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A Statistical RKR Fracture Model for the Brittle Fracture of Functionally Graded Materials

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Abstract

A statistical Ritchie-Knott-Rice (RKR) [1] model for brittle fracture is considered for an FGM containing a slender notch. The FMG is modeled as linear elastic, with its strength described by two-parameter Weibull statistics. The Young's modulus is assumed to vary either linearly or sigmoidally. A compact tension (C(T)) fracture mechanics specimen is analyzed via the finite element method, considering the effect of modulus variation on the near-tip stress state. Results can be characterized by the stress intensity, K. For spatially constant Weibull parameters, the RKR model is used to predict the expected fracture toughness, K_Φ, i.e., the K at which the first flaw failure occurs with probability Φ. For sufficiently high Weibull modulus, the failure occurs essentially at the notch tip. For sufficiently low Weibull modulus ($m < 4$), K_Φ for an FGM is found to vary up to 25% from that of a homogeneous body.

Introduction

Functionally Graded Material (FGM) research is motivated by the need for properties that are unavailable in any single material and the need for graded properties to offset adverse effects of discontinuities for layered materials. Applications such as coatings for turbine blades usually focus on high temperature mechanical properties, although FGM's optimized for electrical properties have also been investigated. For FGM's with compositions including a ceramic or a glass, fracture is an important design limitation. If the FGM is a brittle/brittle composite (Mo/SiO_2 [2], Al_20_3/Si_3N_4 [3]), linear elastic fracture mechanics (LEFM) can be used to characterize such failure, providing that the effects of the gradient in elastic modulus and residual stress are accounted for. The present paper addresses the former effects assuming there are no residual stresses.

Background

The study of fracture mechanics for FMG's has yielded elastic crack-tip stress field solutions for various gradients. It has been determined that, for an FGM with modulus E such that $E(x)=E_o e^{\beta x}$, as $r \to 0$ the stress field varies as [4]

$$\sigma_{ij} = \exp(\beta x)\left[\frac{1}{\sqrt{2\pi r}}(K_I f_{ij}^{I}(\theta) + K_{II} f_{ij}^{II}(\theta))\right], \qquad (1)$$

where K_I and K_{II} are the mode I and mode II stress intensity factors, which have the same interpretation as in homogeneous materials, although their values for a given load will differ from the homogeneous case. The angular functions $f(\theta)$ are identical to those in homogeneous materials, and for locations asymptotically close to the tip of a crack, $\beta x \ll 1$, the stress field for a

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homogeneous material and an FGM are identical. However, the existence of the homogeneous crack-tip field in a material with varying modulus violates the conditions of compatibility in linear elasticity. Therefore, for distances away from the crack-tip (at locations with modulus different from that at the tip) the stress field must take on a different character.

A number of fracture mechanics solutions have also been obtained for FGM's under mixed-mode loading conditions, again with the stress intensity dependent not only on crack length and geometry, but also on the shape and strength of the gradient [4].

Brittle fracture can be dictated by the stress at the crack tip, or some distance away, as in the Ritchie-Knott-Rice (RKR) fracture model [1], which describes the onset of fracture in terms of a critical stress being sustained over a critical distance ahead of the crack tip. One basis for the latter behavior arises because the measured strength of a brittle material is not determined by the highest stress at any point, but rather by the volume over which stress is applied. The resulting statistical nature of the strength of a brittle material has been explicitly incorporated into analyses of the crack problem [5,6]. In these statistical fracture models, the competition of high stresses near the crack tip and increasing sampling volumes away from the crack tip can result in the most probable location of failure initiation being some distance *away* from the tip of the crack.

Given that the RKR fracture model depends on the distribution of stresses ahead of the crack tip, the fracture of an FGM could take place at a K different from that of a homogeneous material with exactly the same distribution of strengths. To explore the implications, a fracture mechanics specimen with various modulus gradients but identical statistical strength parameters has been studied via the finite element method.

Problem Formulation and Numerical Procedures

Consider the total probability of failure, Φ, of a brittle (weakest-link) structure described by two-parameter Weibull statistics

$$\Phi = 1 - \exp\left[- \int_{vol} \left(\frac{\sigma}{\sigma_o} \right)^m \frac{dV}{V_o} \right] \qquad (2)$$

where m is the Weibull modulus, which is a measure of the scatter in strengths. The probability of a failure for a cracked body can be assessed by simply substituting the asymptotic crack tip stresses of Eq. 1. However, this integral is not defined for $m > 4$ due to the strong singularity associated with σ^m. A previous study [5] of fracture of steel used both the linear elastic stress solution (with the integral excluding the plastic zone) and the nonlinear elastic HRR solution (simplifying the blunting region); these assumptions allowed Φ to be calculated explicitly.

