Adaptive Hybrid Mesh Optimization and Refinement

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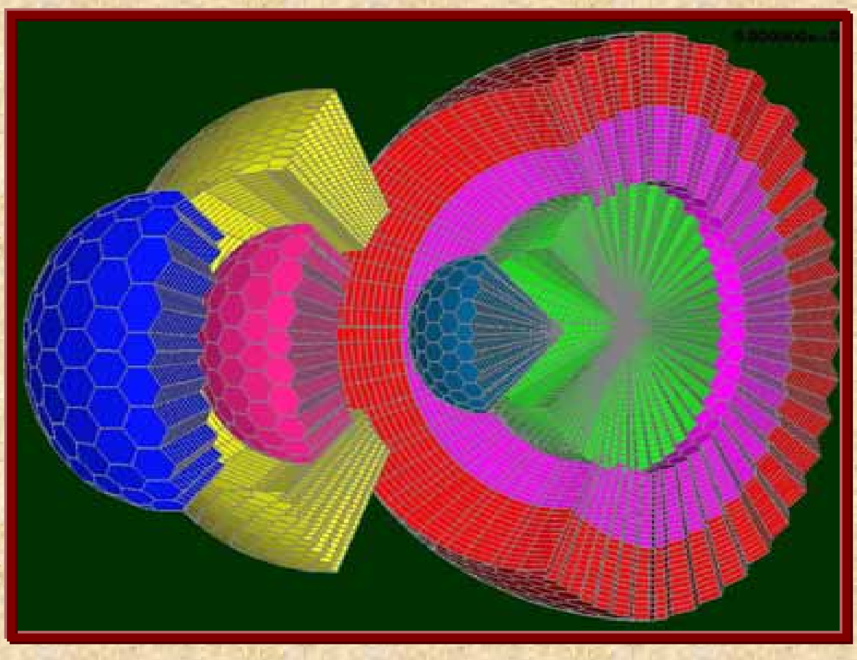
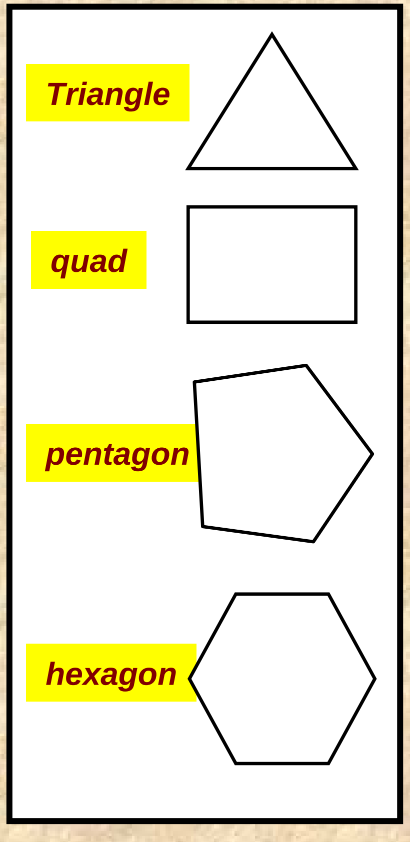
Overview

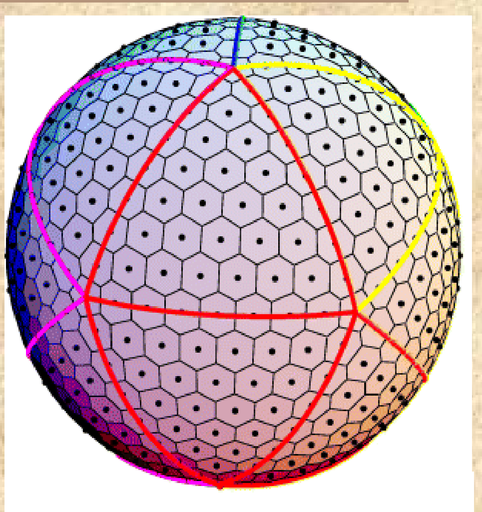
- **Hybrid Mesh Generation**
- **Mesh Quality Requirements**
- ***Hybrid Mesh Optimization***
- ***Adaptive Hybrid Meshing***
- ***Future Development and challenges***

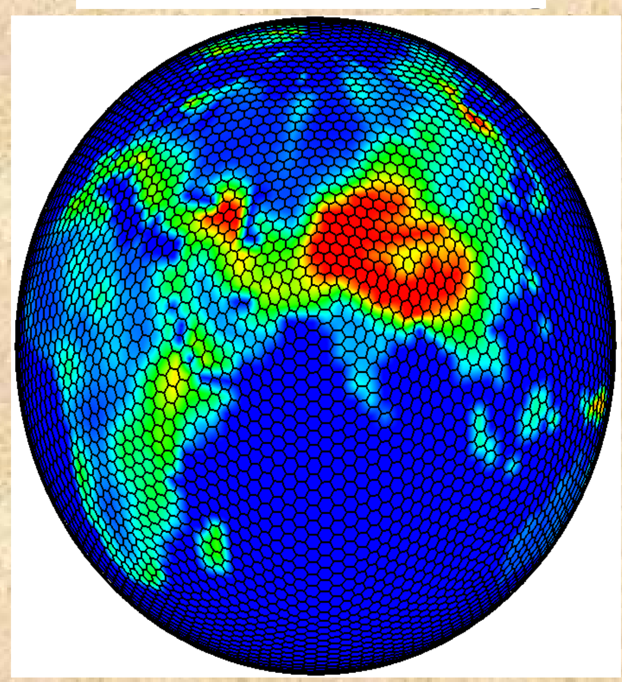


Hybrid Mesh Generation

- *Generation of isotropic or anisotropic structured, unstructured, or hybrid meshes*
- *Mesh cell faces are composed of triangles, quads, pentagons, and hexagons*
- *Mesh optimization algorithms to improve geometric quality measures such as angles, lengths, and areas*
- *Elliptic and algebraic models are used to optimize and redistribute mesh nodes to capture geometric or simulation features*

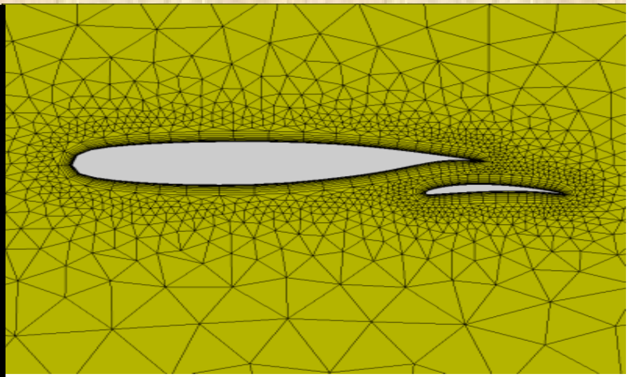


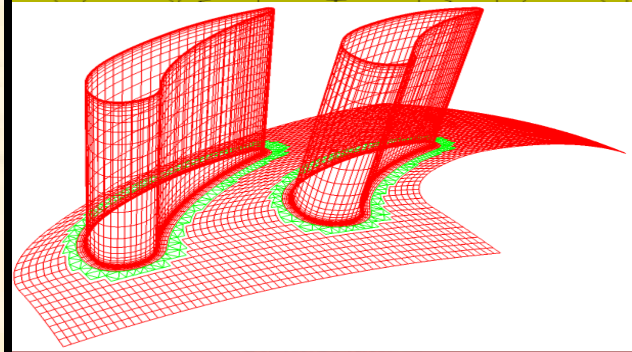


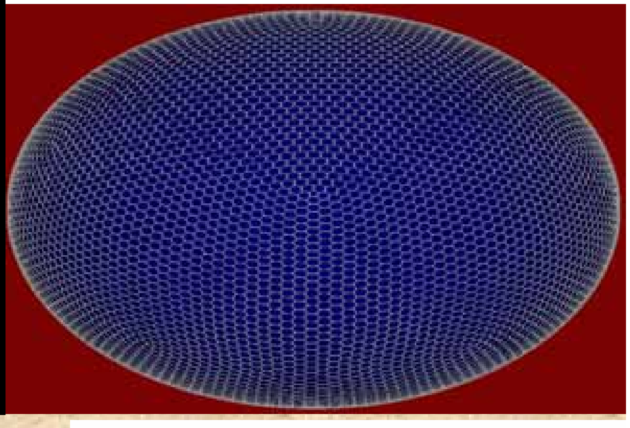


Advantage of Hybrid Meshes

- Mesh cell topology range from hexagons, pentagons, quads, to simple triangles.
- Hybrid meshes combines the advantages of structured and unstructured meshes.
- Icosahedra meshes are best fitted for solving symmetric computational problems.
- Quadrilateral layers close to boundary exhibit good orthogonality and clustering.
- Prismatic and hexahedral elements are best suited for regions of high solution gradients.
- Triangular elements are well suited for the resolution of active geometric features.
- The flexibility of hybrid mesh generation approach allows for automation process.
- Hybrid meshes are less sensitive to stretching and less stiff, so gradients can be better approximated than is in the case of triangular cells.







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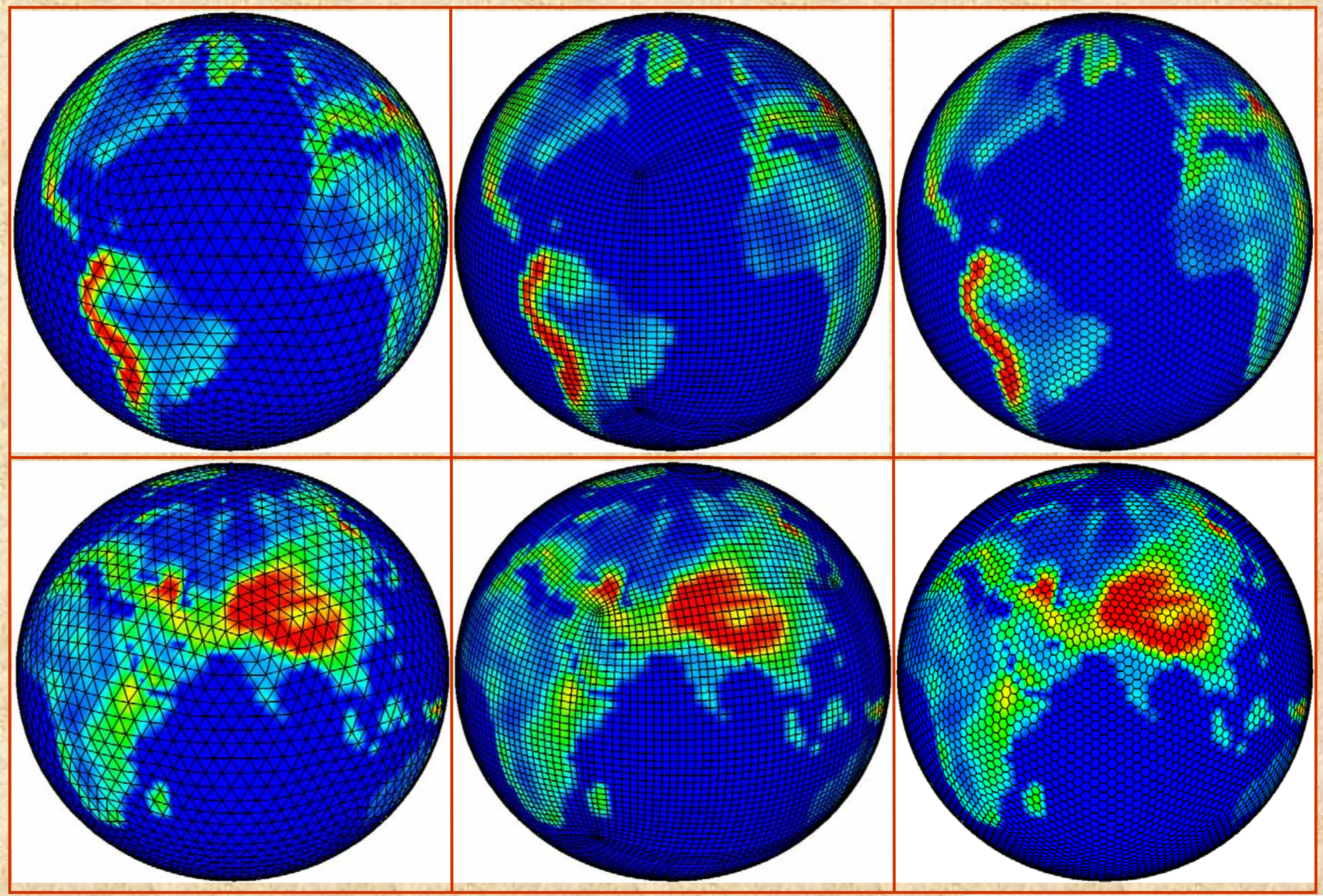
Triangular, Quadrilateral, and Icosahedra Surface Meshes for Climate Modeling



