Don’t open that envelope: solutions using p-boxes

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Are the problems bizarre?

- Well, they're not *that* bizarre. Problems this simple arise in real risk analyses all the time.
- In the real world, uncertainty comes in various flavors (probability distributions and intervals), and we're often asked to combine them together.
- If we can't agree on the answers for these stylized pseudo-problems, what hope do we have for agreement on messier problems?

We took the problems seriously

Answered all the problems

Used one method for all of them

Rigorous (not approximate)

Best possible (couldn’t be any tighter)

…but we cheated on the black box problem

Uncertainty calculi

Worst case analysis
Interval arithmetic
Moment propagation
Probability theory (Monte Carlo, analytic)
Second-order Monte Carlo
Dempster-Shafer theory
Theory of random sets
Probability bounds analysis
Imprecise probability
Fuzzy arithmetic
Hybrid arithmetic
O-theory

Types of uncertainty

- Variability (aleatory uncertainty)
- Incertitude (epistemic uncertainty)
- Vagueness (gradations in definitions)
- Ambiguity, confusion

Variability

- Arises from natural stochasticity
- Variation due to
  - temporal fluctuations in temperature
  - inhomogeneity within materials
  - manufacturing inconsistencies
  - distribution of conditions, etc.
- Not reducible by empirical effort

Incertitude

- Arises from incomplete knowledge
- Ignorance due to
  - limited sample size
  - detection limits
  - possible biases in sampling design
  - use of surrogate data
- Reducible with empirical effort

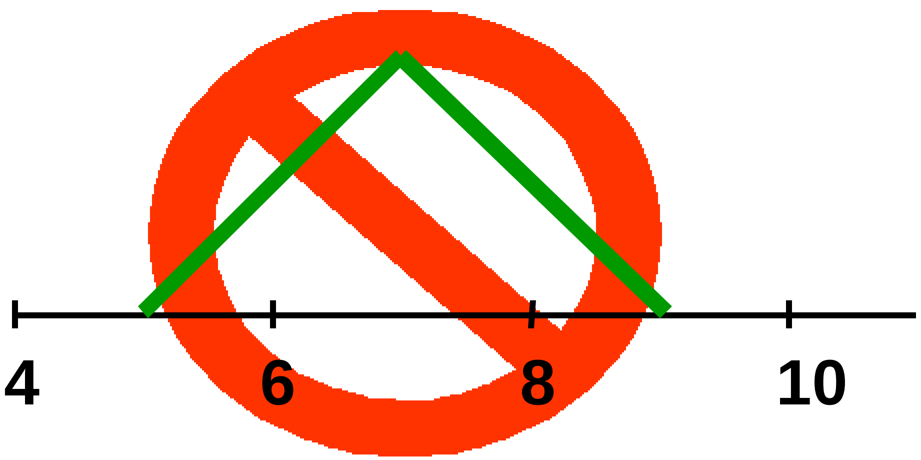
Incertitude

Suppose

A is in [2, 4]

B is in [3, 5]

What can be said about the sum *A*+*B*?

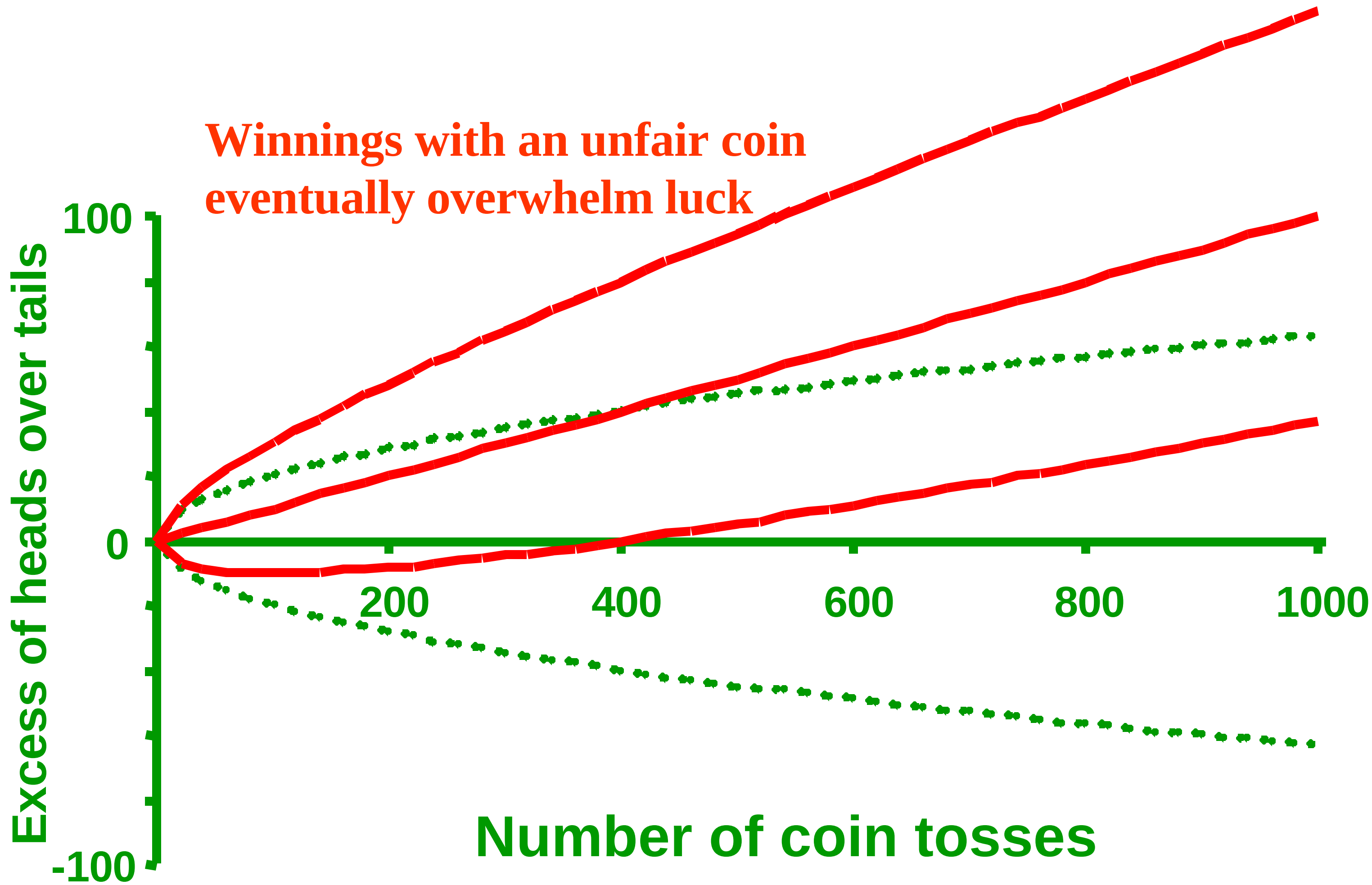


The right answer for risk analysis is [5,9]

Why probability is overconfident

- The approach used in Monte Carlo methods (including 2MC) is reasonable only if the bias in measurement errors is zero
- This seems highly dubious, however, when the measurements are made with the same technique, in the same lab, around the same time, by the same observer, etc.
- Measurements are likely to share biases, so interval analysis methods are needed

Measurement error accumulates



Must be treated *differently*

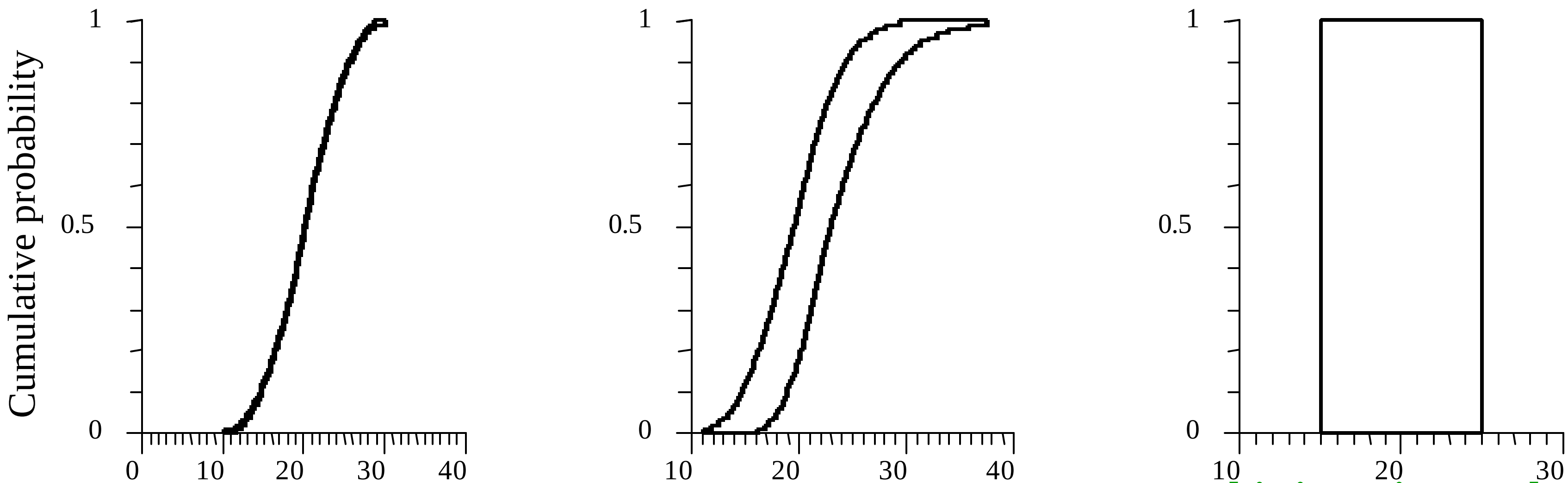
- Variability should be modeled as randomness with the methods of probability theory

- Incertitude should be modeled as ignorance with the methods of interval analysis

- Imprecise probabilities can do both at once

Probability box (p-box)

