IDS 160B: Relativisitic Astrophyics & Cosmology
Problem Set 2 - George F. Smoot - Feb 1995

1 Cosmic Ray Propagation: Isotopes

As discussed in class, for energies in the range of 1 to 10 GeV cosmic rays are measured to have roughly the solar system "universal" abundances except for nuclei just lighter than iron, Fe, (Mn, Cr, Ti, Sc) and the nuclei lighter than CNO (^{3}H, ^{3}He, Li, Be, B). Fe and CNO are abundant nuclei and their lighter neighbors are rare in the solar system but are much more common in the cosmic rays. This suggests fragmentation of the cosmic ray nuclei into lighter nuclei in collisions with the interstellar medium. The interstellar medium is typically 90% hydrogen and most of the rest is helium with about 1% heavier elements so that interaction cross-section is often quoted in hydrogen equivalents.

1.1 Estimate the cross-section for a nucleus to fragment by assuming that the radius of a nucleon (proton or neutron) is about 1.2×10^{-13} cm. For example, the radius of a nucleus of atomic number A is $r = A^{1/3} \times 1.2 \times 10^{-13}$ cm. What do you calculate for ^{56}Fe and ^{12}C? Compare to 7.63×10^{-25} cm^2 and 2.52×10^{-25} cm^2, respectively, that are obtained by other means.

1.2 If the cross-section for Fe to fragment to Mn is 1.21×10^{-25} cm^2, and the ratio of Mn to Fe is 0.1, what is the mean column density through which the Fe passes? The mean column density, ℓ, is the integrated equivalent hydrogen density on the particle path: $\ell = \int n_{\rm H} dl = \langle n_{\rm H} \rangle ct$ where t is the mean travel, or escape time. Column density has units of number per unit area or mass per unit area.

Suggestion: Set up the equilibrium condition equation (the rate of production of Mn is equal to the rate of loss of Mn). Approximate by assuming that the total fragmentation cross-sections are essentially the same for the two elements, Fe and Mn, and that the leakage loss rate for both elements is $1/\tau$.

1.3 If the hydrogen density in the Galactic disk is about 1 cm^{-3}, what is the mean travel time before escape? If the mean hydrogen density in the region bounded by the Galactic halo is about 0.01 cm^{-3}, what would be the mean travel time before escape? If the mean density of the Universe is 10^{-5} cm^{-3}, what would be the mean travel time? There probably is no escape from the Universe and we think that the age of the Universe is roughly 10^{10} years. What does this imply about the Universe as the storage container for cosmic rays?

2 The mean travel time of cosmic rays

The isotope ^{10}Be is radioactive with a lifetime $\tau_{\mathrm{Be}} = 1.6 \times 10^6$ years: i.e. $dN_{\mathrm{Be}}/dt = -N_{\mathrm{Be}}/t_{\mathrm{Be}}$. This and other radioactive isotopes make it possible to determine the mean travel time of the cosmic rays.

2.1 Use the four velocity u^μ derived from the Minkowski metric to determine the time dilation and apparent lifetime, that we, located on or near the Earth, will observe for a particle with energy E and mass m. Use the relation: $p^\mu = mu^\mu = (E, p_x, p_y, p_z)$ to show that $E^2 - p^2 = m^2$ (or, including factors of c: $E^2 - p^2c^2 = m^2c^4$) and thus $E = \gamma m$, where $\gamma = (1 - v^2)^{-1/2}$. Show also that the absolute magnitude of the momentum p is $p = \gamma m v$. Remember that the proper time τ is the time in the particle rest frame, where $d\tau^2 = dt^2 - (dx^2 + dy^2 + dz^2) = \eta_{\mu\nu} dx^\mu dx^\nu$.

2.2 Assume that ^{9}Be and ^{10}Be are produced in the same ratio as the cross-sections for ^{12}C to fragment into them: 4.3×10^{-27}cm^2 and 2.9×10^{-27}cm^2, respectively. Set up the equilibrium equation: creation rate = loss rate, and include the energy dependence of the ^{10}Be lifetime (assume that ^{9}Be is stable). Assume that the mean cosmic ray escape time is 10 million years and that the mean density is 1 Hydrogen atom per cubic centimeter. This is consistent with observations to date. Now use this equation to make a rough plot of the ratio of ^{10}Be to ^{9}Be for energies between 0.1 and 100 GeV per nucleon.

3 Bending of Cosmic Rays by Magnetic Fields

Recall the equation of motion for electromagnetic radiation,

$$d\boldsymbol{p}/dt = q\boldsymbol{E} + \frac{q}{c}\,\boldsymbol{v} \times \boldsymbol{B}$$

(or the equivalent $dp^\mu/dt = (q/m)\,p_\nu F^{\mu\nu}$), and the relation of Problem 2 that $p = \gamma m v$.