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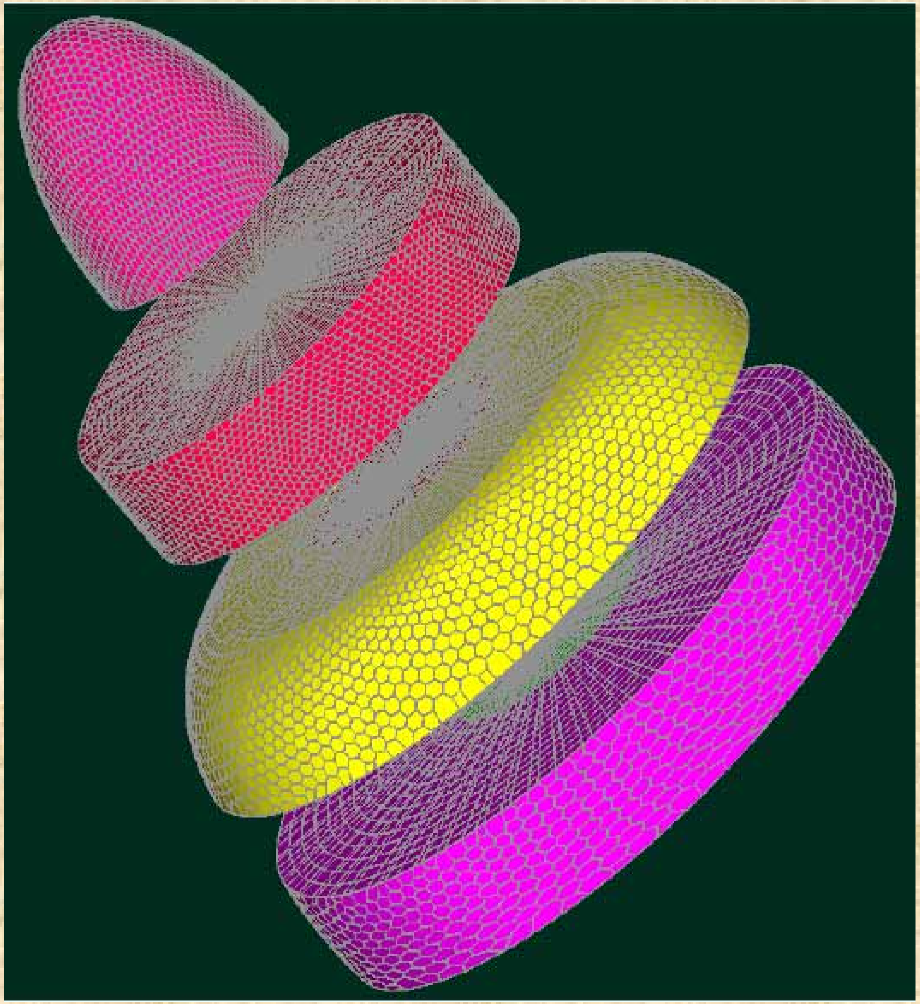
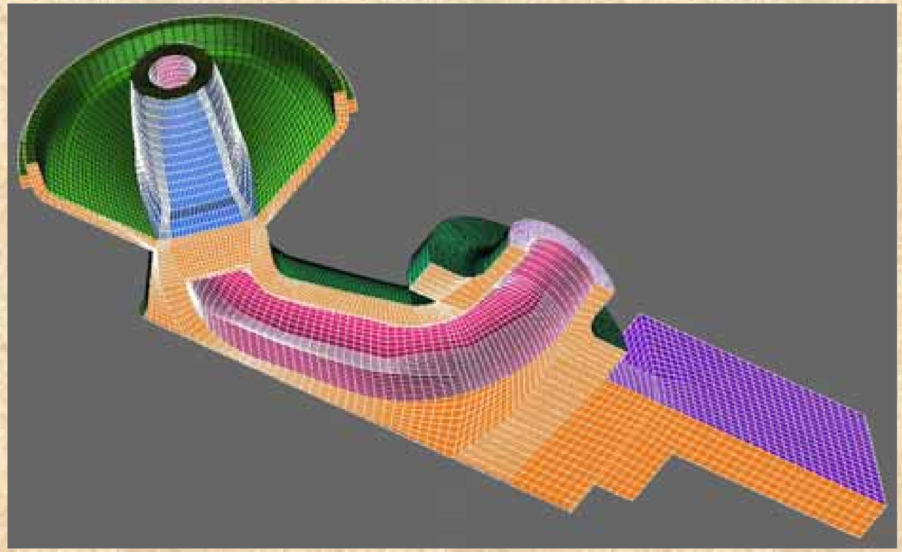
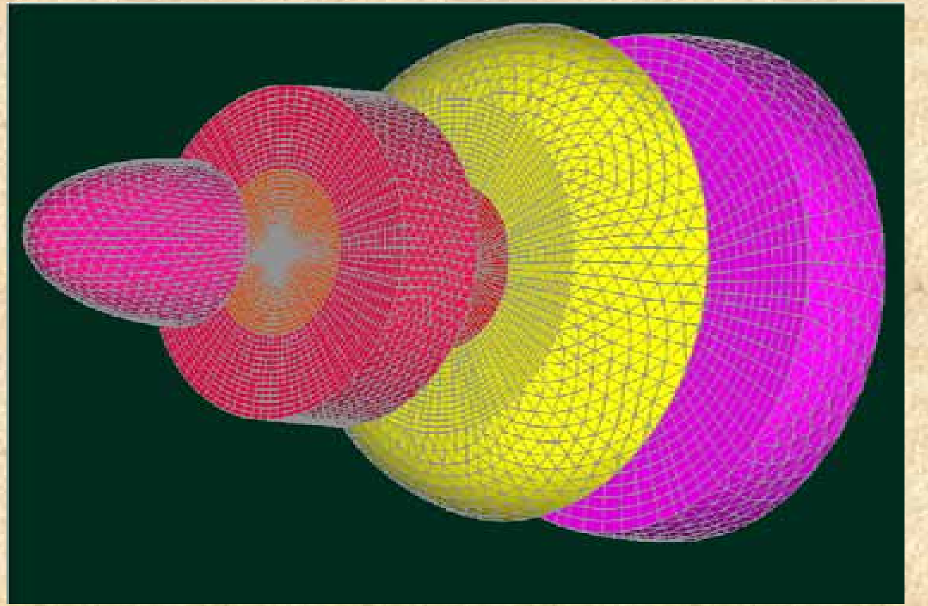
Hybrid Mesh Generation & Optimization







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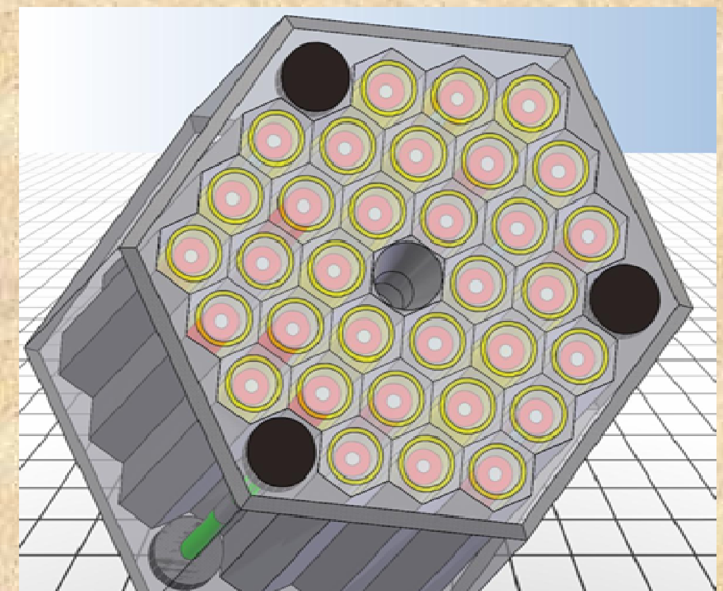
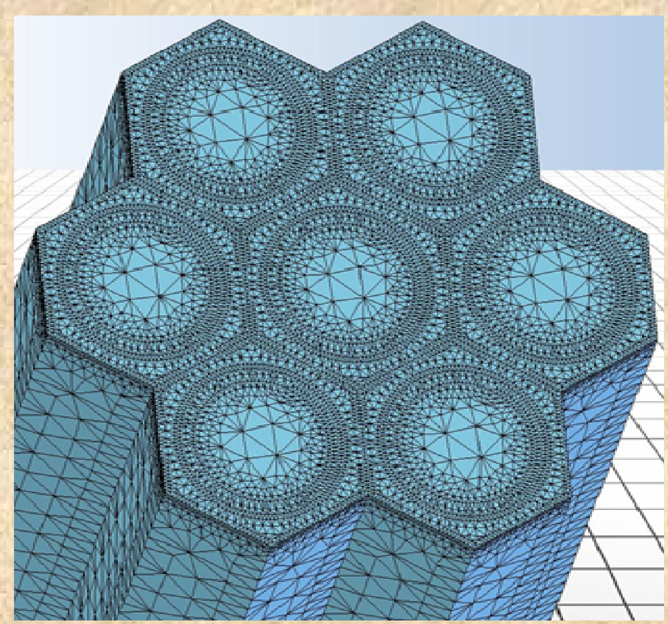
Meshing Challenges – Multiphysics Requirements

- **Different physics models impose distinct mesh requirements**
  - **Thermal-hydraulics in fluid regions**
  - **Thermo-mechanics in solid regions**
  - **Reaction/diffusion in fuel**
  - **Neutronics everywhere**

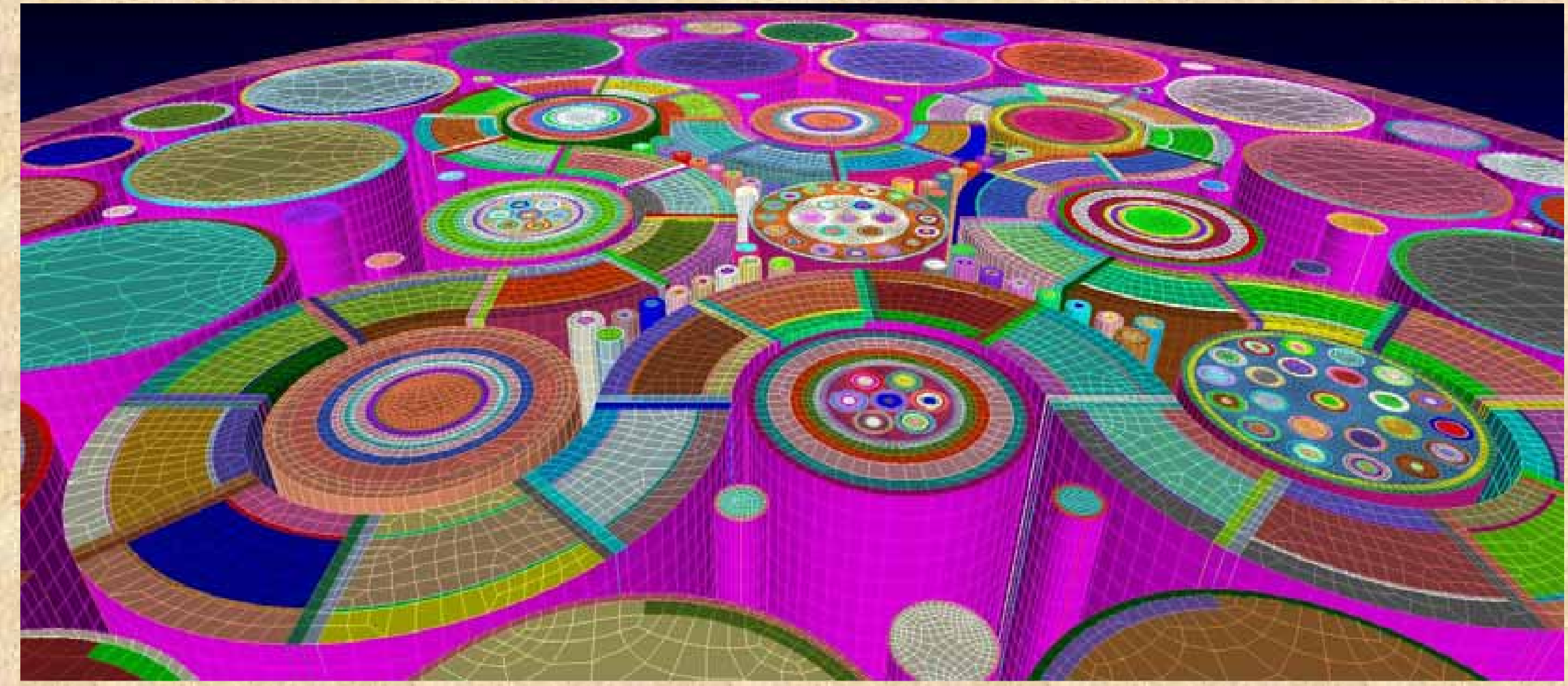
- **Which physics model should define the mesh characteristics?**
  - **A fine, anisotropic mesh is used to capture boundary layer flow and heat flux**
  - **A coarse mesh is needed for neutronics in the coolant channels**





Meshing Challenges – High Quality Elements

- Orthogonal hexahedral elements are preferred by many multiphysics applications
- Anisotropic meshing wire-wrapped pins and fuel geometry will be quite a challenge