- Bounds on a cumulative distribution function (CDF)
- Envelope of a Dempster-Shafer structure (Yager 1986)
- Generalizes probability distributions and intervals



This is an interval, *not* a uniform distribution

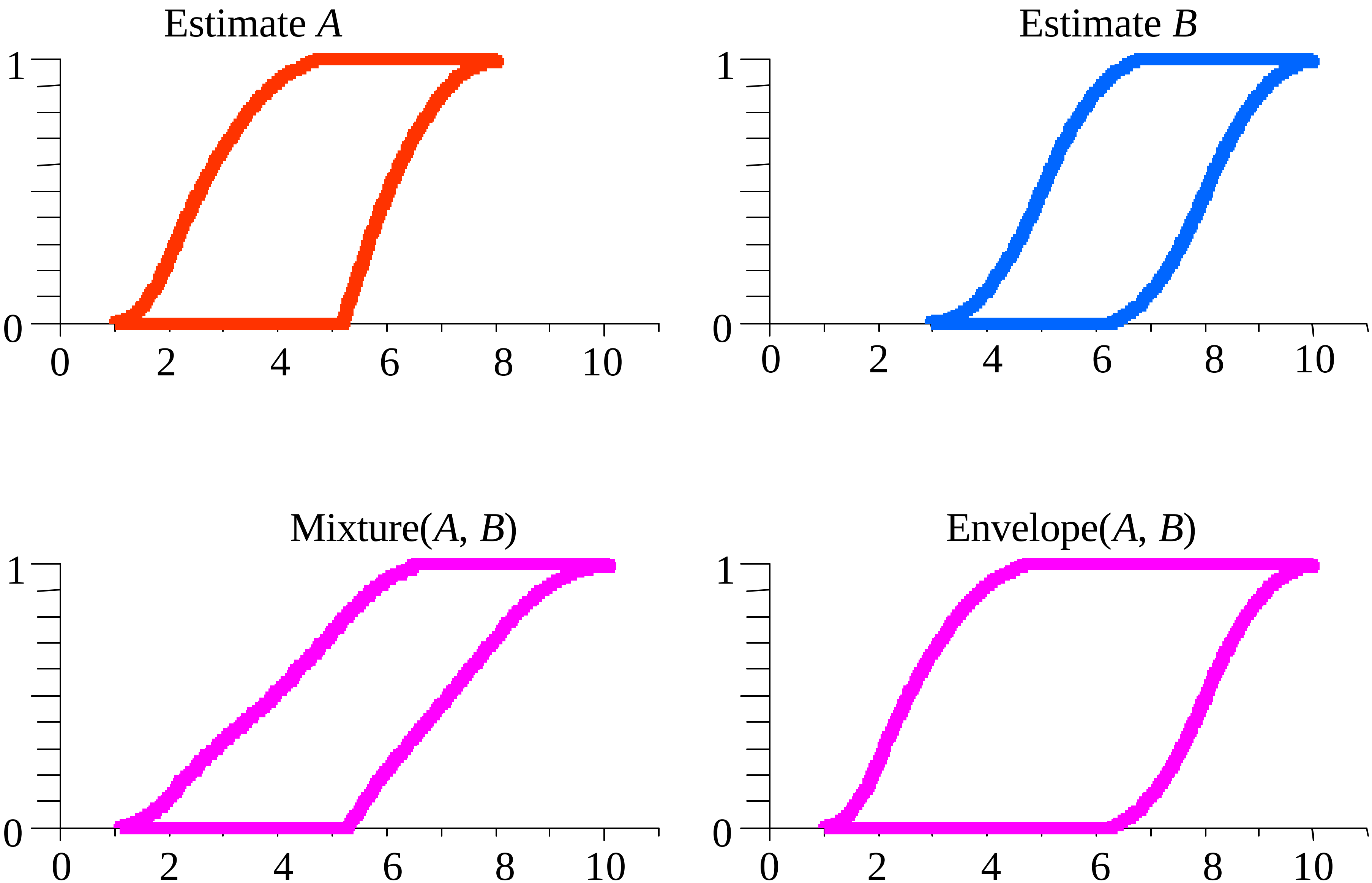
Probability bounds analysis

- P-box arithmetic (+, –, ×, ÷, ^, min, max)
- Transformations (exp, ln, sin, tan, abs, sqrt, etc.)
- Backcalculation (deconvolutions, updating)
- Comparisons (<, , >, , ⊆)
- Logical operations (and, or, not, if, Bonferroni)
- Other operations (envelope, mixture, etc.)

Pooling experts

- Intersection
- Enveloping
- Mixture
- Robust Bayes
- Various averages
- Various versions of Dempster’s rule
- Techniques that account for sample error
- Other aggregation methods (fuzzy, etc.)

These aggregation operators commute with the convolutions of the challenge problems

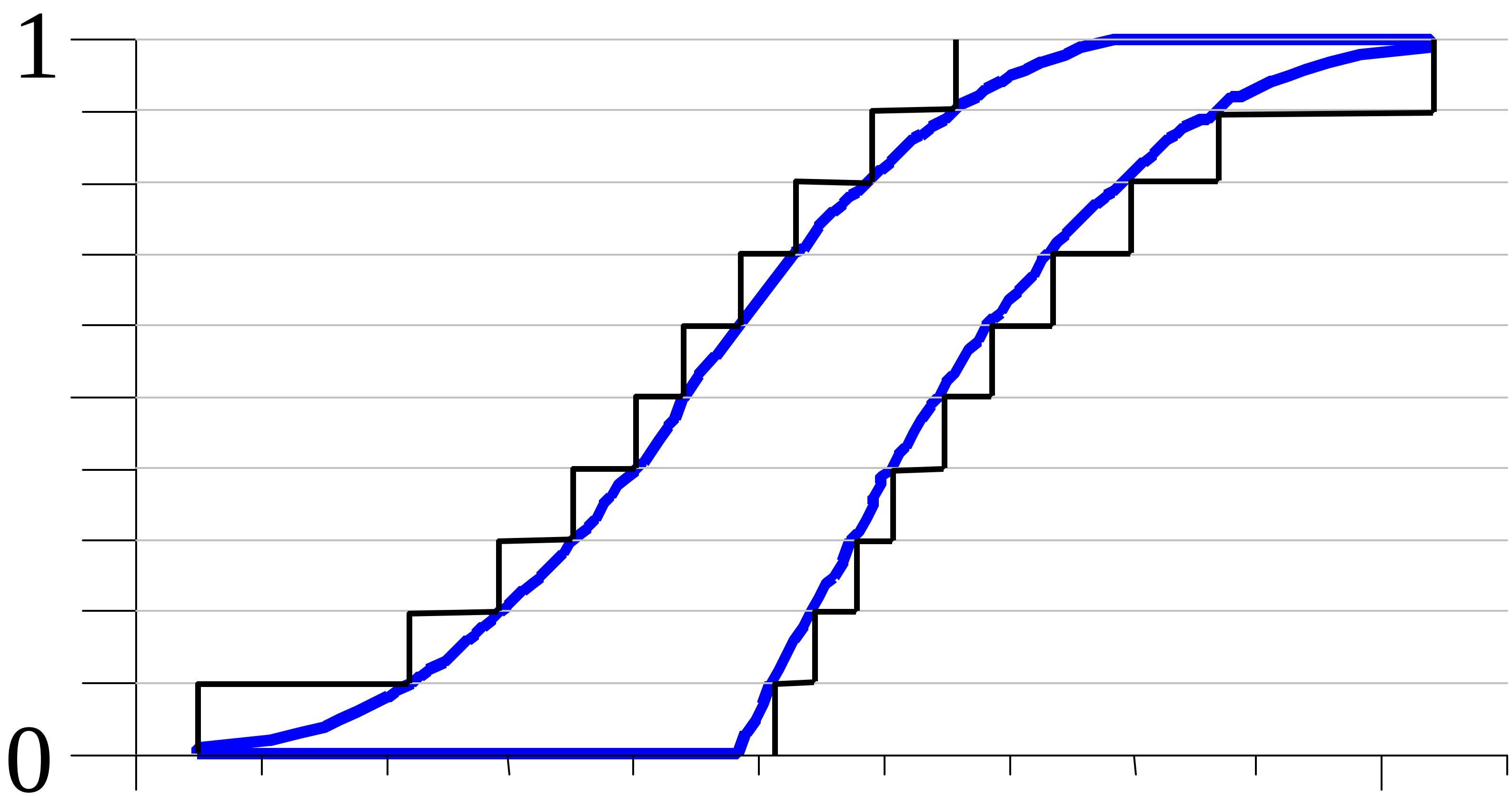


Estimate A
Estimate B
Mixture(A, B)
Envelope(A, B)
1
0
0
2
4
6
8
10

Arithmetic under independence

- Envelopes of CDFs
- Discretization (rigorous, equi-probable)
- Cartesian product
- Condensation
- Simultaneous moment propagation

Discretization (and condensation)



Cartesian product

$A + B$ *independent*	$A \in [1, 2]$ $p_1 = 1/3$	$A \in [2, 4]$ $p_2 = 1/3$	$A \in [3, 5]$ $p_3 = 1/3$
$B \in [2, 4]$ $q_1 = 1/3$	$A+B \in [3, 6]$ prob $= 1/9$	$A+B \in [4, 8]$ prob $= 1/9$	$A+B \in [5, 9]$ prob $= 1/9$
$B \in [3, 5]$ $q_2 = 1/3$	$A+B \in [4, 7]$ prob $= 1/9$	$A+B \in [5, 9]$ prob $= 1/9$	$A+B \in [6, 10]$ prob $= 1/9$
$B \in [4, 6]$ $q_3 = 1/3$	$A+B \in [5, 8]$ prob $= 1/9$	$A+B \in [6, 10]$ prob $= 1/9$	$A+B \in [7, 11]$ prob $= 1/9$

What about other dependencies?