3.1 Find the cyclotron or Larmor frequency, i.e., the orbit frequency and the radius of curvature for a particle with momentum p and charge q. Derive the relationship: $p \cos\theta = 0.3BZR$, with R in meters, B in Tesla (10 kilogauss), Z in units of e (the electron charge), and p in GeV/c.

3.2 The average magnetic field in the Galaxy is 3 μG $= 3 \times 10^{-6}$ Gauss. What is the radius of a 1 GeV/$c = 10^9$eV/c particle with unit charge in the Galaxy? What is its orbital frequency or period?

3.3 Cosmic ray researchers often use the notion of rigidity $\equiv p/q$. The units of rigidity are GV/c so a particle with rigidity of R GV/c would have the same radius of curvature and orbital frequency as a particle with momentum R GeV/c and unit charge. At what rigidity (and for Hydrogen energy, $m_{\mathrm{H}}c^2 =$ 939 MeV) does the radius of curvature become comparable to the scale of the Galaxy, i.e. 30 kpc or approximately 90,000 light years? How does this compare with the "knee" of the CR ray flux at 10^{15} eV?

4 Relativistic Kinematics

The center of mass energy and momentum are useful quantities. They will be used in the next two problems. Use 4-vector algebra for this exercise. The four momentum is $p^\mu = mu^\mu = (E, p_x, p_y, p_z)$. Recall that the norm of a four vector is $\eta_{\mu\nu}p^\mu p^\nu$.

4.1 Consider two particles that have momenta p_1 and p_2, and thus energies E_1 and E_2 appropriate to masses m_1 and m_2. The two particles collide and stick together. What is the energy and momentum of the resulting product? This is the center of mass energy and momentum. Express the answer in terms of the rest masses of the two particles, the product of the two energies, the product of the momenta and the cosine of the angle θ between them.

4.2 To what does this reduce if one of the particles is massless?

4.3 What is the minimum energy of an incident CR proton hitting an interstellar proton (effectively at rest) for the production of antiprotons. Antimatter is always produced with a matching matter partner for conservation of critical quantities such as charge. Thus, the simplest reaction is proton + proton $\rightarrow$ proton + proton + proton + antiproton.

4.4 What is the minimum energy in the ISM (interstellar medium rest frame) for an antiproton produced at or near threshold? Since the flux of CR protons falls so quickly with energy, most antiprotons will be produced by protons of incident energy at and just above threshold. This is relevant because a significant flux of very low energy antiprotons may have been observed in cosmic rays. The argument is that the annihilation (destructive collision of particle and antiparticle) of some as yet undiscovered particle (e.g. a WIMP) can produce antiprotons at these low energies and on up to the mass of the particle.

4.5 Show that the Galilean transformation relating the two coordinates of two reference frames moving relative to each other: $x' = x - vt$ and $t' = t$ is modified to $x' = \gamma(x - vt)$ and $t' = \gamma(t + vx)$. Assume the Minkowski metric in both frames and that proper distances (times) are independent of the inertial reference frame so that the transformation must be a rotation. (The transformations must be linear in x and t otherwise the metric in the primed coordinate would not be quadratic, even if we allowed cross terms in dx' and dt', which we do not in rectangular coordinates).
Hint: Consider the motion of the origin $(t, x) = (0, 0)$ from a primed system moving in the x direction with velocity v after expressing the relation between the primed and unprimed coordinates as a rotation through an angle α. Find the relation between the sines and cosines of α and the velocity v.

5 Scattering on Light

Just as colliding with ISM particles (which are mostly protons in hydrogen atoms and a few % other light nuclei) can fragment a cosmic ray nucleus, colliding with light (photons) can cause effects including fragmentation. Consider light to be made of particle-like photons each with energy $E = h\nu$ and momentum $p = E/c$.

5.1 If the binding energy of a nucleon in a cosmic ray nucleus is E_B, what energy photon is necessary to eject a nucleon for cosmic rays of mass M as a function of the cosmic ray energy E? If the typical nucleon binding energy E_B is about 0.5% of a nucleon mass, express the answer in terms of energy per nucleon (E/A) making the approximation that protons and neutrons have the same mass [at least to the accuracy that the energy per nucleon (E/A) is measured].

5.2 The cosmic microwave background fills the universe with a photon density of about 550 per cubic cm (for 3°K and 415 per cc for 2.73°K) and with a current typical energy of 1/4000 eV. What is the maximum energy that an Fe nucleus can have to travel from the source to us, meaning 1% or less survive the journey. The photo-fragmentation cross-section is similar to the proton fragmentation cross-section. The cross section for a photon to eject a neutron is shown in the accompanying figure. (The proton production is similar to the photoneutron cross-section). A typical scale is $0.5A$ mb (millibarn) $= 0.5 \times 10^{-3} \times 10^{-24} A$ cm^2, where A is the number of nucleons.