In the absence of plasticity an analogous "cut off" dimension must be allowed. A simple assumption is that of a small root radius at the tip of the "crack". A long slender notch will exhibit a K-dominant stress field like that of a crack [7], but near the tip will maintain finite values of stress and allow Eq. 2 to be calculated for all values of m.

The values of the stresses to be used in Eq. 3 were calculated via finite element analysis. A compact tension (C(T)) fracture mechanics sample was modeled with the mesh in Fig. 1a. The near-tip region was analyzed with the mesh in Fig. 1b. This region was analyzed via a boundary layer methodology; the displacement boundary conditions for this mesh were that of the asymptotic crack solution [7]. This requires (1) that the radius of the mesh, R, in Fig 1b be much larger than the notch radius, ρ, (e.g., $R = 60\rho$ has been found to be sufficient) and (2) that the size of this zone is smaller that the K-dominant zone of the C(T) sample, $R \ll a$. This, in turn, rendered the results of the crack-tip calculation (mesh B) independent of the gradient for those examined here with $\beta \ll 1/R$.

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An existing plane strain linear-elastic finite element code FEAP 4.2 [8] was modified such that during integration of the stiffness matrix the elastic constants were evaluated at each of the Gauss points (3 per direction, 9 total). This allowed for quadradic variation in modulus to be modeled within a single element. The Mesh A consisted of 2200 total elements, 8871 nodes, with the near-tip region, Mesh B, comprised of 3000 elements, 12221 nodes.

Stress intensity factor calibrations were performed for each gradient in Mesh B using two methods: 1) direct fitting of the stress field ahead of the crack and 2) through the elastic strain energy release rate, *G*, and the change in load-line compliance with incremental crack growth. The two results were in very good agreement (within 0.5%) for all gradients studied.

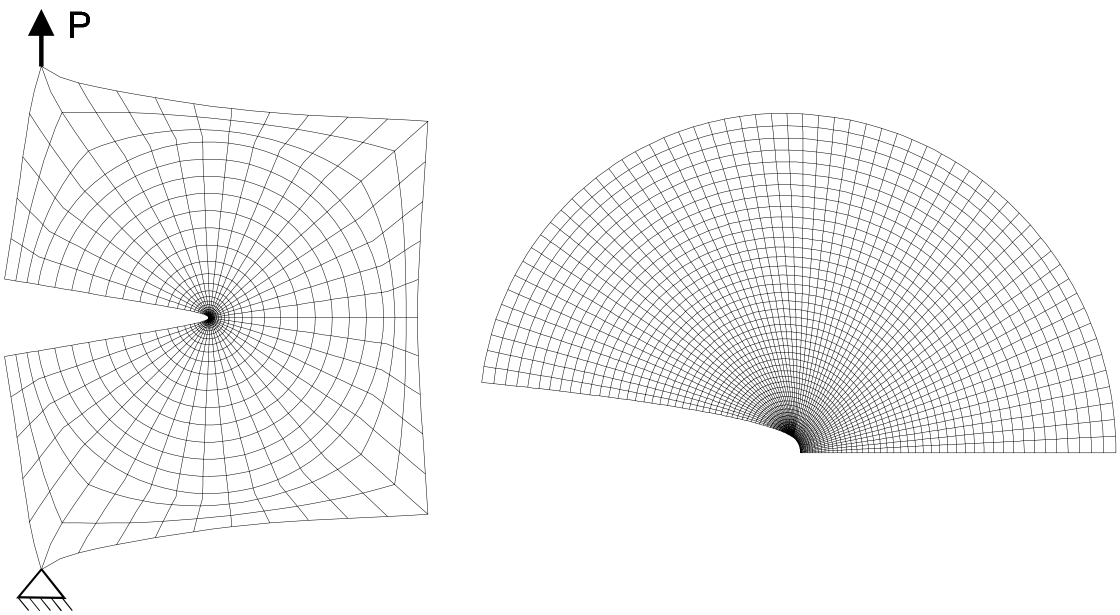

Fig. 1a. The mesh used to study the compact tension (C(T)) fracture mechanics sample

Fig. 1b. Mesh B, used to study the influence of notch root radius. Boundary conditions at $r = R$ were chosen to be those of the homogeneous crack tip solution.