- Independent
- Perfectly positive (maximal correlation)
- Opposite (minimal correlation)
- Specified copula
- Positively associated
- Negatively associated
- Particular correlation coefficient
- Unknown dependence

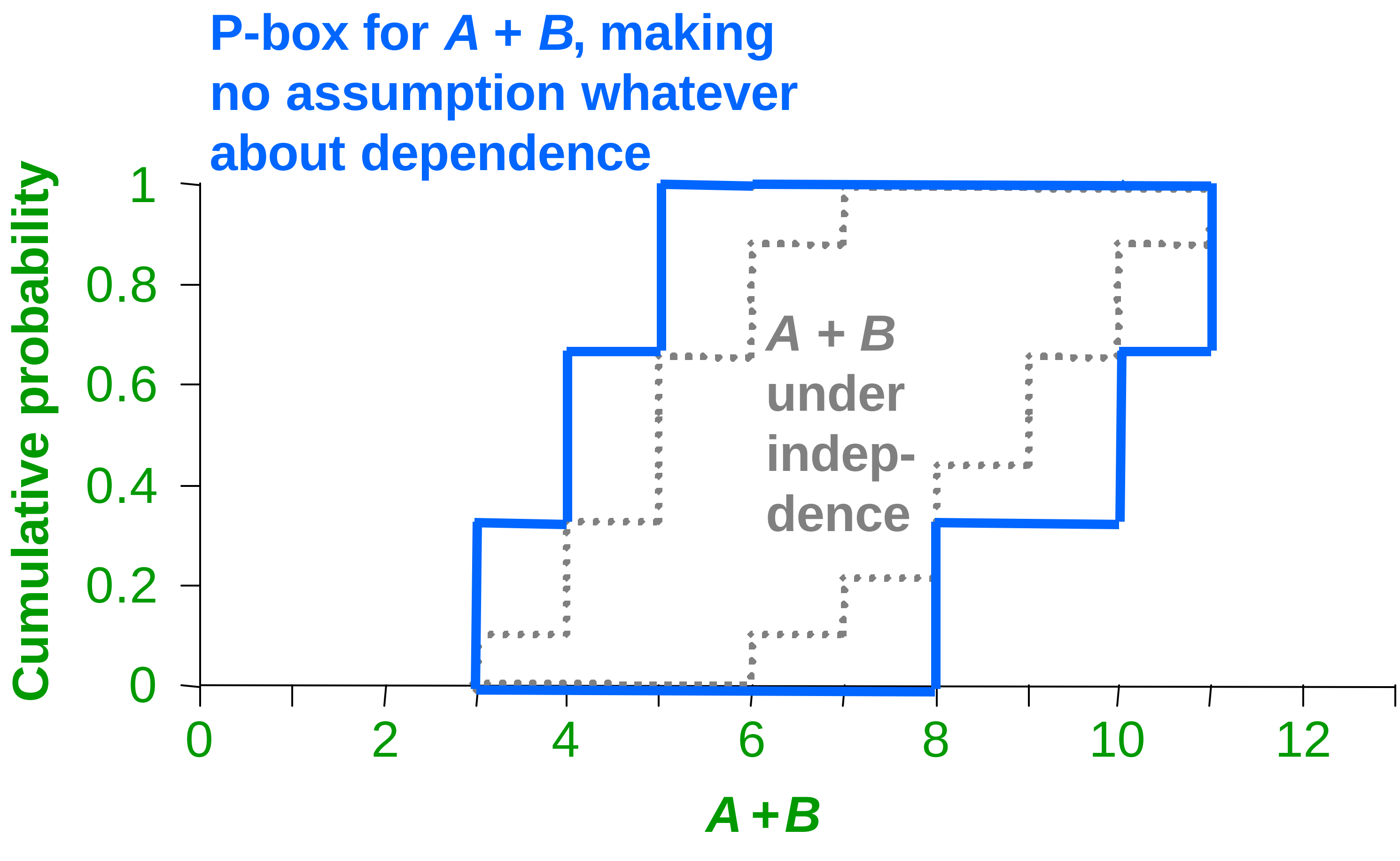
Don’t fully specify the dependence

If dependency is unknown

(Or if, say, only a correlation coefficient can be specified)

- Frank, Nelsen and Sklar gave a way to compute the optimal answer directly from marginals
- Williamson and Downs extended it to p-boxes
- Berleant described a brute-force strategy based on linear programming

Example



P-box for A + B, making no assumption whatever about dependence
A + B under indep- dence
Cumulative probability
1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
12
A+B

Lessons about dependence

- Dependence assumptions can sometimes make a big difference, especially in the tails
- Assuming independence without evidence is wishful thinking
- Counterfactual assumptions are stupid (body size independent of skin surface area)
- Unless the correlation coefficient is close to ± 1, knowing it tells you little extra information about the convolution

Advantages of bounding

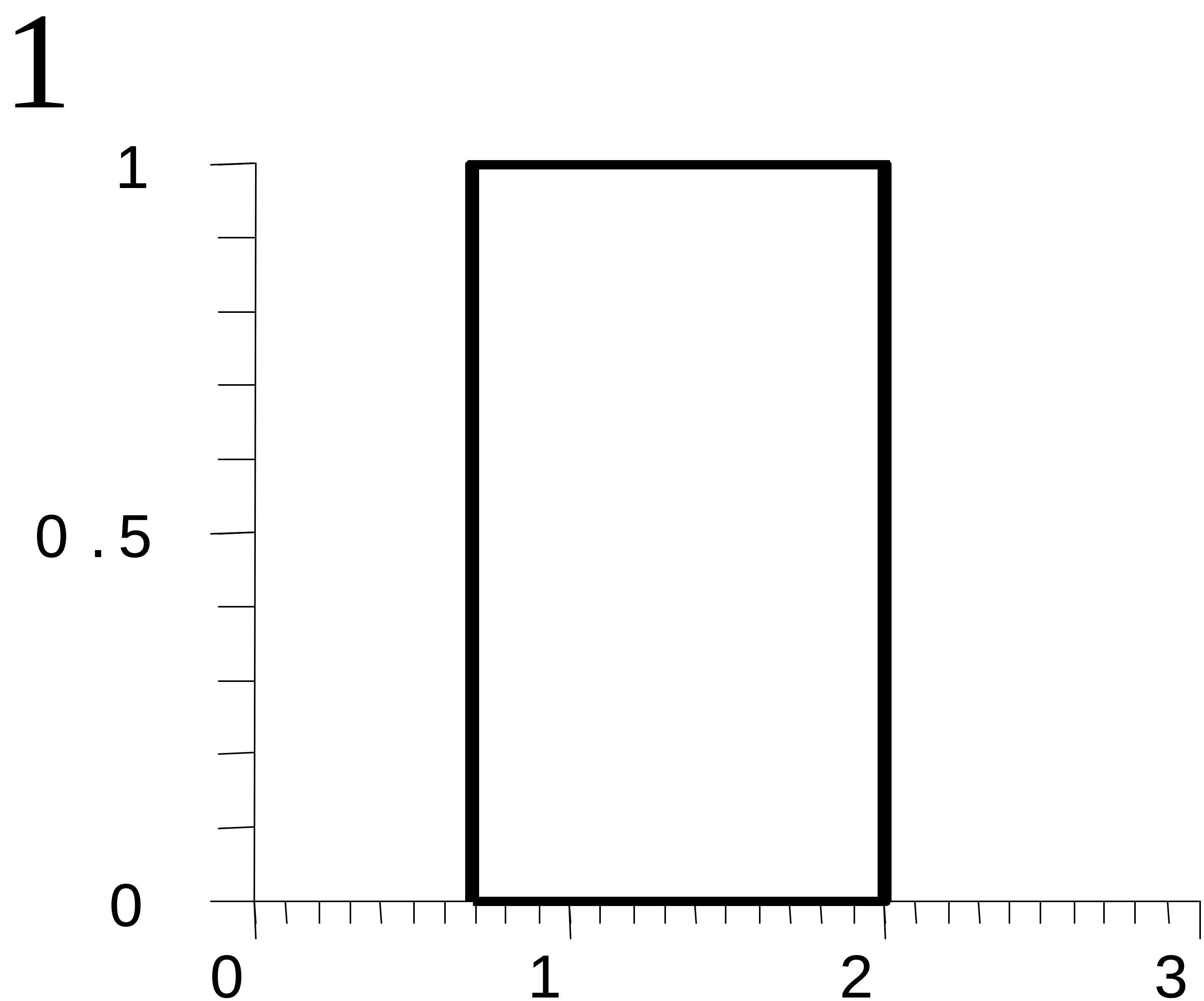
- Rigorous rather than approximate
- Usually much easier to compute (no integrals)
- Possible even when estimates are impossible
- Simple to combine
- Often sufficient to specify a decision

