"Physicists are like 3% of rats."
-Max Zolotorev, Lawrence Berkeley National Laboratory

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1

(a)

As demonstrated in last week's problem set (problem 8), an ideal quarter-wave plate is described by a unitary Jones matrix. This means that the irradiance of the light beam is unaffected by traversing the plate. Also, the mirror is a perfect conductor, so 100% of the light is reflected. So the final wave's irradiance must be the same as the incident wave's.

(b)

The wave, initially polarized along the $\hat{x}$ direction, first passes through the quarter wave plate, whose fast axis is oriented at -45^o with respect to the initial light polarization:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} E_0 \\ 0 \end{bmatrix} = \frac{E_0}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{1}$$

Then the light bounces off a perfectly conducting mirror. This reverses the sign of the Poynting vector, which in turn changes the sign of the B-field relative to the E-field, since (as can be shown from Maxwell's equations):

$$\vec{E} = E_0 \hat{\eta} e^{i\left(\vec{k}\cdot\vec{r} - \omega t\right)}$$
$$\vec{B} = \frac{1}{v}\hat{k} \times \vec{E}, \tag{2}$$

where $v = c/n$ is the phase velocity of light in a medium and $\hat{\eta}$ is the direction of the light's electric field. The light electric field undergoes a phase shift of π upon relection (as can be deduced from the boundary conditions), but for circularly polarized light the direction of rotation (clockwise or counter-clockwise) of the electric field with respect to a fixed coordinate system is preserved. However, we are now viewing it from the opposite direction (since $\vec{k}$ changed sign). Therefore the handedness of polarization has changed upon reflection:

$$\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \frac{E_0}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{3}$$

This result can also be arrived at using Fowles's reflection matrix (page 52).
The beam now passes back through the quarter waveplate, but now the wave sees the fast-axis oriented at +45^o. So we find that:

$$\begin{bmatrix} E''_x \\ E''_y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \cdot \frac{E_0}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = -iE_0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4}$$

In other words the resultant light is linearly polarized in the $\hat{y}$ direction.

Problem 2

First let's derive Fowles' result regarding the acceptance angle α for a fiber-optic cable (pages 46-47).
At the first phase transition as the light enters the fiber-optic cable, we have from Snell's law:

$$\sin\alpha = n_1 \sin\beta. \tag{5}$$

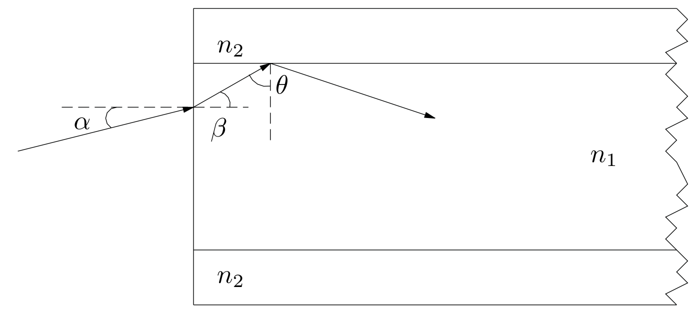

Figure 1

We want $\theta = \pi/2 - \beta$ to be greater than or equal to the critical angle, $\sin^{-1} n$ for total internal reflectance, where $n = n_2/n_1$. With a little trigonometry it can be shown that for these conditions,

$$\sin\beta = \frac{\sqrt{n_1^2 - n_2^2}}{n_1}. \tag{6}$$

Combining Eqs. (5) and (6) yields:

$$\sin\alpha = \sqrt{n_1^2 - n_2^2}, \tag{7}$$

which proves Fowles's assertion.
The next step is to compute the solid angle of light accepted by the fiber-optic cable, given by:

$$\Delta\Omega = \int_0^{2\pi} \int_0^{\alpha} \sin\theta d\theta d\phi = 2\pi(1 - \cos\alpha). \tag{8}$$

Divided by the total solid angle (4π), this yields the fraction T of the total light transmitted by the core to the end of the cable:

$$T = \frac{(1-\cos\alpha)}{2}. \tag{9}$$

Problem 3

A vector field $\mathbf{F}(\vec{r})$ is equal to the curl of a vector potential $\mathbf{A}(\vec{r})$, so we know that the divergence of $\mathbf{F}(\vec{r})$ is zero:

$$\nabla\cdot\mathbf{F}(\vec{r}) = \nabla\cdot(\nabla\times\mathbf{A}(\vec{r})) = 0. \tag{10}$$

Then we know that:

$$\int \nabla\cdot\mathbf{F}(\vec{r})dV = \oint \mathbf{F}(\vec{r})\cdot d\vec{A} = 0 \tag{11}$$

Choose a volume of vanishing thickness $\vec{\delta}$ about a surface of area A (where $\vec{\delta}$ is always normal to the surface), then from Eq. (11) we have that:

$$\mathbf{F}_\perp\left(\vec{r}+\vec{\delta}\right)A - \mathbf{F}_\perp(\vec{r})A + O(\delta) = 0 \tag{12}$$

where $O(\delta)$ indicates a term of order δ, which arises from some finite amount of "flux" of $\mathbf{F}(\vec{r})$ out the sides of the volume.