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Conformal Hybrid Mesh Generation

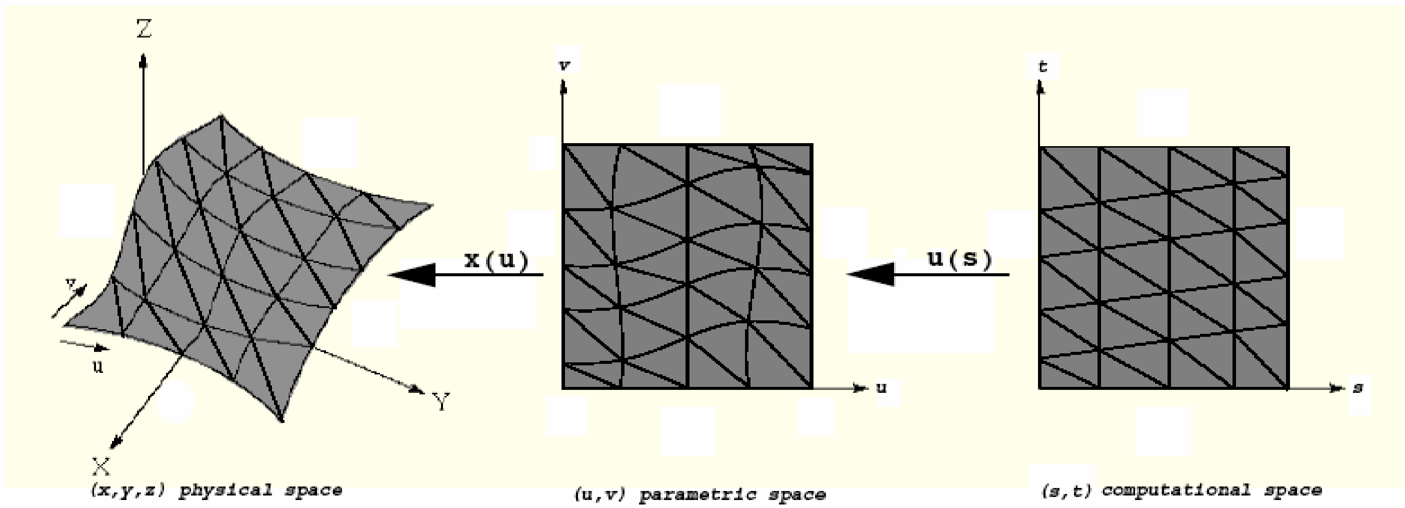
The surface geometry is defined as a mapping $\mathbf{x}(u,v) : I\!R^2 \rightarrow I\!R^3$,

$$\mathbf{x} = (x,y,z) = \Big(x(u,v), y(u,v), z(u,v)\Big)$$

The surface mesh is defined as a composite mapping $\mathbf{x}(s,t) : I\!R^2 \rightarrow I\!R^3$,

$$\mathbf{x}(\mathbf{s}) = \mathbf{x}(\mathbf{u}) \circ \mathbf{u}(\mathbf{s})$$

where $\mathbf{u} = (u,v)$, and $\mathbf{s} = (s,t)$.



Mapping from computational space to physical space via parametric space

Adaptive Hybrid Surface Meshing

Given a C^2 mesh quality function, *e.g.*, cell area, $\phi \equiv \phi(\mathrm{s})$ in the (s, t) domain. To smoothly equidistribute ϕ over the mesh, we minimize the $\mathcal{L}^2$ norm of the gradient-weighted quality function

$$\int \rho ||\nabla_{\mathbf{s}} \phi||^2 \mathrm{d}S,$$

where ρ is the mesh point density function and $\mathrm{d}S = ||\mathrm{x}_u \times \mathrm{x}_v|| \mathrm{d}u \mathrm{d}v$.

We solve Euler-Lagrange equation whose solution minimizes this functional is

$$\nabla_{\mathbf{s}} \cdot \rho \nabla_{\mathbf{s}} \phi = 0.$$

The density function ρ is chosen as the

- Area-gradient equidistribution, where mesh point density ρ is proportional to the *rate of change of the area function* ϕ:

$$\rho \propto ||\nabla \phi||$$

- Curvature-weighted adaptive spacing, where mesh points are concentrated in areas of large curvature:

$$\rho \propto ||\mathcal{K}\sqrt{\phi}||,$$

- Solution-weighted adaptive smoothing, where mesh points are concentrated in areas of large gradient of the solution:

$$\rho \propto ||\nabla \psi \sqrt{\phi}||,$$

To transform previous Poisson equation $\nabla_{\mathrm{s}} \cdot \rho \nabla_{\mathrm{s}} \phi = 0$ from the computational domain (s, t) to the parametric domain (u, v), we apply coordinate transformation to the second-order differential equation for surfaces

$$\nabla_{\mathrm{s}} \cdot \rho \nabla_{\mathrm{s}} \phi = \frac{1}{J}\left\{ \frac{\partial}{\partial u} \frac{\rho \mathrm{g}_{22} \phi_{\mathrm{u}} - \rho \mathrm{g}_{12} \phi_{\mathrm{v}}}{J} + \frac{\partial}{\partial v} \frac{\rho \mathrm{g}_{11} \phi_{\mathrm{v}} - \rho \mathrm{g}_{12} \phi_{\mathrm{u}}}{J} \right\}$$

The density function ρ is chosen as the linear combination of geometric and solution parameters

$$\rho = \lambda_a + \lambda_\kappa \left(|\mathcal{K} \sqrt{\phi}| \right) + \lambda_s \left(|\nabla \psi \sqrt{\phi}| \right),$$

where the positive weights λ_a, λ_κ, and λ_s satisfy $\lambda_a + \lambda_\kappa + \lambda_s = 1$.

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Under this transformation, the parametric variables become the independent variables, so that so that the equation can be expressed in the form

$$\nabla_{\mathrm{u}} \cdot \Phi = 0$$

where the vector function Φ is given by

$$\Phi = \frac{\rho}{J}\left(\mathrm{g}_{22}\phi_{\mathrm{u}} - \mathrm{g}_{12}\phi_{\mathrm{v}}, \mathrm{g}_{11}\phi_{\mathrm{v}} - \mathrm{g}_{12}\phi_{\mathrm{u}}\right)$$

The Jacobian J of the surface mapping $\mathrm{x}(\mathrm{u})$ is given

$$J = ||\mathrm{x}_u \times \mathrm{x}_v|| = \sqrt{\mathrm{g}_{11}\mathrm{g}_{22} - \mathrm{g}_{12}^2}$$

where $\mathrm{g}_{11} = \mathrm{x}_{\mathrm{u}} \cdot \mathrm{x}_{\mathrm{u}}$, $\mathrm{g}_{12} = \mathrm{x}_{\mathrm{u}} \cdot \mathrm{x}_{\mathrm{v}}$, and $\mathrm{g}_{22} = \mathrm{x}_{\mathrm{v}} \cdot \mathrm{x}_{\mathrm{v}}$.

Solution Adaptive Mesh Optimization

We choose a cell-centered finite volume discretization to solve for ϕ in the parametric domain Ω. The PDE can be re-written in integral form

$$\int_{\Omega} \nabla \cdot \Phi \mathrm{d}A = 0$$

by Gauss's theorem this equivalent to

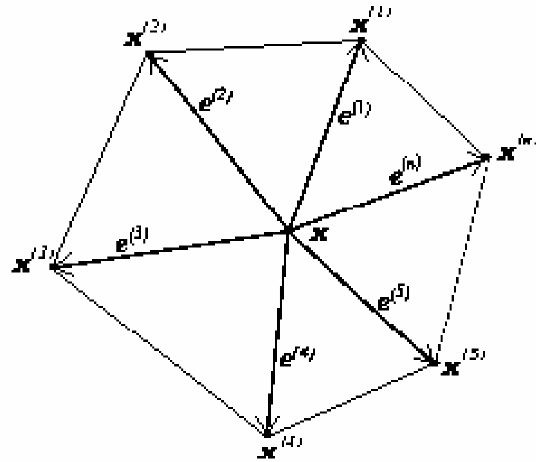
$$\int_{\partial\Omega} \Phi \cdot \mathrm{d}\mathbf{l} = 0$$

where $\mathbf{l} = l\hat{\mathbf{t}}$ is edge length l times the unit tangent vector $\hat{\mathbf{t}}$ pointing away from Ω.

In particular, for a given cell C_i, this equation can be expressed as

$$\sum_{\text{edges } j} \int_{E_{ij}} \Phi \cdot \mathrm{dl} = 0$$

where E_{ij} is the is the current cell edge.



The semi-discrete form of the integral equation is given by

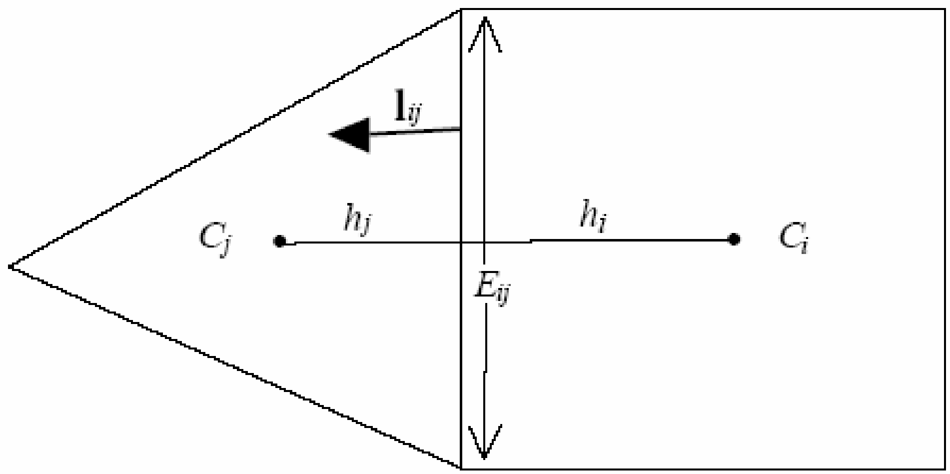
$$\sum_{\text{edges } j} \Phi_{E_{ij}} \cdot \mathrm{l}_{ij} = 0$$

where $\Phi_{E_{ij}} = \frac{1}{l_{ij}} \int_{E_{ij}} \Phi \cdot \mathrm{dl}$ is the average value of Φ on the edge E_{ij} with length l_{ij}.

The value of $\Phi_{E_{ij}}$ on any edge is computed using a cell centered first-order approximation

$$\Phi_{E_{ij}} \approx \frac{h_i \Phi_{C_i} + h_j \Phi_{C_j}}{h_i + h_j}$$

where C_i and C_j are the cells sharing E_{ij}, and h_i, h_j are the distances from the cell centers to the edge, and Φ_{C_i} , Φ_{C_j} are the average value of Φ in the cells C_i and C_j.



Definition of quantities on edge E_{ij} of cell C_i and shared by C_j

To be able to calculate the vector function

$$\Phi_{C_i} = \left(\frac{g_{22}\phi_u - g_{12}\phi_v}{J}, \frac{g_{11}\phi_v - g_{12}\phi_u}{J} \right)_{C_i}$$

in the cell C_i, we need to compute $\nabla\phi_{C_i} = (\phi_u, \phi_v)_{C_i}$. The metrics g_{11}, g_{12}, g_{22}, and J are evaluated from the surface definition $\mathbf{x}(\mathbf{u})$ at cell center.

Again we use Gauss's theorem

$$\begin{aligned}\nabla\phi_{C_i} &= \frac{1}{A_i}\int \nabla\phi \, dA \\ &= \frac{1}{A_i}\int \phi \, d\mathbf{l} \\ &= \frac{1}{A_i}\sum_{\text{edges } j} \phi_{E_{ij}} \mathbf{l}_{ij}\end{aligned}$$

where A_i is the area of cell C_i and $\phi_{E_{ij}}$ is the average value at of ϕ on E_{ij} and is approximated from values at cell centers

$$\phi_{E_{ij}} = \frac{h_i\phi_{C_i} + h_j\phi_{C_j}}{h_i + h_j}$$

Therefore,

$$\nabla\phi_{c_i} = \frac{1}{A_i}\sum_{\text{edges } j} \frac{h_i\phi_{C_i} + h_j\phi_{C_j}}{h_i + h_j}\, \mathbf{l}_{ij}$$

Solution Adaptive Mesh Smoothing

To relocate the new node $\mathbf{u} \in \cup_{i=1}^{n} C_i$ we minimize the following functional

$$\mathcal{F} = \sum_{\text{cells } i} \left(\Delta\phi_i^t(\mathbf{u}) - \Delta\phi_i^s\right)^2$$

$$\text{subject to } \; \Delta\phi_i^s = \max\left\{\Delta\phi_i^s, -\mu\phi_i\right\}$$

where $\Delta\phi_i^s = \phi_i^s - \phi_i$, $\Delta\phi_i^t(\mathbf{u}) = \phi_i^t(\mathbf{u}) - \phi_i = J_i(A_i^t(\mathbf{u}) - A_i)$, and $\mu \in [\frac{1}{2}, 1)$.