Best-practice methods

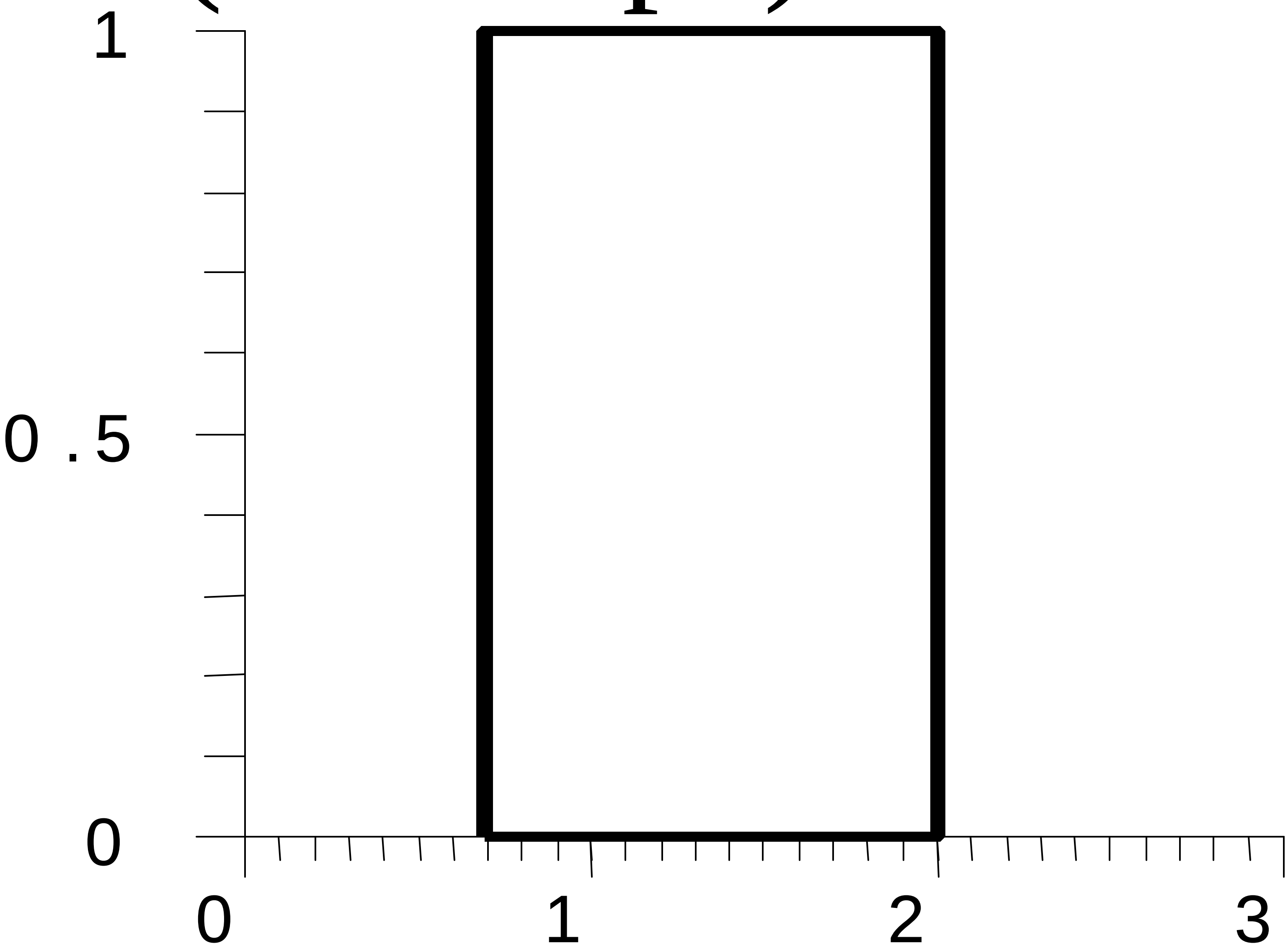
- Are transparent about computations
- Make no unjustified assumptions
- Graphically illustrate state of knowledge
- Don’t confound variability and incertitude
- Assess relative contributions of each

Answers to challenge problems

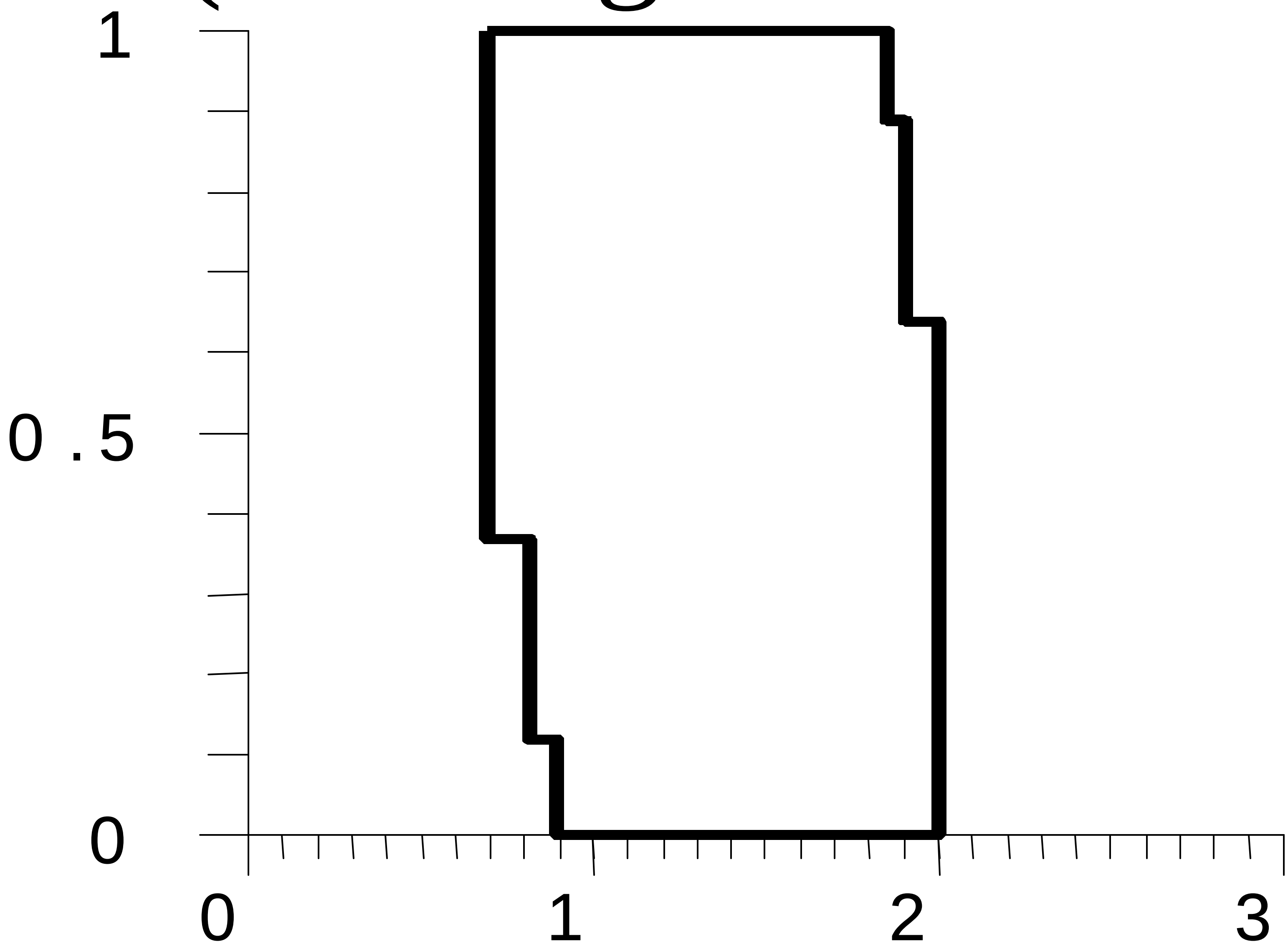
- Plots show bounds on the complementary (inverted) cumulative distribution functions
- Ordinate is probability (zero to one)
- Abscissa is the value of function $(a+b)^a$, except for problem B where it's D_s
- We don't make any assertion about the right aggregation operator to use, so we used several (but just show mixture answer)


1
1
0.5
0
0
1
2
3

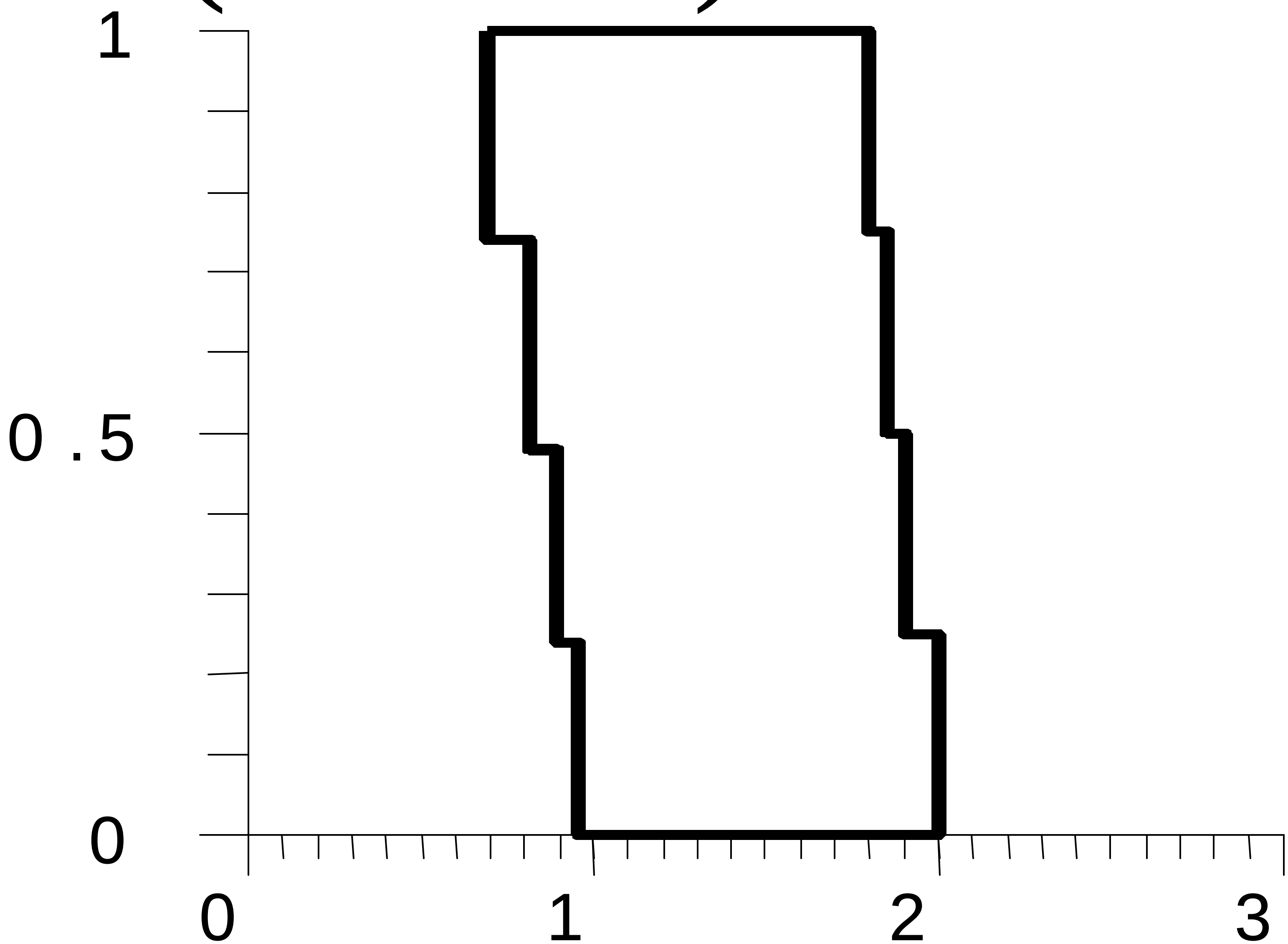
2a (envelope)