5.3 Photoproduction of Pions. A pion (π meson) cannot be created until the energy in the center of mass exceeds that of the proton by the mass of the pion (134.98 and 139.57 MeV for neutral and charged pions respectively). Note the figures show that the cross-section for the photoproduction of pions rises sharply with energy peaking at about 0.5 mb (millibarn) $= 0.5 \times 10^{-27}$cm^2 at the Δ^+ (1430 MeV) resonance: photon + proton $\rightarrow \Delta^+$. which decays to proton and a π^0 or a neutron and a π^+.

(a) What is the energy of a CR proton necessary to produce a Δ^+ colliding with CMB photons? What would be the energy for starlight, $E \sim 1$ eV?

(b) What is the maximum distance that one would expect to have that proton travel? That is, when 99% of such protons would interact and lose energy?

(c) What is the approximate mean degradation (fractional energy loss) in energy per interaction?

(d) If the mean density of galaxies is 10^{-75} per cubic cm, from about how many galaxies could we receive cosmic ray protons of this energy?

6 The limits of Astronomy

Just as particles can interact with photons to produce other particles, photons can interact with photons to produce particles. The lightest particles that a pair of colliding photons can produce is an electron + antielectron (positron) pair which each have identical masses of 511 keV.

6.1 What is the formula for the center of mass energy of two colliding photons. (Use the formula from Problem 4 or rederive a formula for two massless particles.)

6.2 What is the minimum photon energy necessary for e^+e^- pair production if the other photon is a typical CMB photon? How different is the necessary mean energy? (A rough number is all right—consider right angle scattering.) If there is a significant infrared background radiation ($20°\text{K} < T < 300°\text{K}$), what would be the energy?

6.3 If the peak pair production cross-section is 30 mb $= 3 \times 10^{-29}$cm, what is the mean distance such an energetic photon can travel. That is, what is the mean free path and the 1% survival distance.

6.4 The Earth's ionosphere has a free electron densities of the order of 10^4—10^6 cm^{-3} and a magnetic field of about 0.3 Gauss. The plasma frequency is $\omega_{\mathrm{p}}^2 = 4\pi n_{\mathrm{e}} e^2/m$ and the magnetic precession frequency $\omega_{\mathrm{B}} = eB/mc$. Photons with frequency lower than the plasma frequency are reflected. The addition of a magnetic field modifies that situation by allowing one circular polarization to pass in some bandwidths with a strong angular dependence. Compute the plasma frequency and thus the cut-off frequency and photon energy in eV. What are the plasma and magnetic precession frequency values for the ionosphere and for the interstellar medium for densities of about 1 per cc and the 3 microgauss field? What energy and wavelength correspond to the ISM plasma frequency?

6.5 Because the magnetic fields and plasma are randomly distributed they make an effective low frequency/energy cutoff. The cosmic microwave background and possible infrared background create an upper limit for photon astronomy just as particle astronomy (Cosmic Ray observations) has a low energy cut off set by the solar winds and magnetic field and a high energy cut off. Light and particles also have the limit that they cannot travel further than about ct where t is the age of the Universe. (As we will see the expanding of space will carry things somewhat further but the distance is still on this scale. Another effect also cuts off the distance on a similar scale.) Thus the range of photons coming to us is limited both in frequency space and physical space. Thus what we can investigate is restricted.

If each photon can carry only one bit of information, then the darkness of the night sky, that is the limited total energy flux incident on the Earth, implies that the total information about this restricted space is limited. Assume that the total energy density of radiation near the Earth, neglecting the solar flux, is on the order of 1 eV per cubic cm and it is roughly equally divided per logarithmic interval from the plasma cut off to the high energy cut off. How many photons or bits are incident per second per square cm? How does this compare with the flux of CMB photons which should be about one quarter of the total energy?

6.6 Clearly every photon does not carry unique information. Many phenomena are multimodal. That means that they show up in many different wavelength bands (i.e. emit photons in many different quantum states or modes). A good example is blackbody (Planckian spectrum) radiation in which the information carried by all the photons is that the body is optically thick and is in thermal equilibrium at a temperature T. Then, it is the deviations from Planckian and the angular dependence that are interesting for radiation from the Sun and the CMB. Though the flux of CMB photons is great, the information contained by them, if the spectrum is Planckian, is in the angular dependence. In principle the same temperature variation should appear at every frequency. Thus we must introduce the idea of the multimodality and the information content of a message. Information content for astronomy is clearly limited by entropy of the emission and thus the emitter. In the early 1980's Martin Harwit reviewed the information and discovery history in astronomy and made a prediction of the future discovery rates for major (cosmos-shaking) discoveries. The history and his prediction made in 1983 are shown in the attached figure. Thus the concept that this is a golden age of astronomy as so many more wavelengths have opened up. We will bear this in mind during the course and return to this later as time permits. *Extra Credit:* If you believe that Harwit is wrong, how is he wrong and what experiment can you do to open a new set of major discoveries? (If you get a sufficiently good answer to this, you will get an A in this course and if you do the experiment successfully you will get a permanent job, prestige, and a Nobel prize!)