Minimizing this functional leads to solving $\nabla\mathcal{F} = 0$ for $\mathbf{u}$. This is equivalent to solving this algebraic system for u and v:

$$\begin{pmatrix} \sum_i J_i^2 a_i^2 & \sum_i J_i^2 a_i b_i \\ \sum_i J_i^2 a_i b_i & \sum_i J_i^2 b_i^2 \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} = \begin{pmatrix} \sum_i J_i a_i \Delta\phi_i^s \\ \sum_i J_i b_i \Delta\phi_i^s \end{pmatrix}$$

where $\Delta u = u - u_c$ and $\Delta v = v - v_c$.

The updated node position on the surface is evaluated using $\mathbf{x}(\mathbf{u})$.

$$\begin{aligned} A_i^t(\mathbf{u}) &= \frac{1}{2}||(\mathbf{u}_i - \mathbf{u}) \times (\mathbf{u}_{i+1} - \mathbf{u})|| \\ &= \frac{1}{2}\Big(a_i u + b_i v + c_i\Big), \end{aligned}$$

where the coefficients $a_i = v_i - v_{i+1}$, $b_i = u_{i+1} - u_i$, and $c_i = u_i v_{i+1} - u_{i+1} v_i$.

Mesh Quality Optimization

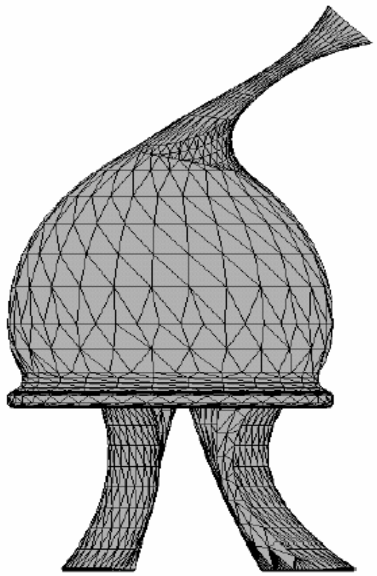
- ***Mesh Smoothing***
  - ✓Mesh point densities are equidistributed on the surface
  - ✓Mesh lines should be smooth to provide continuous derivatives
  - ✓Mesh cells should have areas that vary smoothly across the surface
  - ✓Adjacent facets of the surface mesh have normals adjusted to vary more gradually
- ***Mesh Orthogonality***

  Excessive mesh skewness (non-orthogonal intersection of mesh lines) should be avoid, since it sometimes increase truncation errors
- ***Mesh Reconnection***

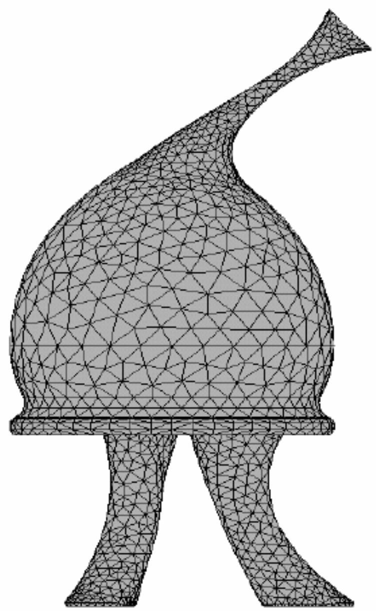
  Edge flipping/swapping to improve mesh cell aspect ratios
- ***Mesh Adaptation***

Mesh points should be closely spaced on the physical surface where large numerical errors are expected

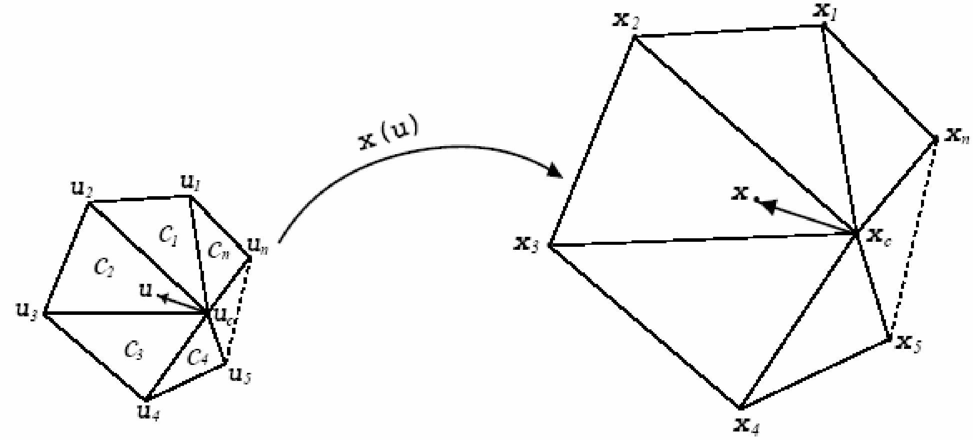
Mesh Optimization

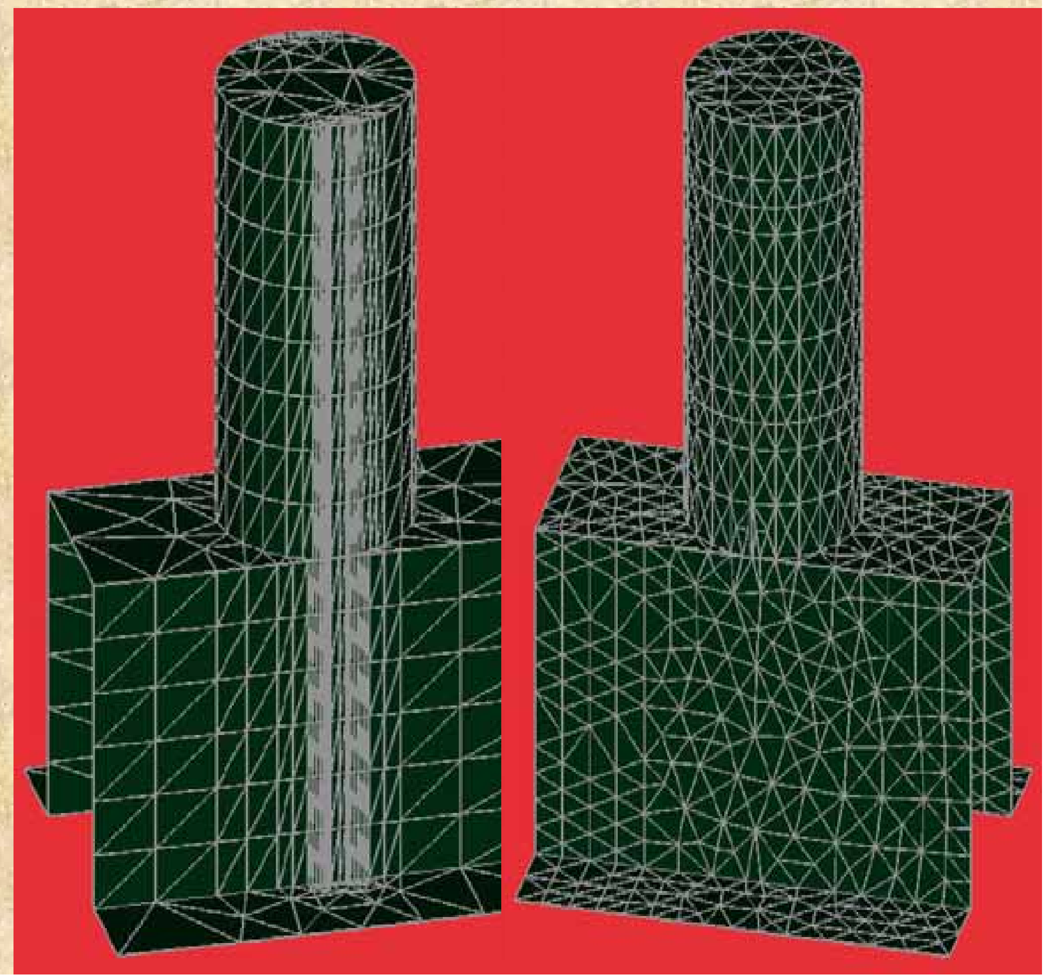


Initial anisotropic triangular mesh on parametric surfaces

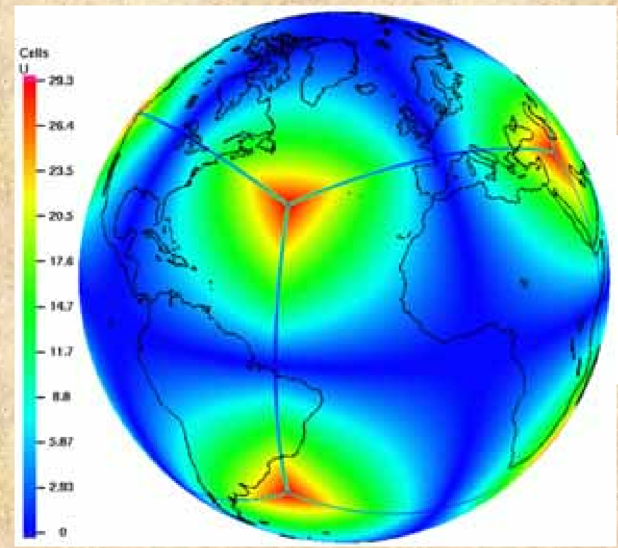


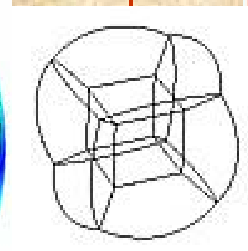
Elliptic isotropic smoothed surface mesh with reconnection