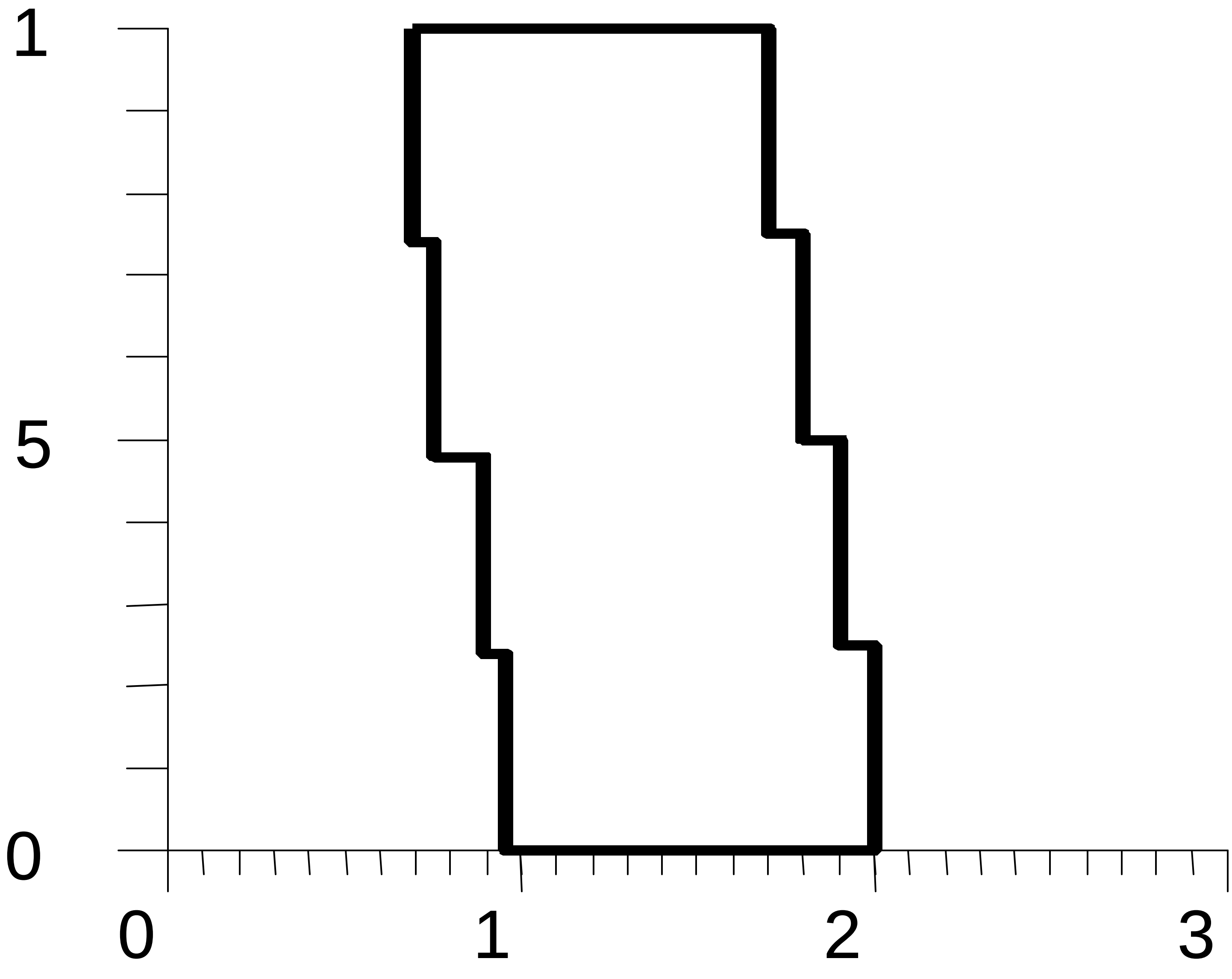
2a (Kolmogorov-Smirnov)



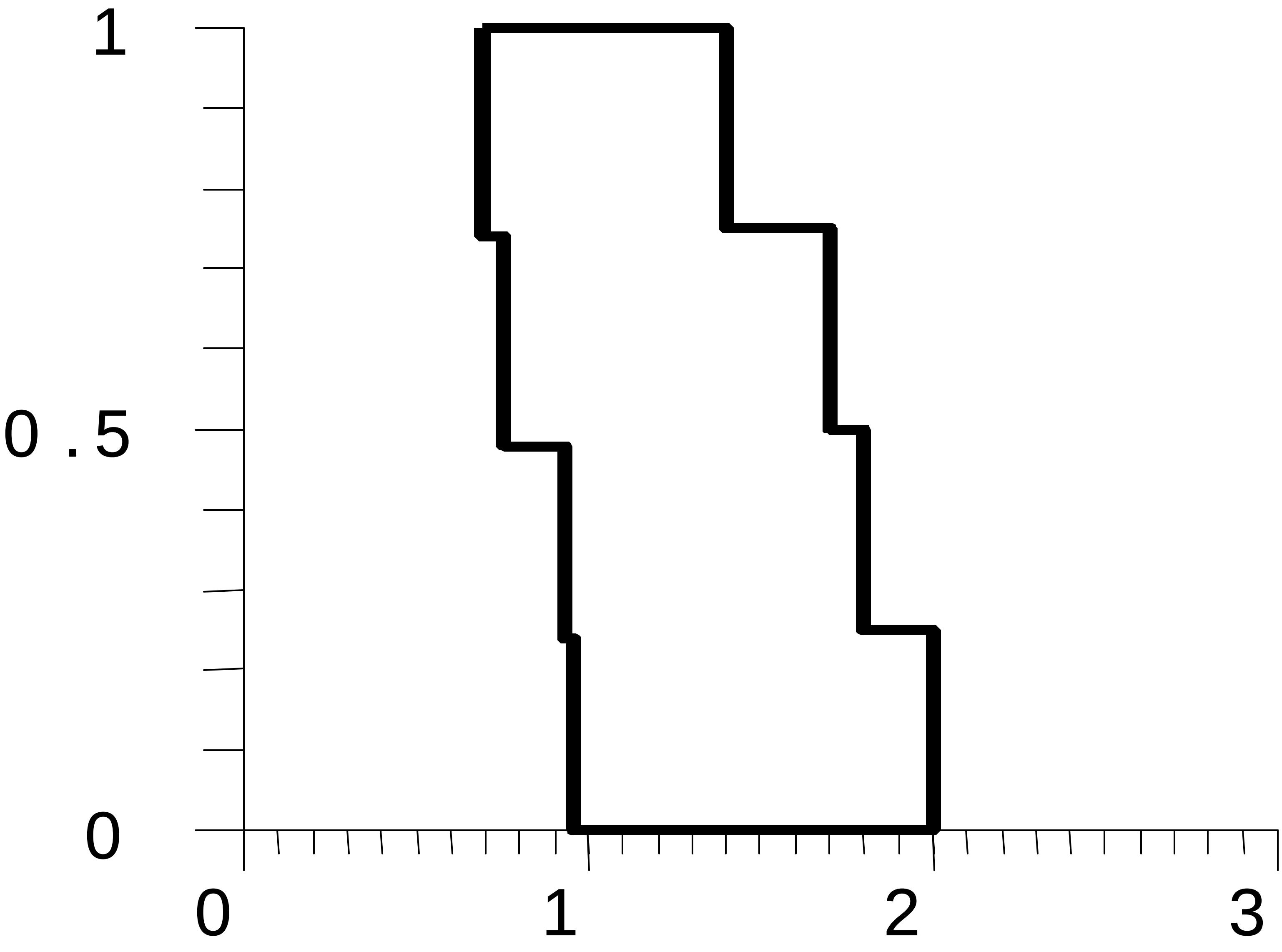
2a (mixture)