The results from each finite element calculation were used to calculate the total failure probability

$$\Phi = 1 - \exp\left[-\frac{bK^m}{V_o \sigma_o^m} \left[\int_{mesh\ \mathrm{A}} \bar{\sigma}_I^m \, dA + \int_{mesh\ \mathrm{B}} \bar{\sigma}_I^m \, dA \right] \right] \tag{3}$$

where $\bar{\sigma}_I$ are the maximum principal stresses per unit K and b is a reference thickness [5].

Each integral was calculated as the sum over all elements, viz.

$$\int_{vol} \left(\frac{\sigma}{\sigma_o} \right)^m \frac{dV}{V_o} \approx \frac{b}{\sigma_o{}^m V_o} \sum_j^{elems} \left[\sum_i^{Gauss\ pts} \left(\sigma_I^m \right)_i J w_i \right]_j \tag{4}$$

where J is the Jacobian of the mapped element (calculated at the Gauss points), w_i's are the weights for Gauss-Legengre quadrature.

For a given failure probability, the associated stress intensity, denoted K_Φ, can be calculated. Choosing $\Phi = 1/2$, Eq. 3 can be rearranged to:

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$$K_\Phi = \left[\frac{\ln(2)\sigma_o{}^m V_o / b}{\int\limits_{mesh\ A} \bar{\sigma}_I^m \, dA + \int\limits_{mesh\ B} \bar{\sigma}_I^m \, dA} \right]^{1/m} \tag{5}$$

This allows direct calculation of the fracture toughness from only three simple assumptions:

- linear elasticity
- weakest-link failure with strengths characterized by a two-parameter statistics
- geometry with small notch root-radius.

It is unlikely that this would accurately characterize the precise value of the fracture properties for a real material, which would likely depend on a more complex interaction of the crack with the microstructure and the improved statistics required for incremental volumes which approach zero. However, this calculated K_Φ can be used to explore the effects of a gradient when compared to the toughness of a homogeneous material.

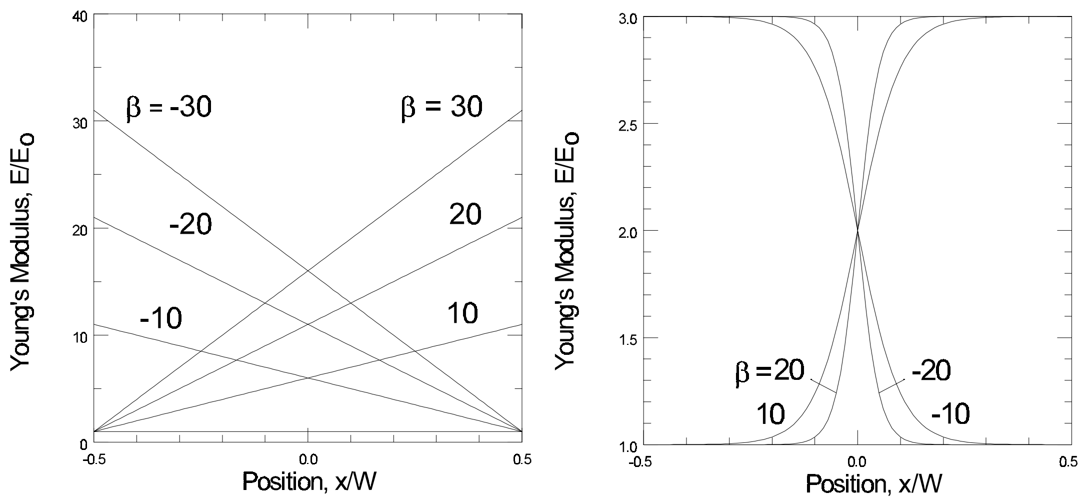


Fig. 2. (a) Linear and (b) hyperbolic tangent variations in modulus studied. In all cases, the crack tip was located at x = 0.

Two gradient shapes were studied in addition to the homogeneous case: first, linear with $E(x)/E_o = \beta x + |\beta| W/2 + 1$, where $\beta = \pm 10, \pm 20, \pm 30$, and second a sigmoidal, $E(x)/E_o = \tanh(\beta x) + 2 * \tanh(\beta W) * |\beta| / \beta$, with $\beta = \pm 10, \pm 20$. Plane strain was assumed and Poisson's ratio was taken as a constant, $\nu = 0.3$. Two notch radii were studied, $\rho/W = 10^{-7}$ and 10^{-6} (if W~cm, then ρ ~1 nm and 10 nm). Finite element calculations were performed for a simplified compact tension sample, Fig 1, with a single crack length, $a/W = 0.5$, and with $W = 1$.