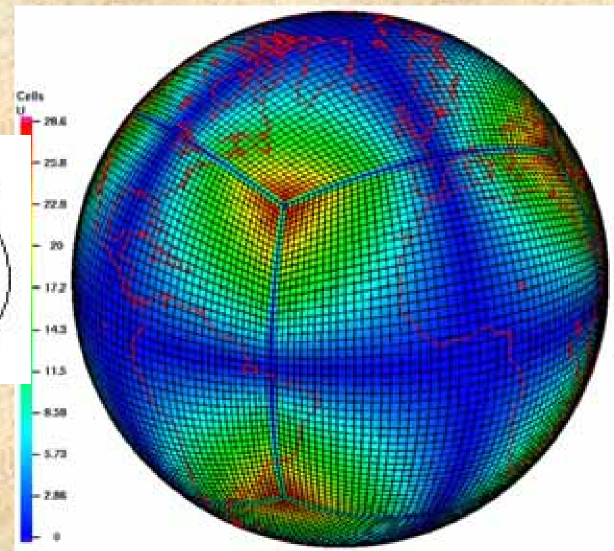


Orthogonal Meshing

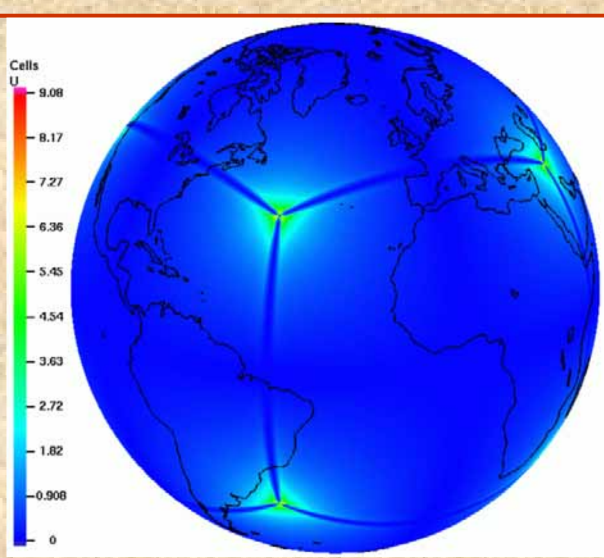


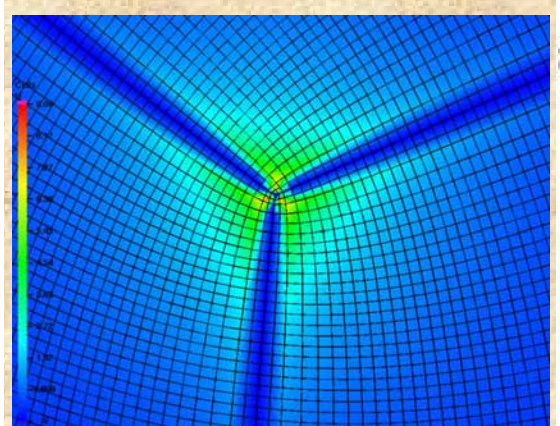


Angle deviation field from orthogonality of an initial surface mesh

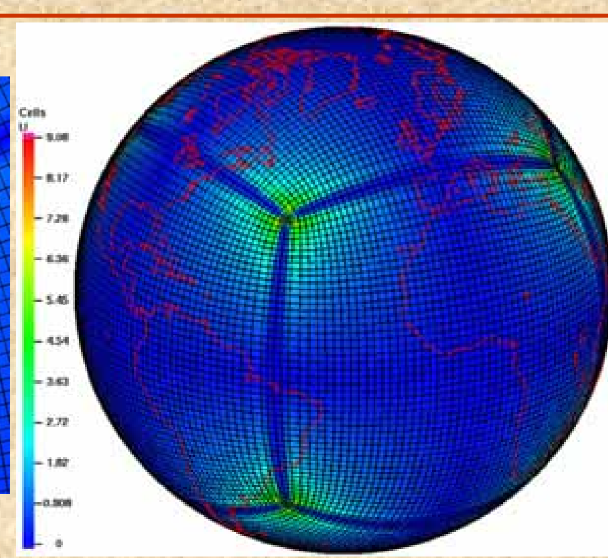


Initial algebraic surface mesh with 90 degree deviation field





A close-up of an orthogonal surface mesh



Angle deviation field from orthogonality an elliptic surface mesh

Orthogonal elliptic surface mesh with 90 degree deviation field

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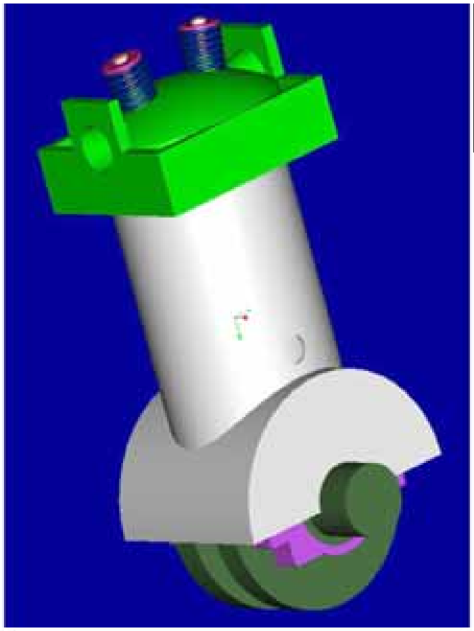
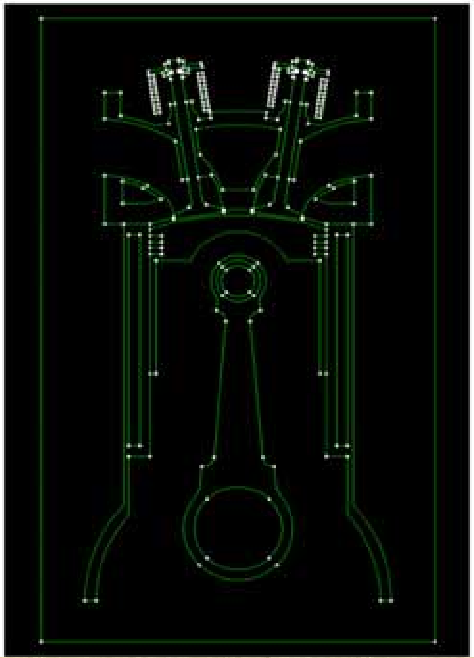
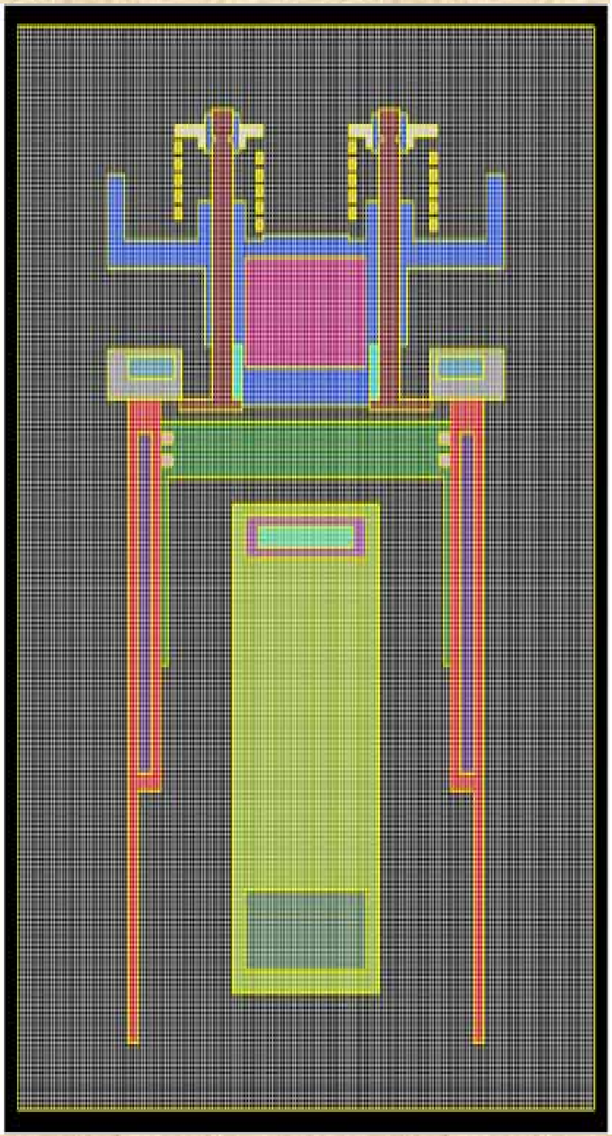
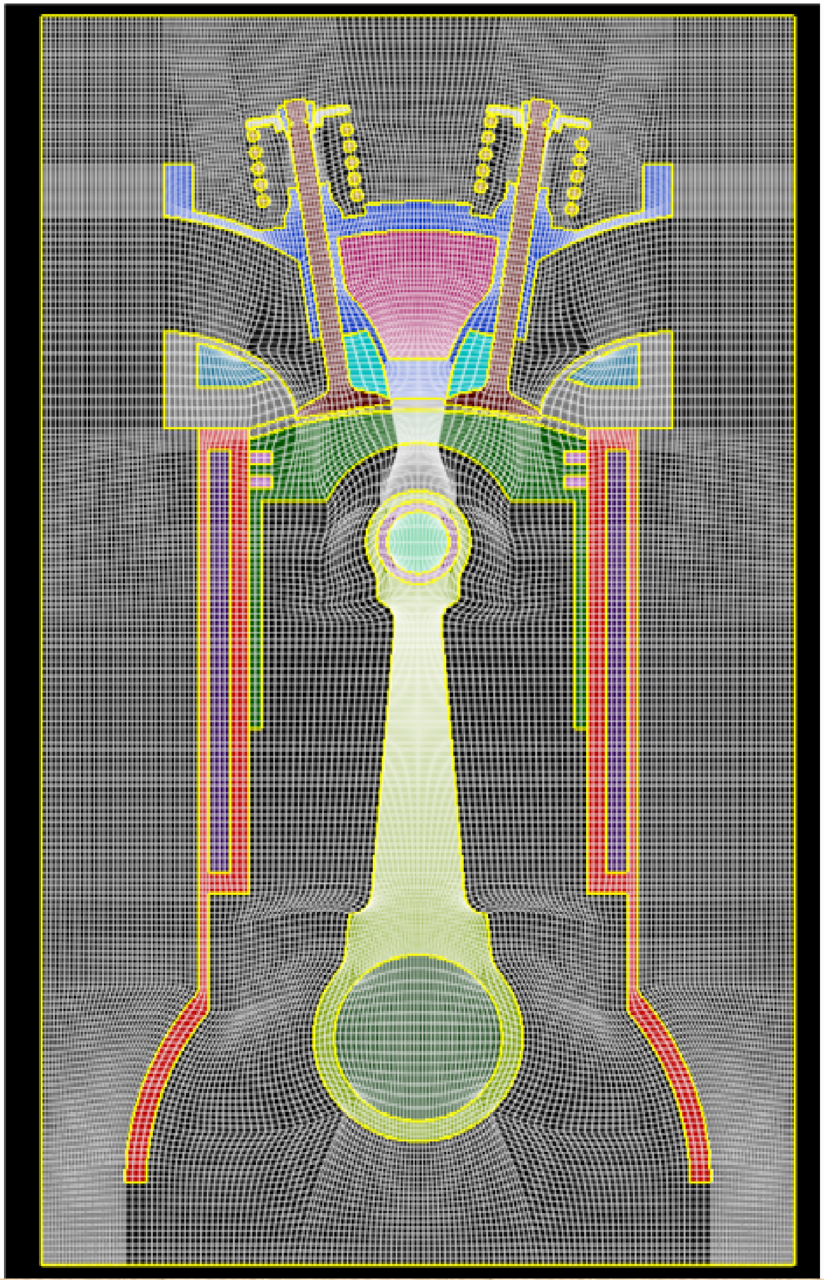


Mesh Alignment







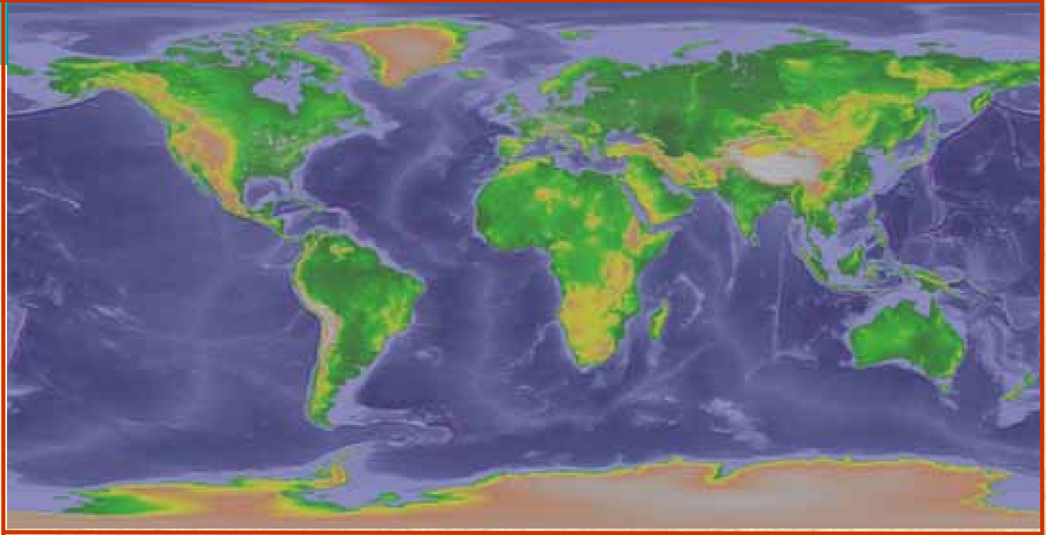
Multiphysics Mesh Adaptation

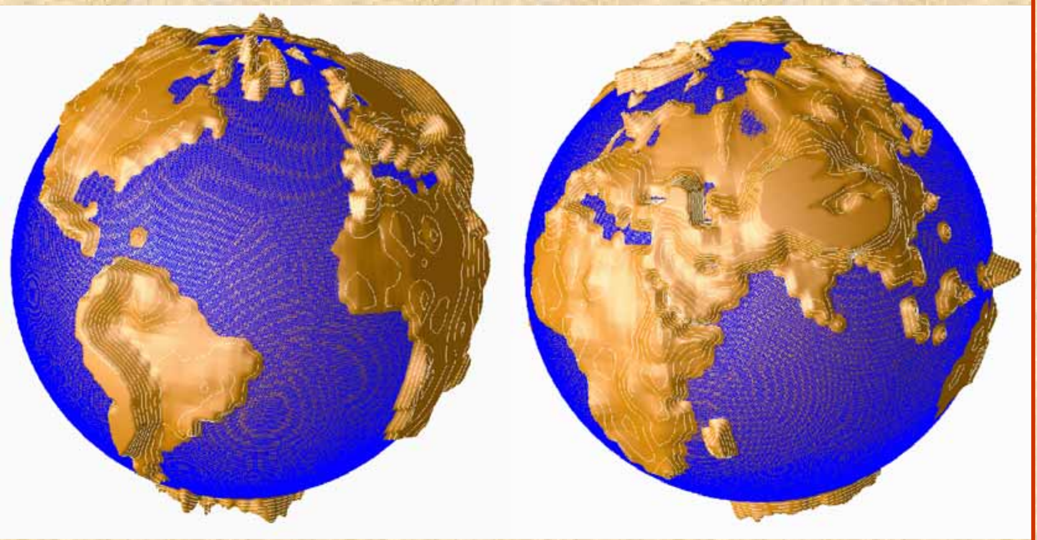
- The “amount” of mesh at a given location should be selected to resolve the smallest physics length scale at that point
- Too little mesh results in a locally-incorrect solution, too much slows the calculation needlessly
- The quality of the solution also depends on other mesh characteristics, however:
  - Element shapes and connectivity
  - Smoothness and “impedance” requirements
  - Element orthogonality requirements
  - Anisotropic elements to match anisotropic physics
  - Boundary representation requirements
  - . . .
- Robust adaptation may be useful for other purposes beyond ensuring proper sampling