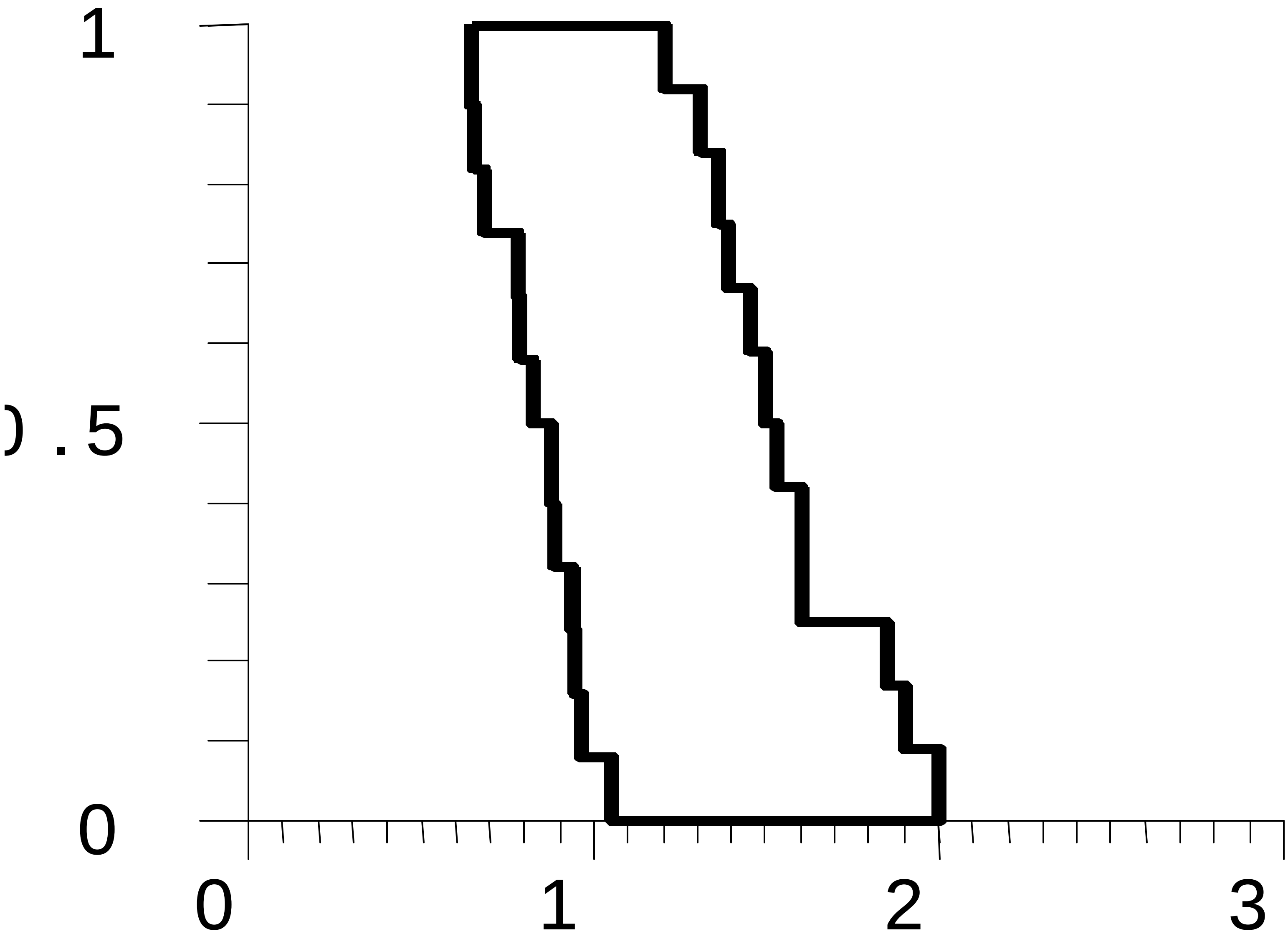
2b



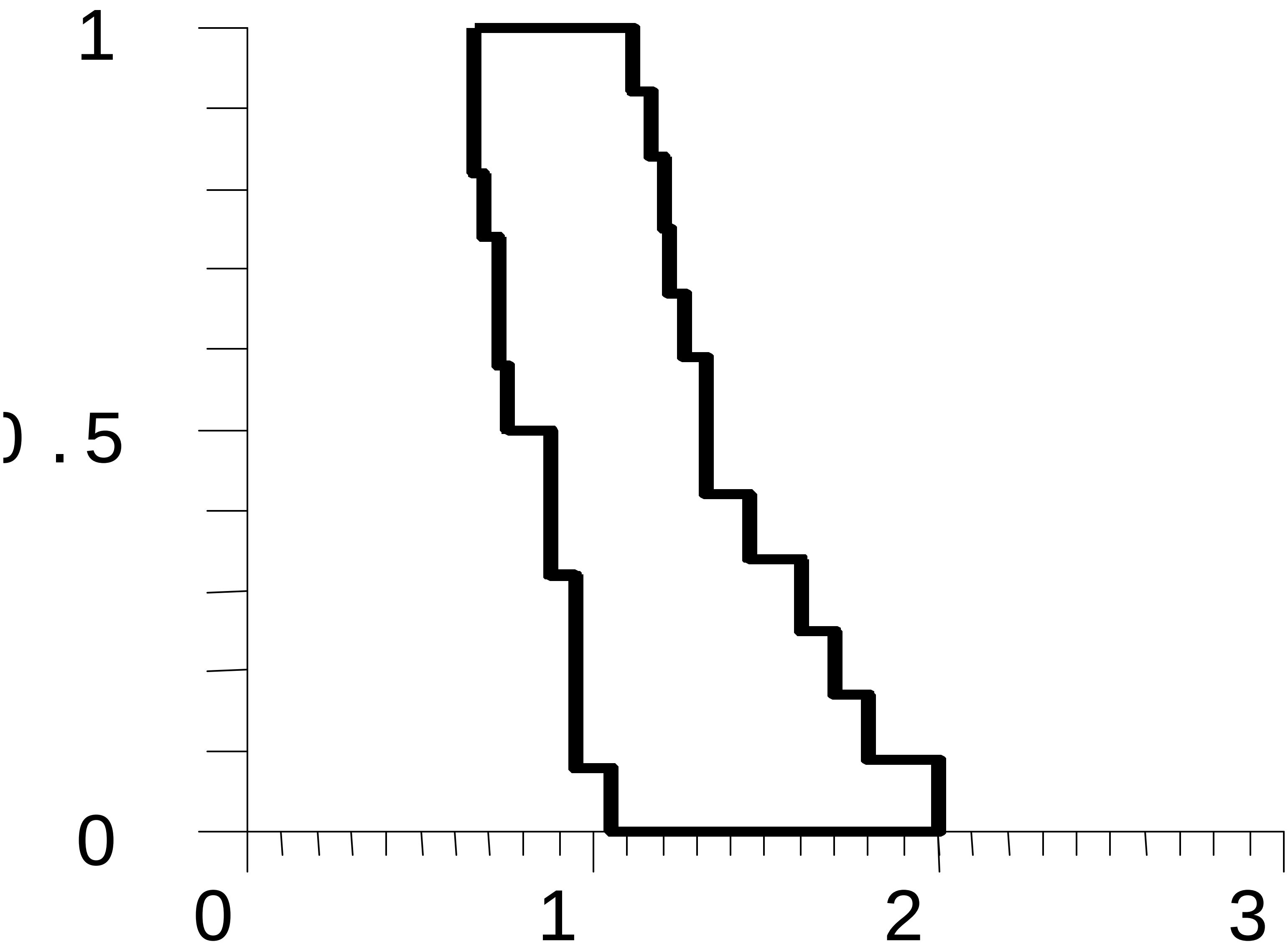
2c



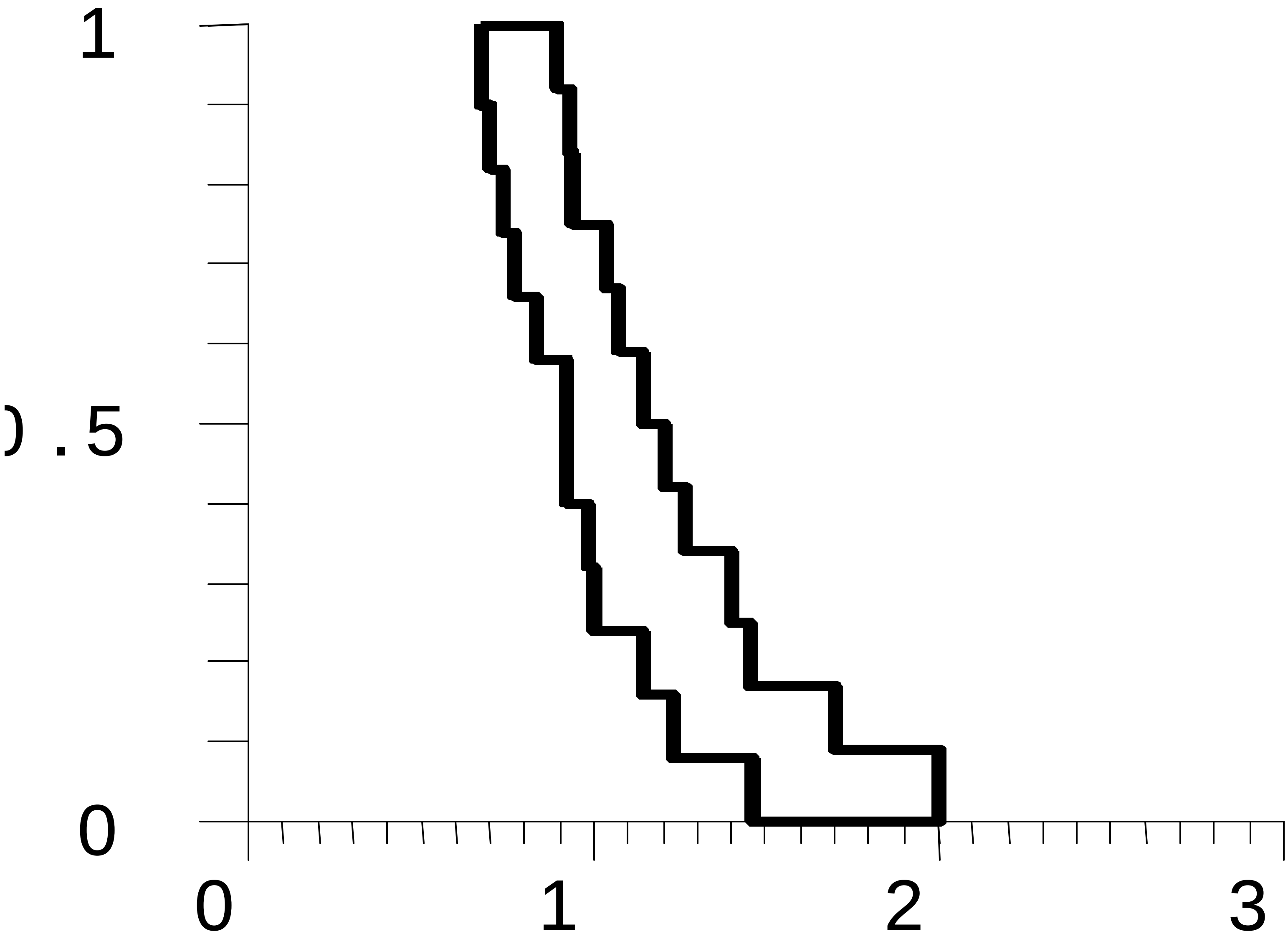
3a



3b



3c