The condition for continuity of $\mathbf{F}_\perp(\vec{r})$ is that for each $\epsilon > 0$, there exists a $\delta > 0$ such that $|\mathbf{F}_\perp\left(\vec{r}+\vec{\delta}\right) - \mathbf{F}_\perp(\vec{r})| < \epsilon$. From Eq. (12) we know that:

$$|\mathbf{F}_\perp\left(\vec{r}+\vec{\delta}\right) - \mathbf{F}_\perp(\vec{r})| = \frac{O(\delta)}{A}. \tag{13}$$

Given any $\epsilon > 0$, clearly we can choose δ to make:

$$\frac{O(\delta)}{A} < \epsilon. \tag{14}$$

Therefore $\mathbf{F}_\perp(\vec{r})$ is continuous.

Problem 4

From Strovink's treatment of reflection/refraction at a plane interface between insulators, we have for normal incidence:

$$\frac{E_r}{E_i} = \frac{Z_2 - Z_1}{Z_1+Z_2}$$
$$\frac{E_t}{E_i} = \frac{2Z_2}{Z_1+Z_2}, \tag{15}$$

where E_r, E_i, E_t are the reflected, incident and transmitted electric field amplitudes, respectively, and

$$Z_{1,2} = \sqrt{\frac{\mu_{1,2}}{\epsilon_{1,2}}}. \tag{16}$$

In particular, for the ferromagnetic material described, $Z_2 \to \infty$ while Z_1 is finite, so:

$$\frac{E_t}{E_i} = \frac{2Z_2}{Z_1+Z_2} \to 2. \tag{17}$$

Problem 5

(a)

Plane waves propagating in the $\pm z$ directions must satisfy Eqs. (15) at the interfaces ($z=0$ and $z=L$). At either interface we have, since $Z_2 \to 0$,

$$\frac{E_t}{E_i} = \frac{2Z_2}{Z_1+Z_2} \to 0. \tag{18}$$

Therefore $\mathbf{E}$ vanishes in the material, i.e. $\mathbf{E}$=0 for $z < 0$ and $z > L$. So at the interfaces, $\mathbf{E}$=0.

The magnetic fields of the plane waves propagating in the $\pm z$ directions between the materials must satisfy $\mathbf{E}_+\times\mathbf{H}_+ = -\mathbf{E}_-\times\mathbf{H}_-$ (i.e. the Poynting vectors of right- and left-traveling waves must be oriented in opposite directions). So while the electric fields cancel ($\mathbf{E}_+ = -\mathbf{E}_-$) at the interfaces the magnetic fields must add ($\mathbf{H}_+ = \mathbf{H}_-$)! Also we have the boundary condition $\mathbf{H}_\parallel = \mathbf{H}'_\parallel$, so that if $\mathbf{H}$ is finite on one side of the interface, it must also exist on the other side. So there can be components of $\mathbf{H}$ everywhere.

(b)

Our requirements from part (a) set up a standing wave, where the components of $\mathbf{E}$ and $\mathbf{H}$ are π out of phase. The wave is time independent in order to assure that $\mathbf{E}$=0 at the interfaces for all times t, so we can postulate:

$$\mathbf{E} = \vec{E_0}\sin kz, \tag{19}$$

which works so long as $k = N\pi/L$ where N is an integer. So we get the condition on angular frequency from $k = \omega/c$, which implies

$$\omega = \frac{N\pi c}{L}. \tag{20}$$

Problem 6

We have two regions as shown in Figure 2, with k_1 and k_2 in each (defined as in the problem, where they are dependent on the potential V and the particle's total energy U, which is conserved, U = T + V). In region 1 we have the wavefunctions:

$$Ae^{i(k_1x-\omega t)} + Be^{-i(k_1x+\omega t)}, \tag{21}$$

and in region 2 we have a transmitted wavefunction:

$$Ce^{i(k_2x-\omega t)}. \tag{22}$$

Continuity of the wavefunctions across the boundary (x=0) demands:

$$A + B = C. \tag{23}$$

Since $\frac{\partial\psi}{\partial x}$ is also continuous:

$$k_1(A - B) = k_2C. \tag{24}$$

Substituting Eq. (23) into (24), we get:

$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2} \tag{25}$$

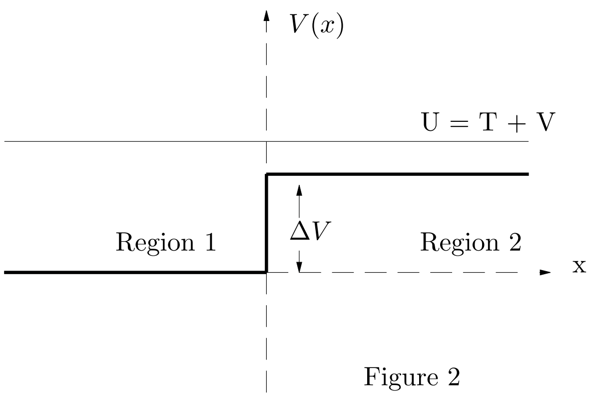

Figure 2

If we assume $n \propto k$, then we get the formula for normal reflection of an EM wave at a dielectric interface with $\mu = \mu_0$:

$$\frac{B}{A} = \frac{n_1 - n_2}{n_1 + n_2} \tag{26}$$

Problem 7

Fowles 3.1

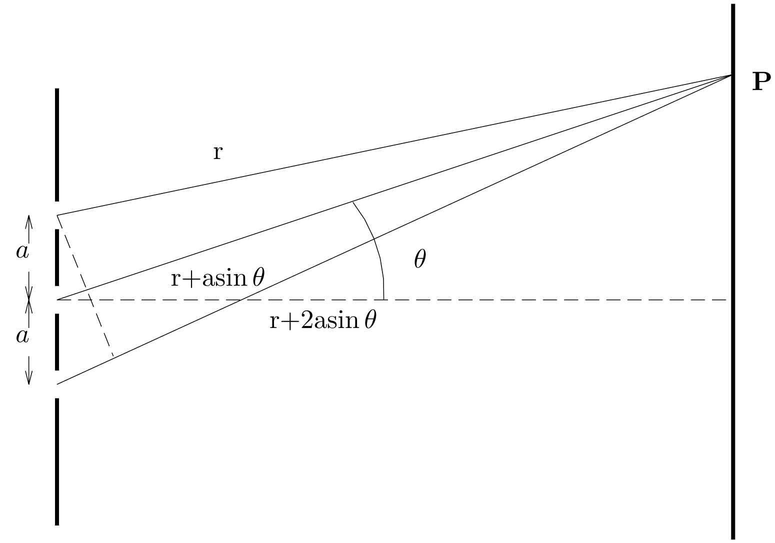

Figure 3

The sum of the amplitudes of the waves A coming from the slits at point P (see Figure 3) are given by the proportionality:

$$A \propto e^{ikr} + e^{ik(r+a\sin\theta)} + e^{ik(r+2a\sin\theta)} \tag{27}$$

or,

$$A \propto e^{ikr}\left(1 + e^{ika\sin\theta} + e^{2ika\sin\theta}\right) \tag{28}$$

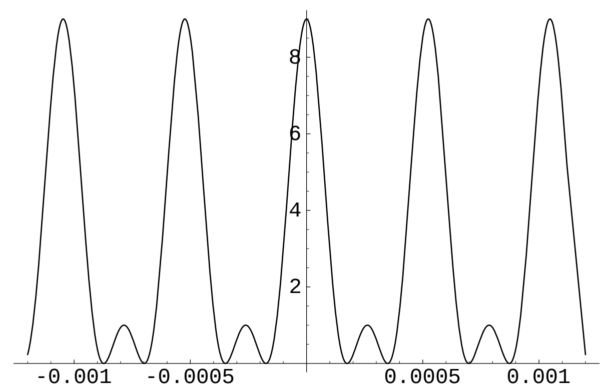

Figure 4

The interference pattern is given by the norm square of the amplitude:

$$I(\theta) \propto |A|^2 \propto (1 + \cos(ka \sin\theta))^2 \tag{29}$$

The pattern that you get when you plot this function depends on what value you choose for ka. Let's take $a = 1$ mm and $k = 12{,}000$ mm^{-1}, then our pattern is shown in Figure 4 as a function of θ.

Problem 8

Fowles 3.6

Light passes through the gas cell twice, so the optical path difference d_{op} is given by:

$$d_{op} = c\Delta t = \frac{2l}{c/n} - \frac{2l}{c} = 2l(n-1) \tag{30}$$

n changes as the gas fills the cell, and since $I \propto 1 + \cos(2\pi d_{op}/\lambda)$, a new fringe appears every time $d_{op} = 1/2$. Thus the total number of fringes N is given by:

$$N = 2\frac{2l(n-1)}{\lambda} = \frac{4l(n-1)}{\lambda}. \tag{31}$$

Plugging in the suggested values gives us $N = 203$ fringes.