Results and Discussion

K-calibrations for various gradients are displayed in Fig. 3. For cracks growing into a more compliant material (negative β), there is an amplification of the stress intensity factor K in comparison with the homogeneous case. Conversely, the crack tip is shielded when the crack is growing into a stiffer material. Thus, for comparing the different gradients, failure probabilities can be calculated for the same applied load, P, or for the same applied K. The formulation preceding outlines the calculation of Φ for the same K. Failure comparisons based on loads can be inferred from the results of Fig. 3 and indicate that cracks growing into positive β gradients are expected to be tougher than homogeneous or negative β materials.

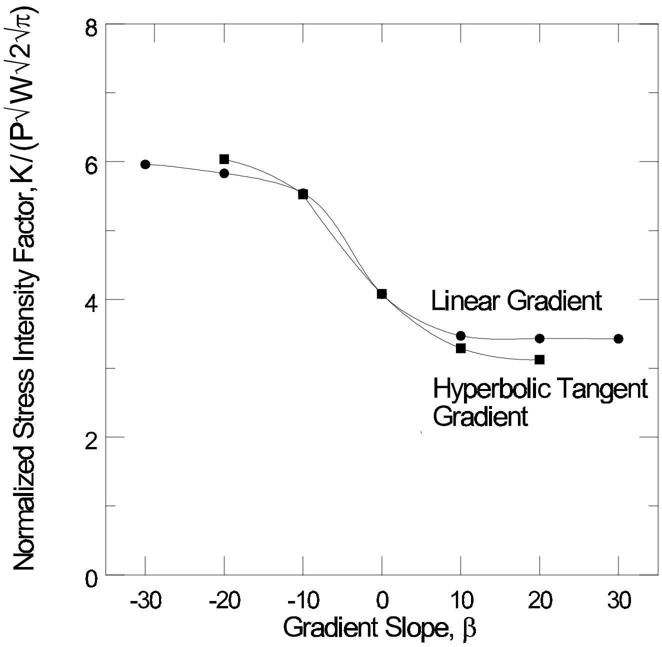
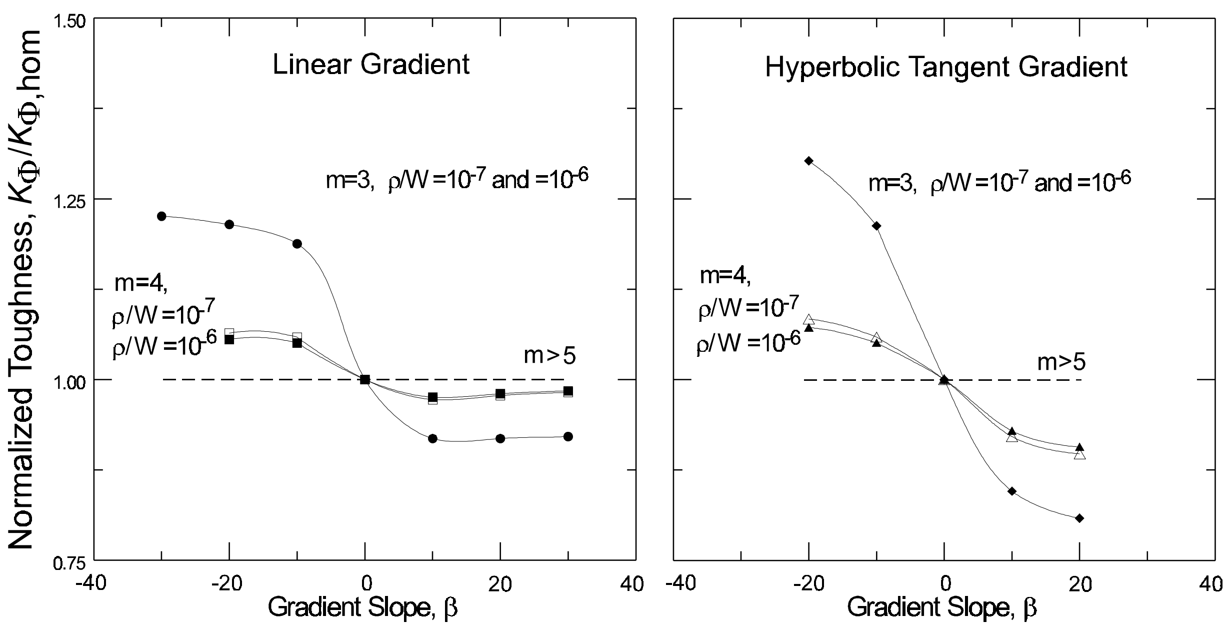


Fig. 3. Stress intensity factor calibration for the gradients studied. The shielding effect on the stress intensity factor for cracks growing into stiffer materials is evident.