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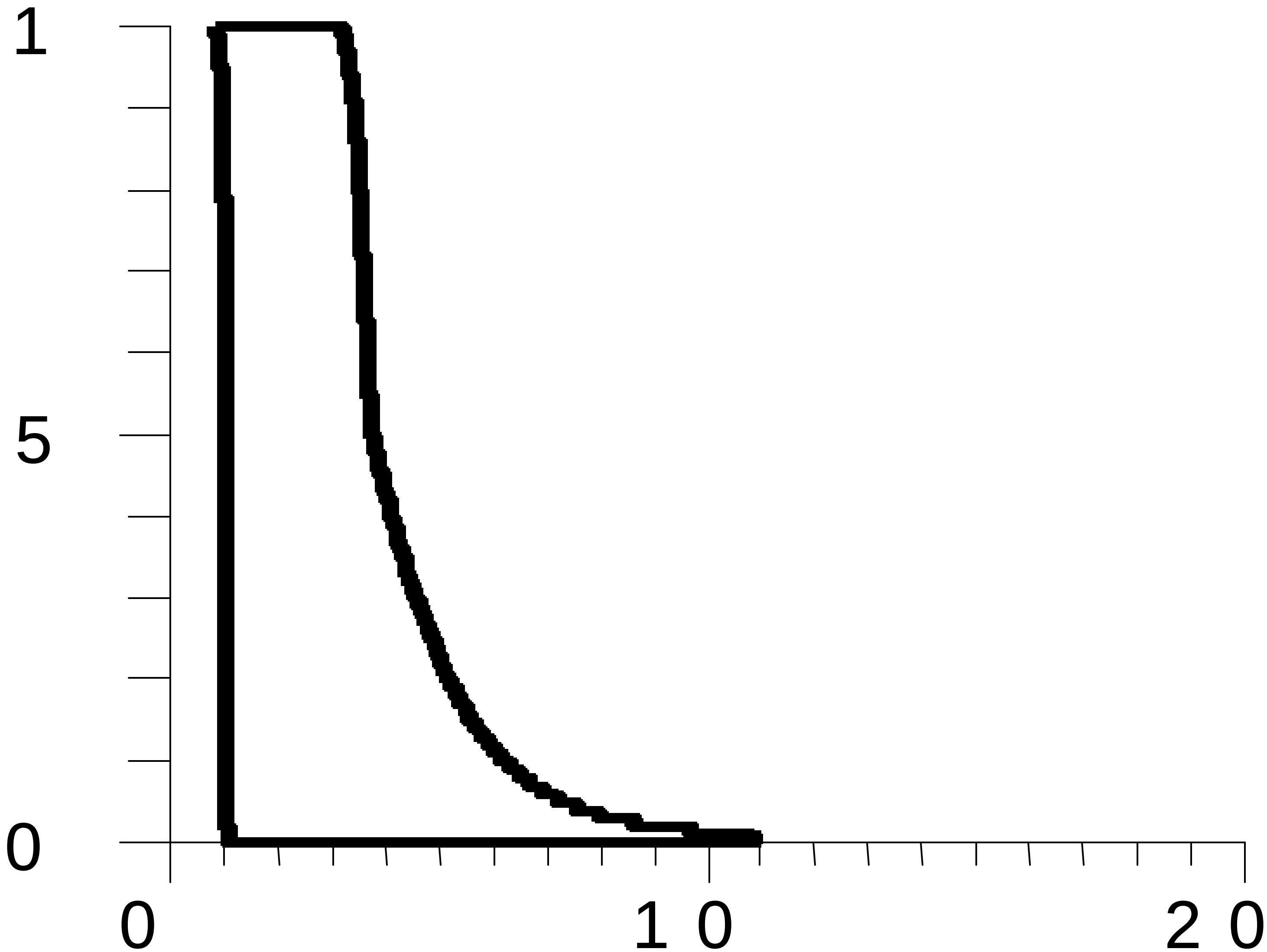
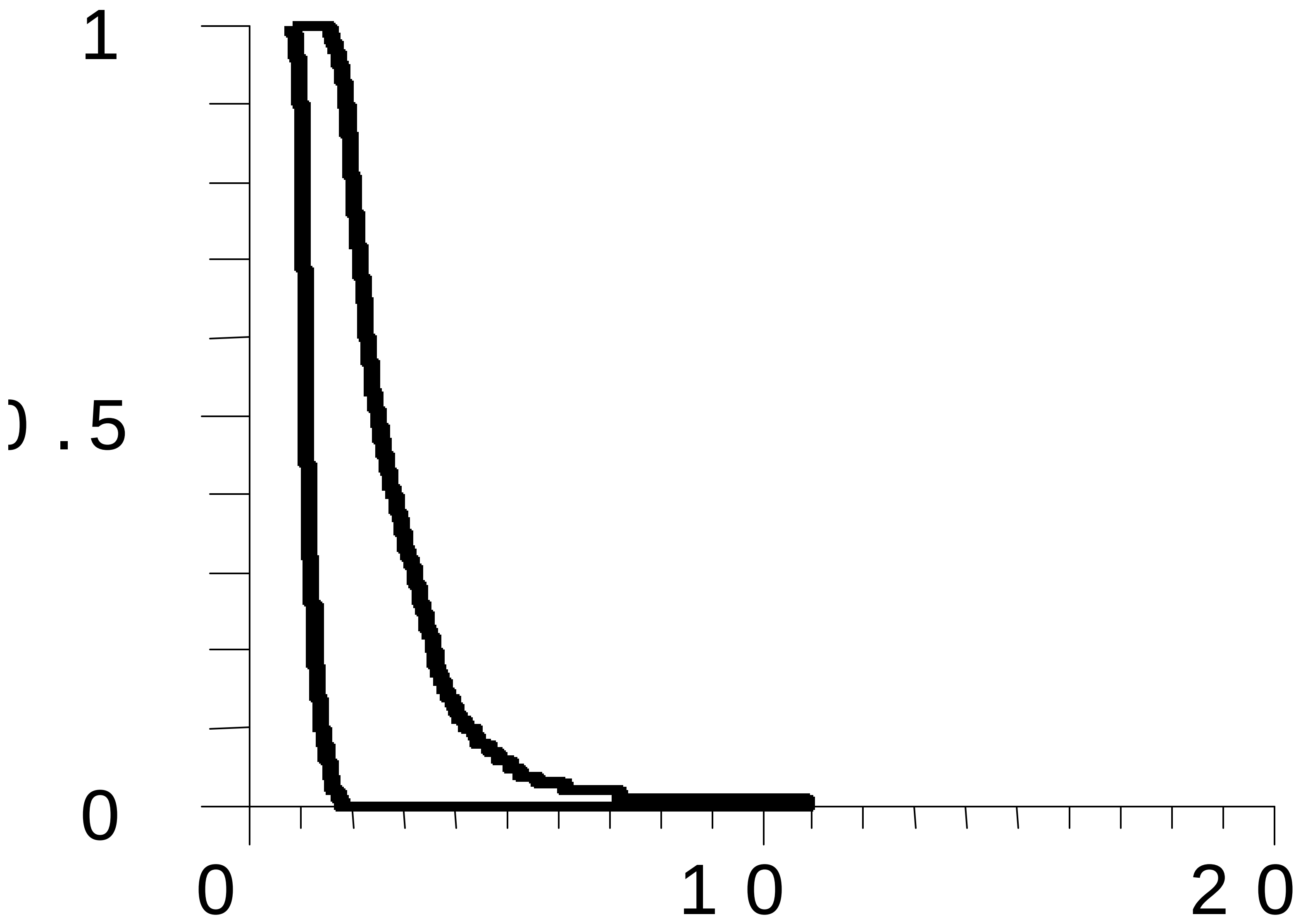
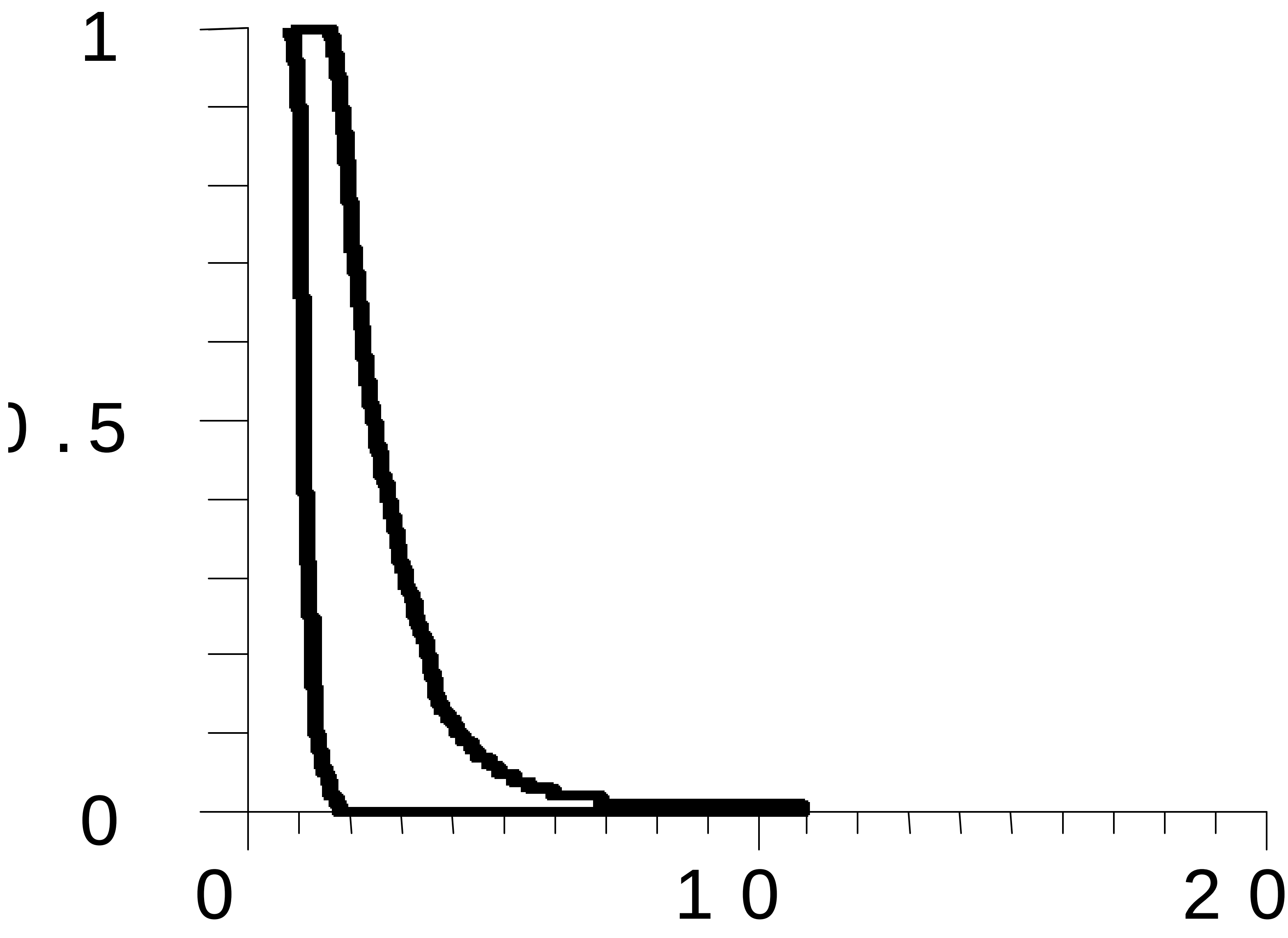