ADAPTIVE SHALLOW ATMOSPHERE SIMULATION

Why orography and how is used in climate codes

- **Orography is the average height of land, measured in geopotential meters**
- **Orography plays an important role in determining the strength and location of the atmospheric jet streams**
- **The orographic impact is most pronounced in the numerical simulation codes for the detailed regional climate studies**
- **Crucial parameter for prediction of many key climatic dynamics, elements, and moist physics, such as rainfall, snowfall, and cloud cover**
- **The phenomena of climate variability are sensitive to orographic effects can be resolved by the generation of finer meshes in regions of high altitude**
- **Resolving orography produces a more accurate prediction of wetter or dryer seasons in a particular regions**
- **Orography defines the lower boundary in general circulation models**





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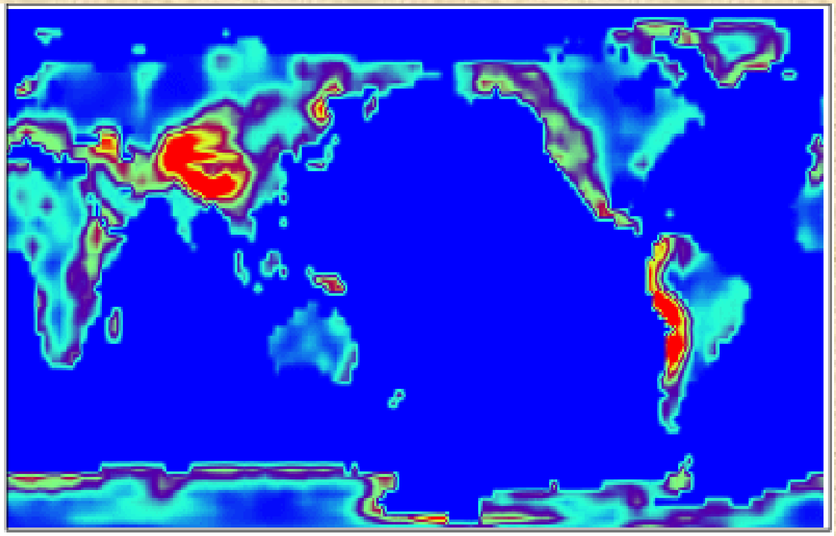
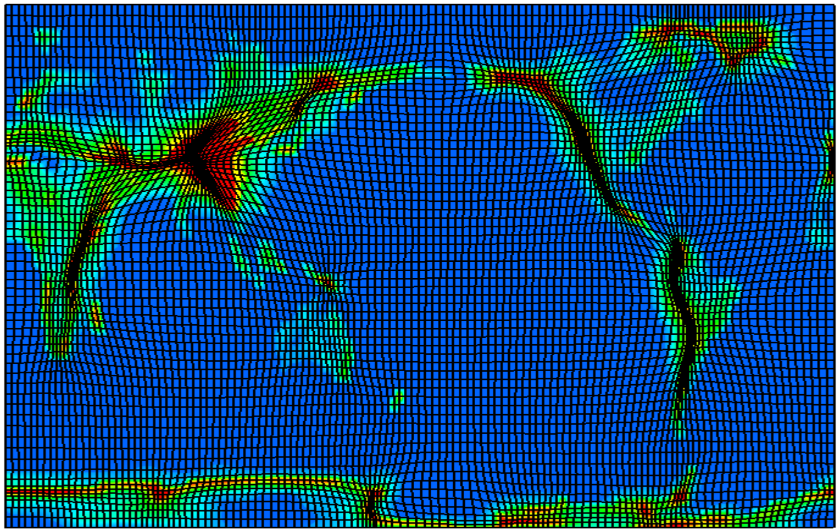
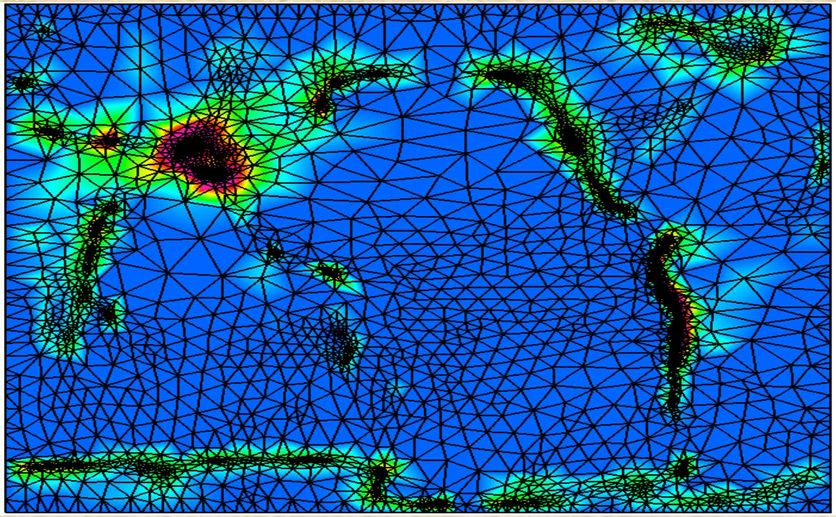
r-Adaptive Hybrid Mesh Generation for Climate Modeling



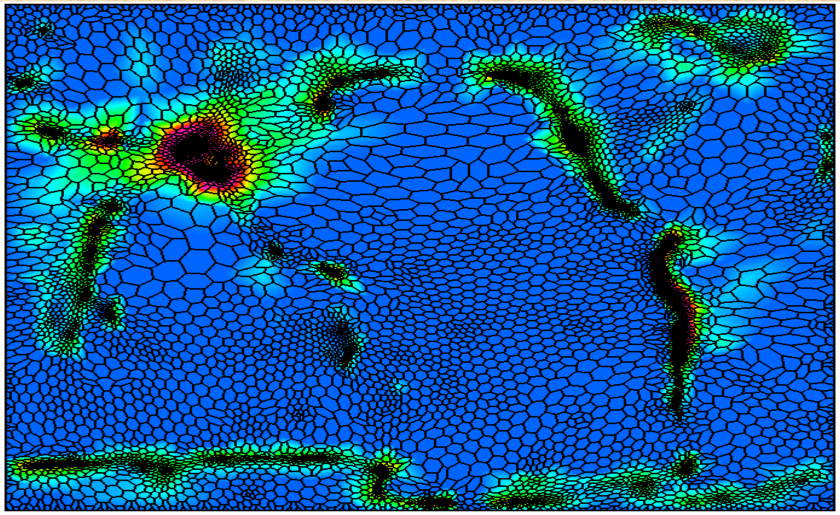
Earth's planar orography field exhibiting global high altitude regions



Structured adapted planar mesh based on orographic field data



Unstructured adapted planar mesh based on orographic field data



Hybrid adapted planar mesh based on orographic field data

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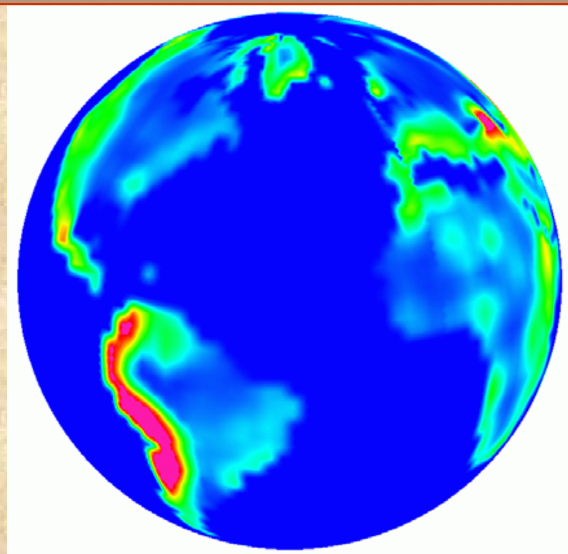
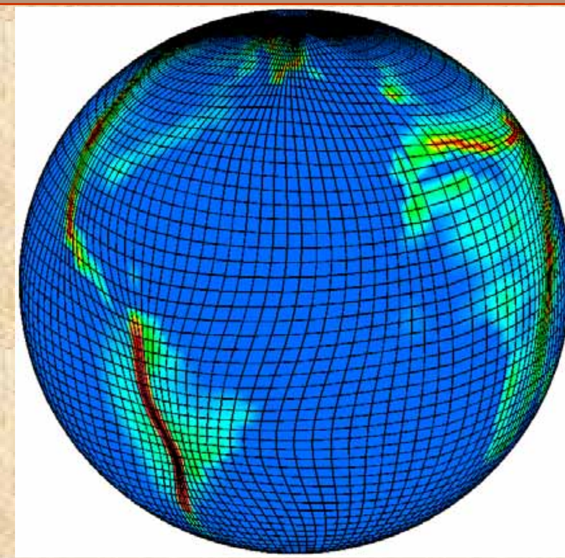
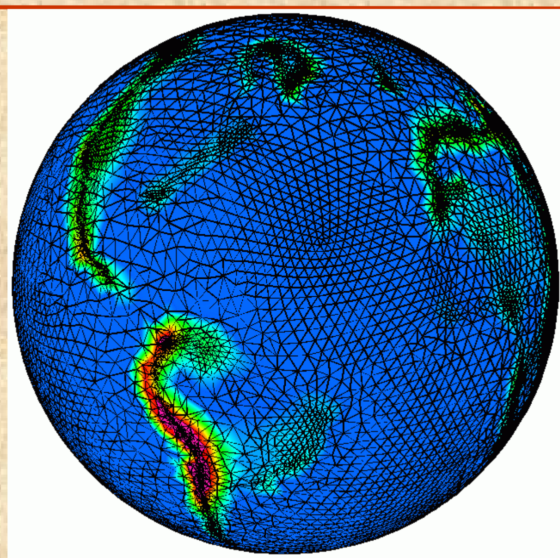
r-Adaptive Hybrid Mesh Generation for Climate Modeling



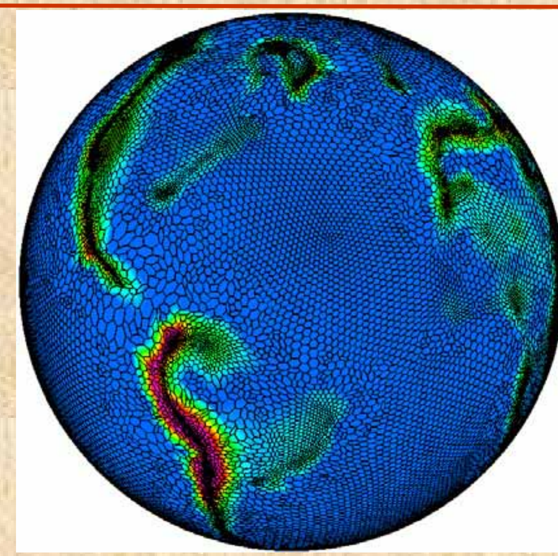
Orography field exhibiting high altitude regions of *Andes and Rockies*



Structured adapted spherical mesh based on orographic field data



Unstructured adapted geodesic mesh based on orographic field data



Hybrid adapted geodesic mesh based on orographic field data

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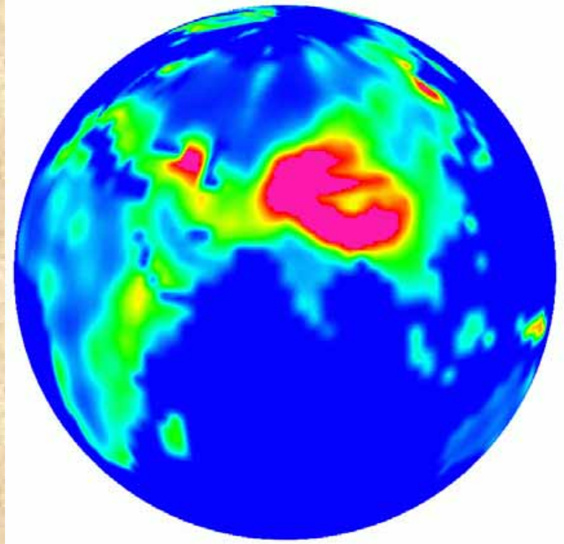
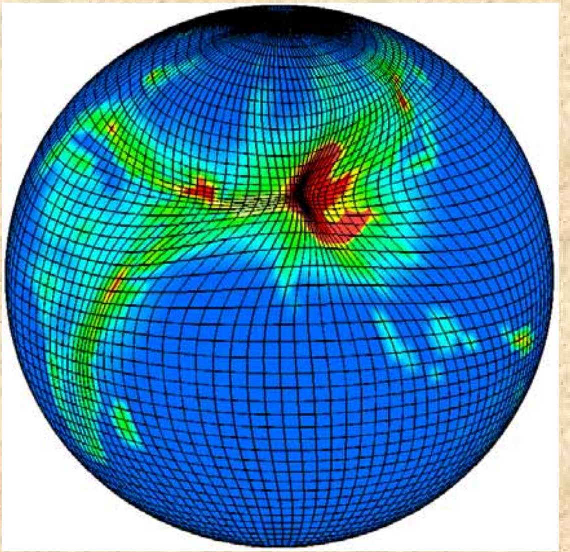
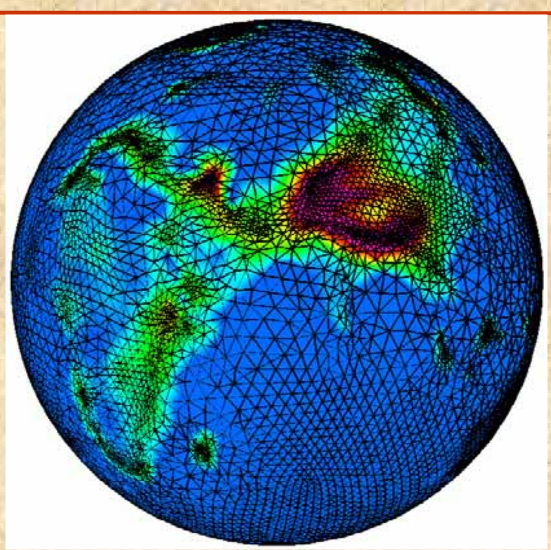
r-Adaptive Hybrid Mesh Generation for Climate Modeling