(a) (b)

Fig. 4. The measurable fracture toughness K_Φ for specimens with (a) linear gradients and (b) sigmoidal gradients in modulus. For both shapes, there is an effective toughening for cracks growing into more compliant material. For Weibull moduli of 5 and greater, the effect is zero.

The failure probabilities compared on a per-*K* basis for various gradients are depicted in Fig. 4. They indicate that for gradients with negative β, the expected fracture toughness is greater than that of a homogeneous material. For positive gradient slope, the expected K_Φ is less than for a homogeneous material.

The results clearly show the role of the gradient in the probability of failure (fracture) for low Weibull modulus, *m*. The differences between different gradients vanish for m ≥ 5. Although these are low moduli in comparison to traditional monolithic ceramics, bend strength tests of a SiO_2/Mo FGM [2] measure a Weibull modulus of 3.8. Thus, it seems that for brittle composites, these moduli can be realistic and predictions of this model can be expected to be exhibited.

The trends of increasing influence of the gradient with decreasing *m* correspond to the most probable location of fracture initiation moving away from the crack tip, where the form of the stress solution deviates from the classical form. The insensitivity of the results to ρ is an expression of this trend. Over the range of parameters studied, for Weibull modulus $m = 3$ the results were insensitive to notch-tip radius ρ and substantially affected by gradient slope β. For Weibull modulus $m \geq 5$, the results were insensitive to β but strongly affected by ρ.

Concluding Remarks

The stress intensity factor and thereby the failure probability are reduced by a positive β gradient for samples experiencing the same load, *P*. Conversely, for materials with Weibull modulus $m \leq 4$, negative β gradients will be tougher than homogeneous materials for comparisons made on a per-*K* basis.

These results indicate that although *K* is a valid scaling parameter for the stress field near the crack tip for an FGM, the dependence of the stress field ahead of the crack on the gradient in *E* renders K_Φ to be an inaccurate predictor of failure if the fracture initiation occurs away from the tip. These effects arise because the stresses ahead of the crack tip and therefore the failure probability are higher with a positive gradient than would exist for a homogeneous material at a given *K*.

Although the source of the trends here is the modulus, similar effects could be expected from the higher-order terms in the Williams expansion. Such terms exist even in homogeneous materials and their influence would indicate a geometry dependence to fracture toughness for any such low Weibull modulus material. Thus, it should be appreciated that the present results depend in detail upon the actual specimen size and geometry.

The framework used in this study allows for the incorporation of residual thermal stresses, mixed-mode loading, and strength parameters, *m* and σ_o, that vary with location. In addition, the problems addressed here concern only the crack growing in the direction of the gradient. The influence of these effects on crack path stability also needs to be discerned.

References

1. R. O. Ritchie, J. F. Knott and J. R. Rice: *J. Mech. Phys. Solids*, 1973, vol. 21:6, pp. 395-410.
2. A. P. Tomsia, R. M. Cannon *et al*: *Evaluation of Mo-SiO_2 Continuous FGM's*, Technical Report No. LBNL-40886, Lawrence Berkeley National Laboratory, 1997.
3. M. Gopal: Ph.D. Thesis, University of California at Berkeley, 1997
4. F. Erdogan: *Composite Engineering*, 1995, vol. 5:7, pp. 753-770.
5. T. Lin, A. G. Evans and R. O. Ritchie: *J. Mech. Phys. Solids*, 1986, vol. 34:5, pp. 477-497.
6. D. Curry and J. F. Knott: *Metal Science*, 1979, vol. 13:6, pp. 341-345.
7. H. Tada, P. C. Paris and G. R. Irwin: *The Stress Analysis of Cracks Handbook*, 2nd ed., Paris Productions, Inc., St. Louis, 1985.
8. O. C. Zienkiewicz and R. L. Taylor: *The Finite Element Method*, McGraw-Hill, New York, N.Y., 1987.

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