5a



5b



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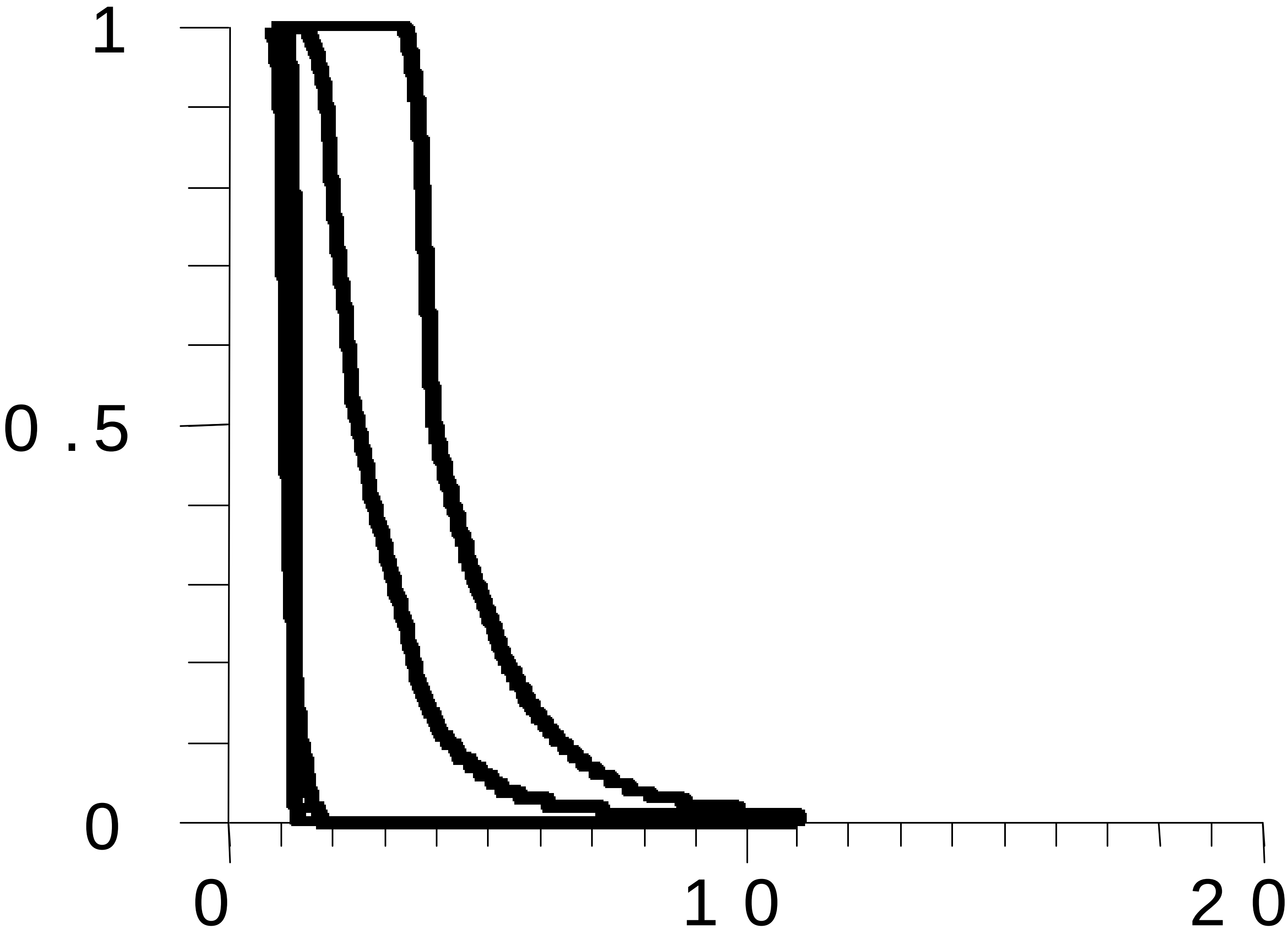

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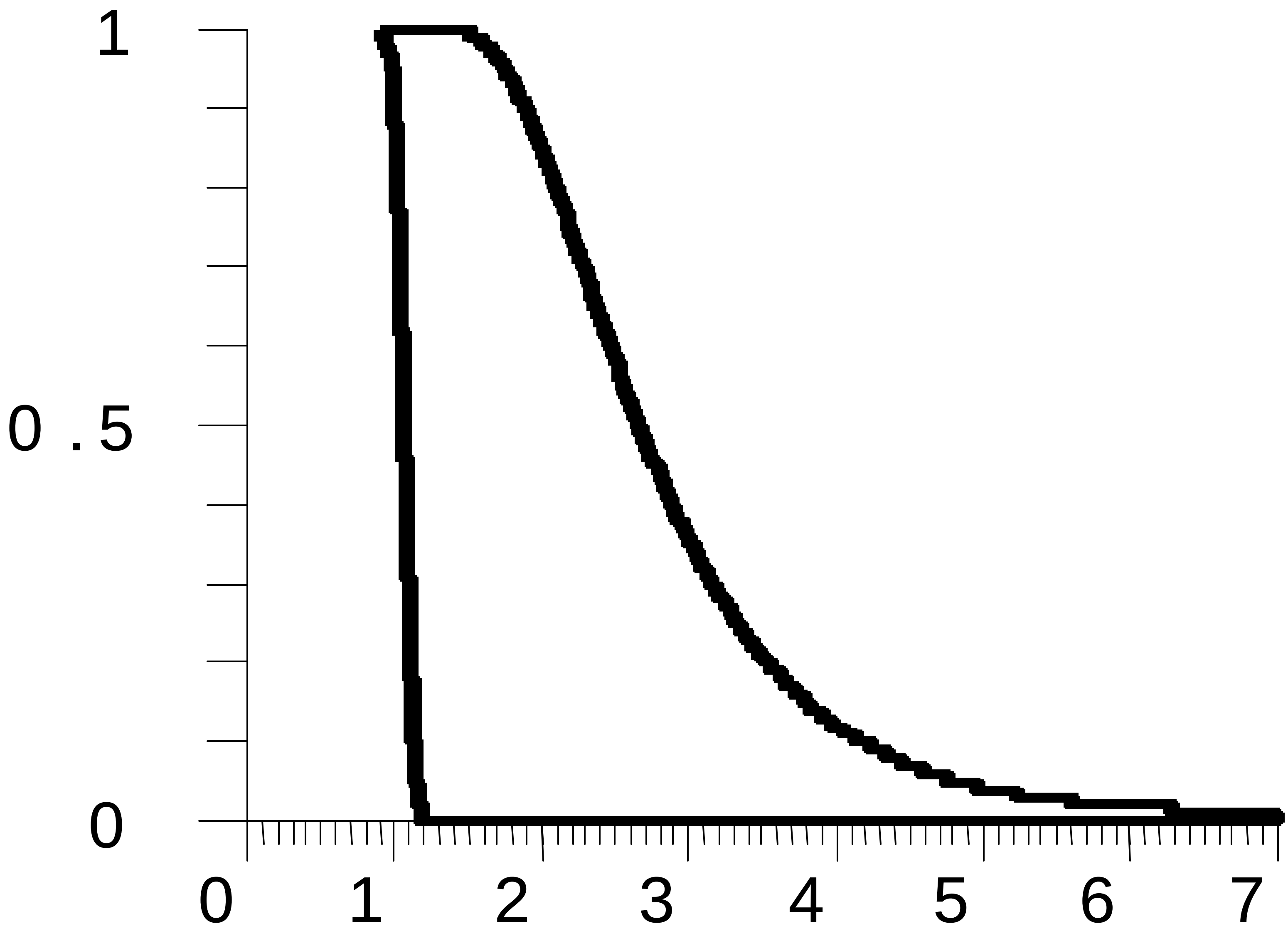

B