,l}$.

# The Convergence Result

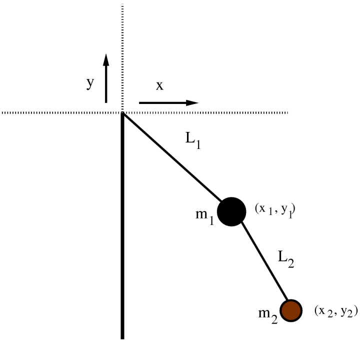
**Assumptions** $(1/2 \leq \alpha \leq 1,\ \gamma \in [0,1])$**.**

- The mass matrix M is a constant, symmetric positive definite matrix.
- The constraint data $\widetilde{\nu}(q), \widetilde{n}(q), \widetilde{D}(q)$ is sufficiently smooth.
- The number of collisions is uniformly upper bounded as $h \to 0$. All collision are plastic.
- The external forces $k_1(t, q, v)$ increase at most linearly with the velocity and position. The approximations $\widetilde{k}_{1q}$, $\widetilde{k}_{1v}$ are bounded.
- The friction cone $\mathcal{FC}(\cdot)$ is uniformly pointed.

**Conclusion.** There exists a subsequence $h_k \to 0$, such that:

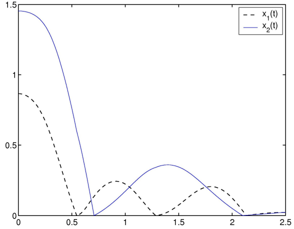
- $q^{\gamma,h_k}(\cdot) \to q(\cdot)$ uniformly in $[0, T]$,
- $v^{\alpha,h_k}(\cdot) \to v(\cdot)$ pointwise a.e. in $[0, T]$,
- $dv^{\alpha,h_k}(\cdot) \to dv(\cdot)$ weak$^*$ as Borel measures in $[0, T]$, and every $(q(\cdot), v(\cdot))$ is a solution of the corresponding MDI.

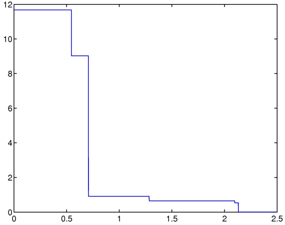
# Double Pendulum with a Long Wall




$m_1 = m_2 = 1,\ L_1 = L_2 = 1$
$(x_1^0, y_1^0) = (\sin(\pi/3), -\cos(\pi/3))$
$(x_2^0, y_2^0) = (x_1^0, y_1^0) + (\sin(\pi/5), -\sin(\pi/5))$
$(\dot{x}_1^0, \dot{y}_1^0) = (0, 0),\ (\dot{x}_2^0, \dot{y}_2^0) = (0, 0)$
$\mathrm{Rest.Coeff}:\ e_1 = e_2 = 0.1$. Final Time: $T_f = 2.5(s)$

# Double Pendulum: Numerical Results



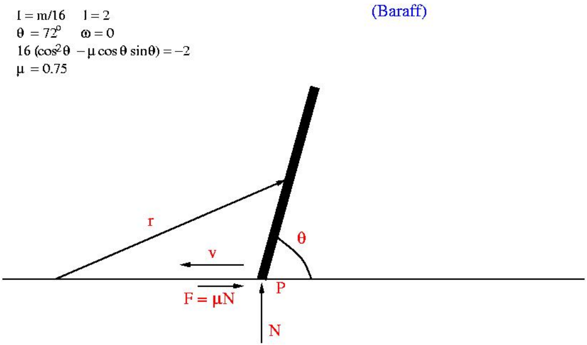



| $h$ | $E_h$ | $R_{0,h}$ | $E_{1,h}$ | $R_{1,h}$ | $E_{2,h}$ | $R_{2,h}$ |
|---|---|---|---|---|---|---|
| $2^{-5}$ | 2.38e-03 | N/A | -1.27e-03 | N/A | -3.18e-03 | N/A |
| $2^{-6}$ | 6.14e-04 | 3.88e+00 | -3.14e-04 | 4.04e+00 | -8.29e-04 | 3.84e+00 |
| $2^{-7}$ | 1.54e-04 | 3.97e+00 | -7.82e-05 | 4.01e+00 | -2.14e-04 | 3.88e+00 |
| $2^{-8}$ | 3.95e-05 | 3.91e+00 | -1.95e-05 | 4.00e+00 | -5.48e-05 | 3.90e+00 |
| $2^{-9}$ | 1.01e-05 | 3.92e+00 | -4.88e-06 | 4.00e+00 | -1.37e-05 | 3.99e+00 |
| $2^{-10}$ | 2.42e-06 | 4.17e+00 | -1.22e-06 | 4.00e+00 | -3.44e-06 | 3.99e+00 |
| $2^{-11}$ | 6.15e-07 | 3.93e+00 | -3.05e-07 | 4.01e+00 | -8.60e-07 | 4.00e+00 |

# Conclusions

- We have shown convergence in the MDI sense for a family of linearly implicit time-stepping schemes.
- For the first time, the convergence analysis includes joint constraints.
- Inovative choice of the velocity sequence.
- Future work: Full cones (non-approximated), iterative schemes, extending the convergence theorem.

# Painleve's Paradox (1895)



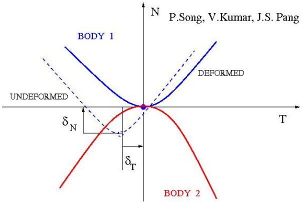

Contact constraint: $y_P \geq 0$

$$\ddot{y}_P = \left(-g + \frac{l}{2}\dot{\theta}^2 \sin\theta\right) + N\left(\frac{1}{m} + \frac{l^2}{4I}(\cos^2\theta - \mu \sin\theta\cos\theta)\right)$$

Baraff: $\ddot{y}_P = -g - \frac{N}{m}$

Painleve Paradox: No classical solutions!

# Modelling Contacts. Compliant Contact Models -Stiff!




- The state is augmented with local deformations and their derivatives, $\delta = (\delta_N, \delta_T)$, $\dot{\delta} = (\dot{\delta}_N, \dot{\delta}_T)$.
- Normal ($\lambda_N$) and tangential ($\lambda_T$) forces depend on $\delta$ and $\dot{\delta}$. For a linear viscoelastic model (the Kelvin–Voigt model),

$$\begin{aligned}\lambda_N &= K_N\delta_N + C_N\dot{\delta}_N\\ \lambda_T &= K_T\delta_T + C_T\dot{\delta}_T\end{aligned}$$

- Compliance in the tangential direction will result in continuity of $\lambda_T$ in situations like "stick–slip" transitions.