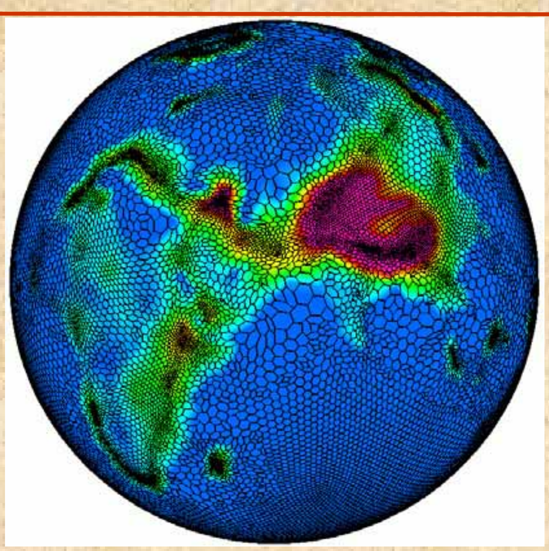
Orography field exhibiting high altitude regions of *Himalayas and Alps*



Structured adapted spherical mesh based on orographic field data



Unstructured adapted geodesic mesh based on orographic field data



Hybrid adapted geodesic mesh based on orographic field data

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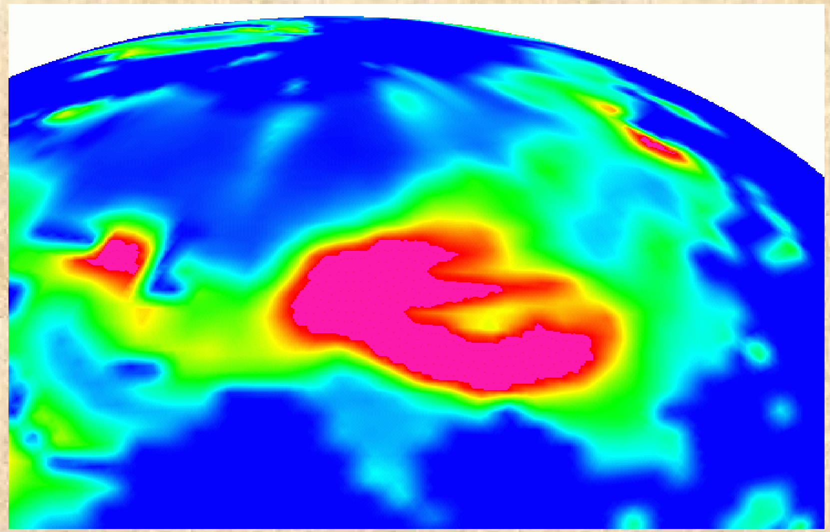
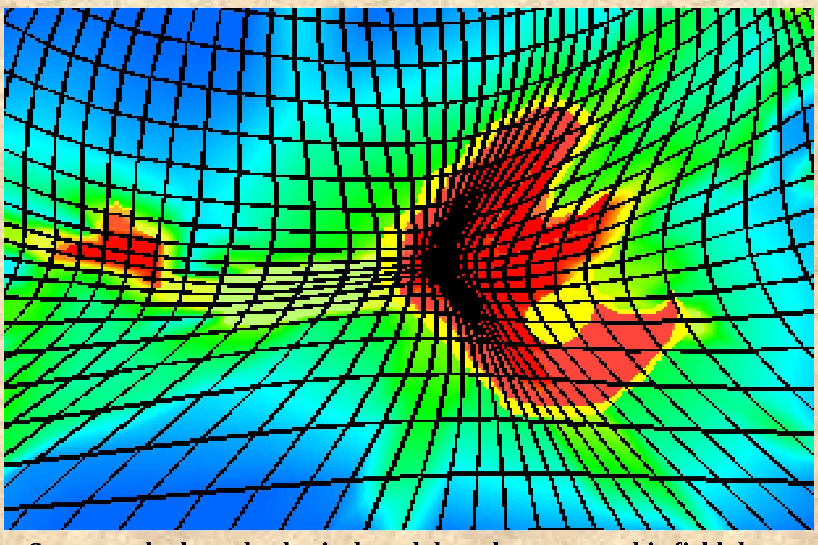
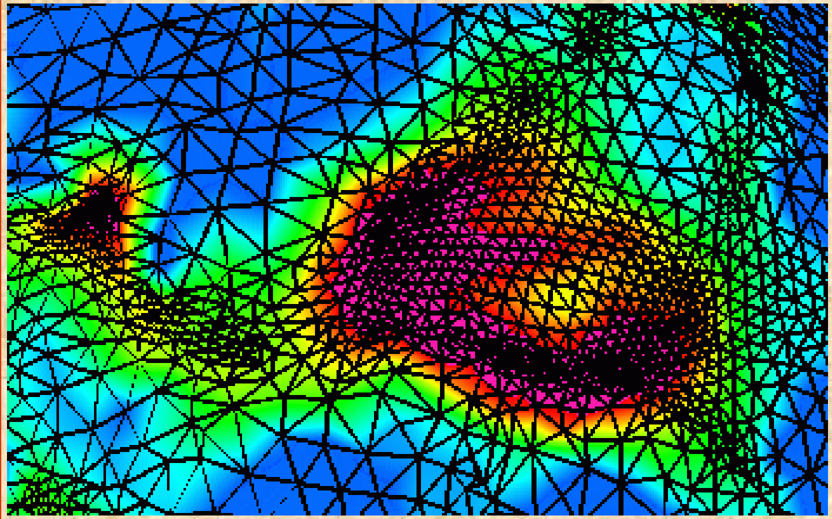
Close-up View of r-Adaptivity



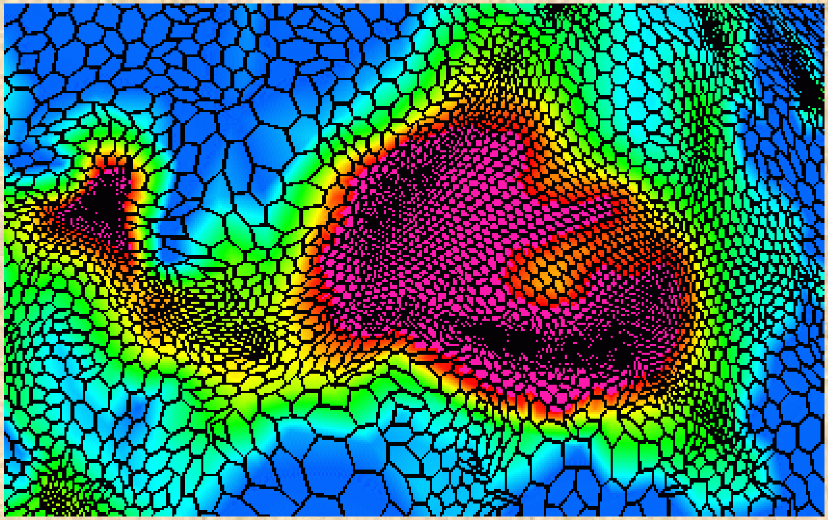
Orography field exhibiting high altitude regions of *Himalayas and Alps*



Structured adapted spherical mesh based on orographic field data



Unstructured adapted geodesic mesh based on orographic field data



Hybrid adapted geodesic mesh based on orographic field data

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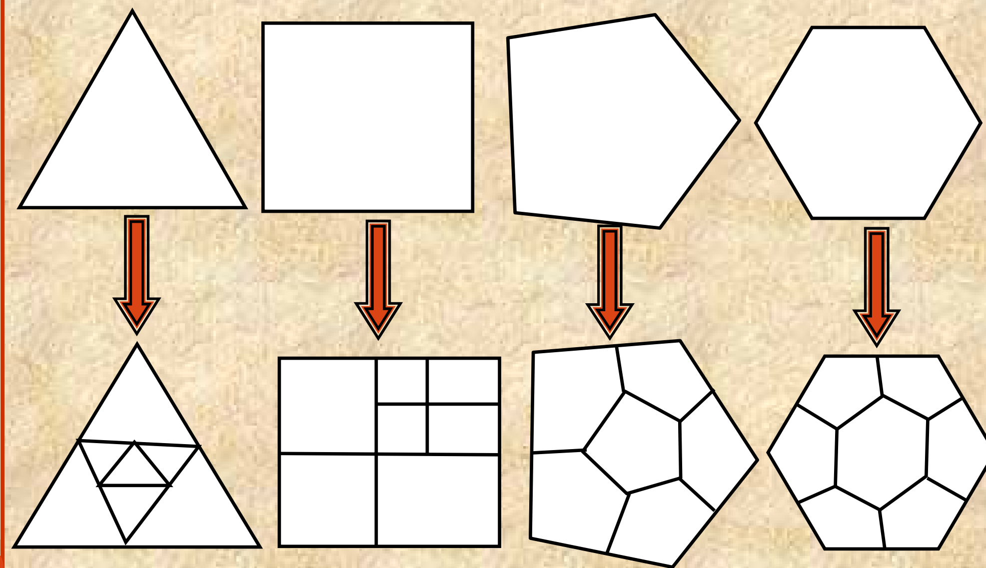
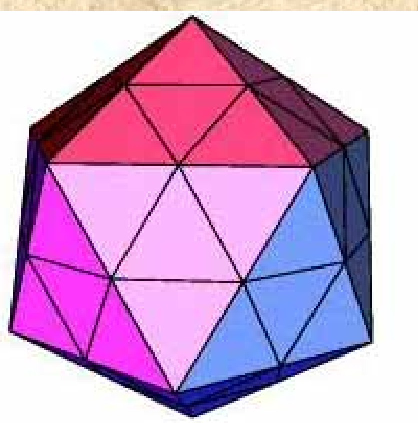
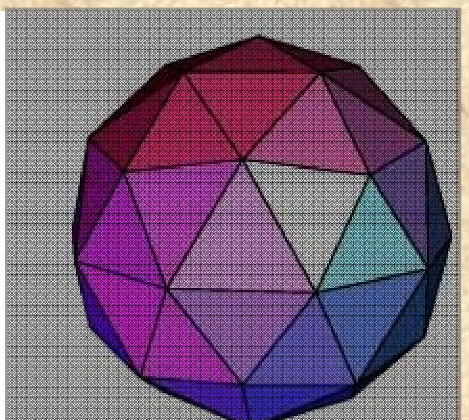
h-Adaptive Hybrid Mesh Generation for Climate Modeling

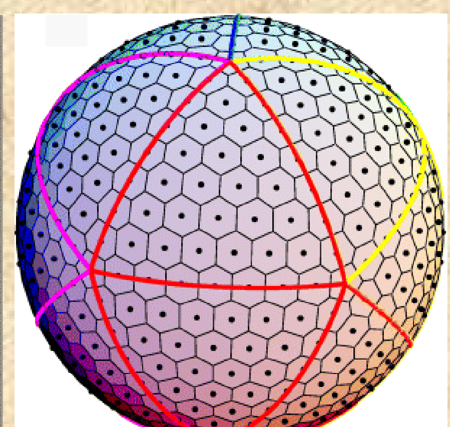
- Given a an adaptive solution or geometric field with large error gradient
- Generate an initial course isotropic or anisotropic geodesic surface mesh
- Mesh cell topology range from triangles, quads, pentagons, to hexagons
- Solution based adaptation and redistribution of mesh nodes to minimize solution error
- Mesh is adaptively refined until mesh density function $\boxed{\rho}$ is less than prescribed tolerance or convergence

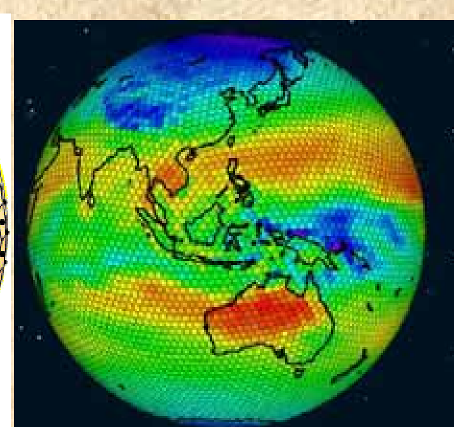
$$\rho = \lambda_a + \lambda_\kappa \left(|\mathcal{K}\sqrt{\phi}| \right) + \lambda_s \left(|\nabla \psi \sqrt{\phi}| \right)$$



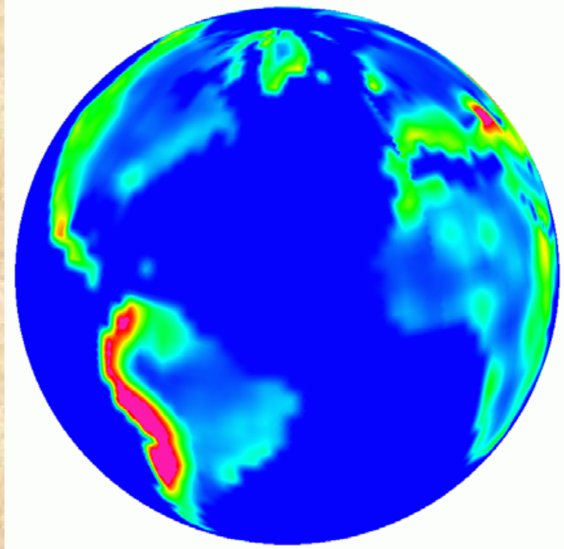




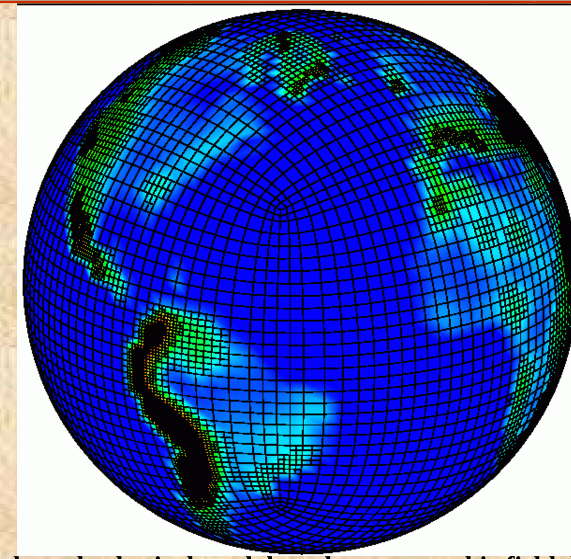




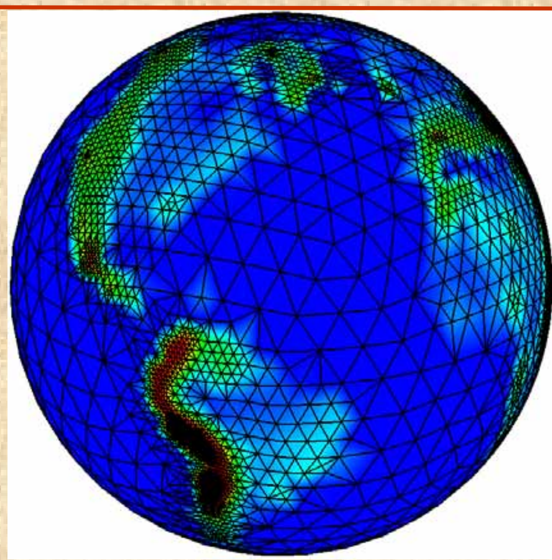
h-Adaptive Hybrid Mesh Generation for Climate Modeling



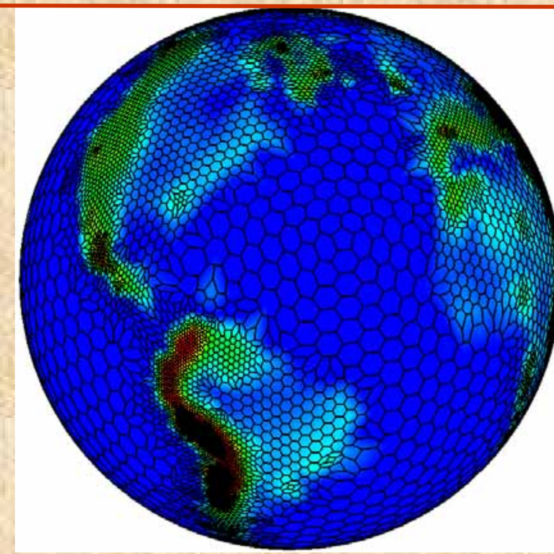
Orography field exhibiting high altitude regions of *Andes and Rockies*



Quad adapted spherical mesh based on orographic field data



h-Adapted unstructured geodesic mesh based on orographic field data



h-Adapted hybrid geodesic mesh based on orographic field data

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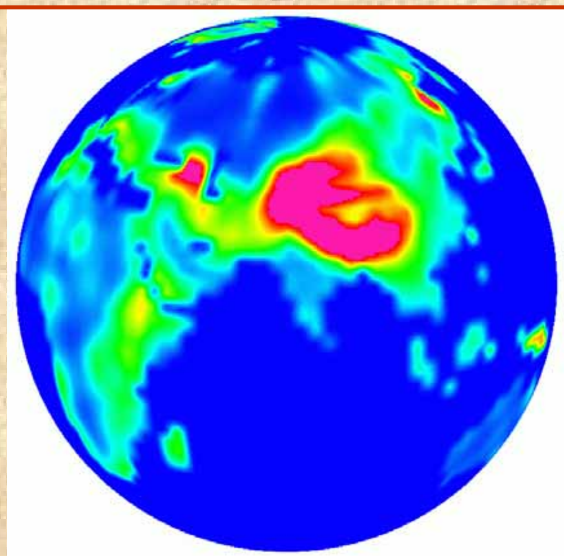
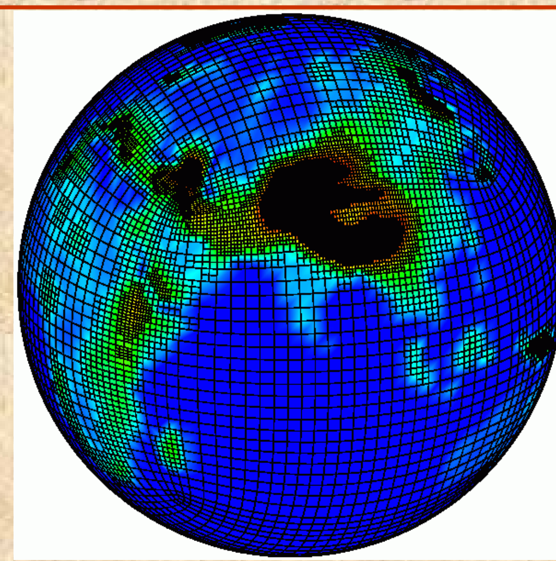
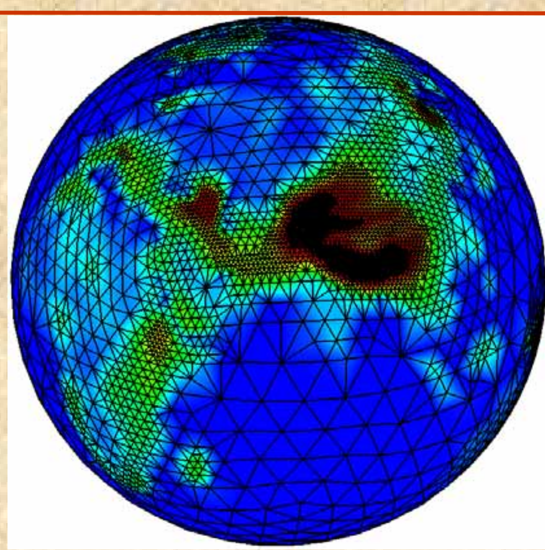
h-Adaptive Hybrid Mesh Generation for Climate Modeling



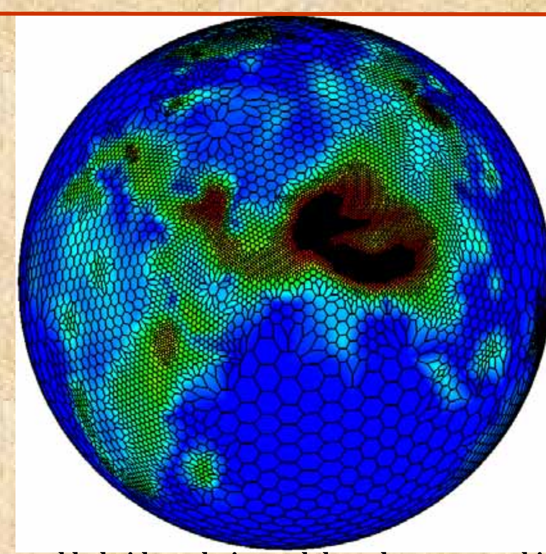
Orography field exhibiting high altitude regions of *Himalayas and Alps*



Structured adapted spherical mesh based on orographic field data



h-Adapted unstructured geodesic mesh based on orographic field data



h-Adapted hybrid geodesic mesh based on orographic field data

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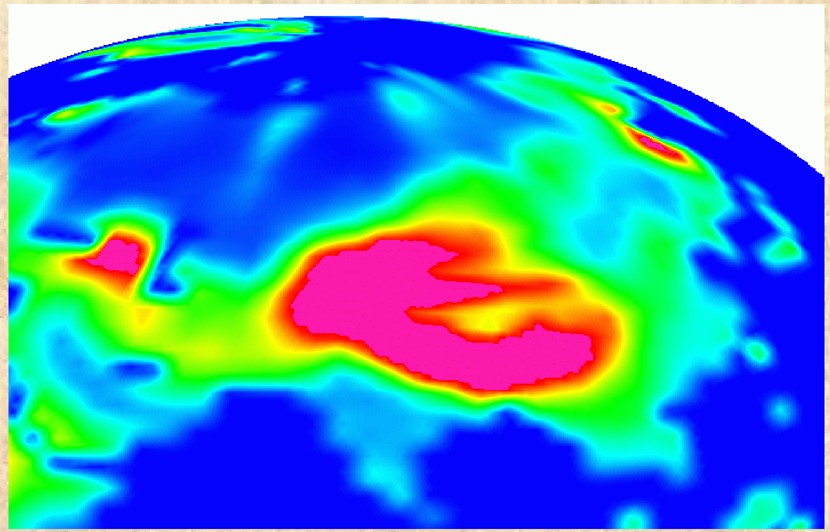
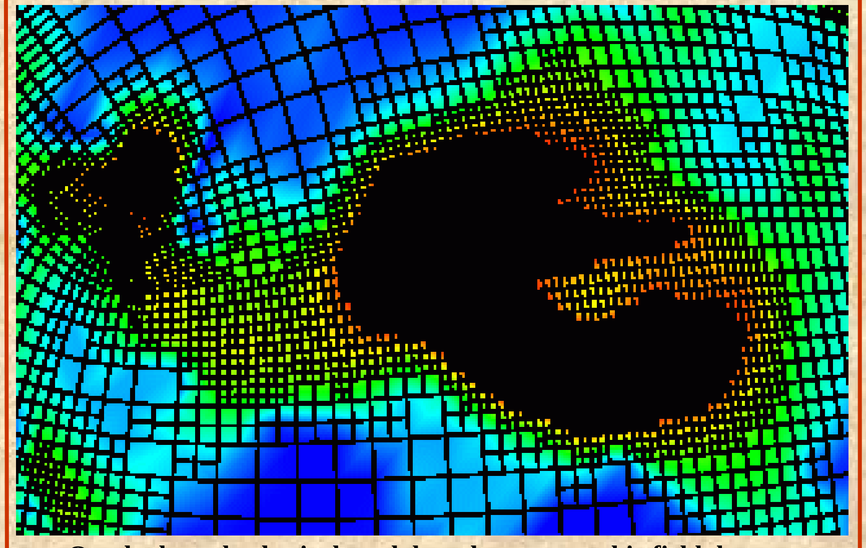
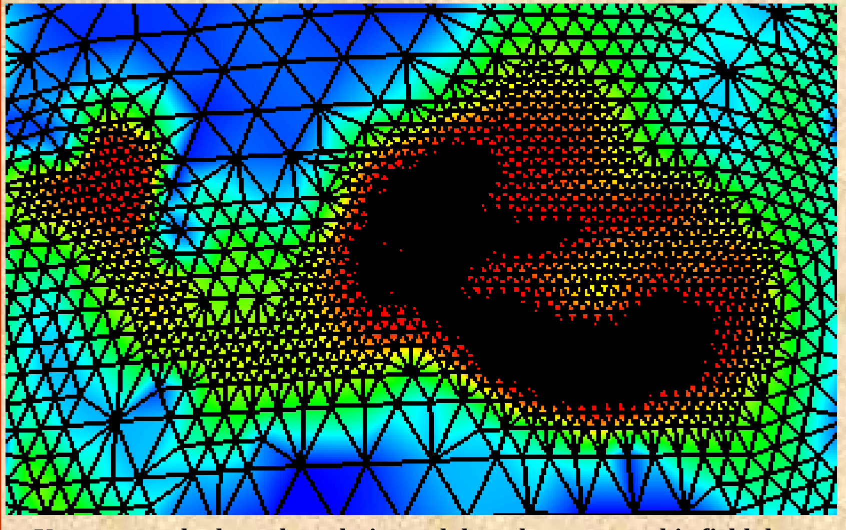
Close-up View of h-Adaptivity



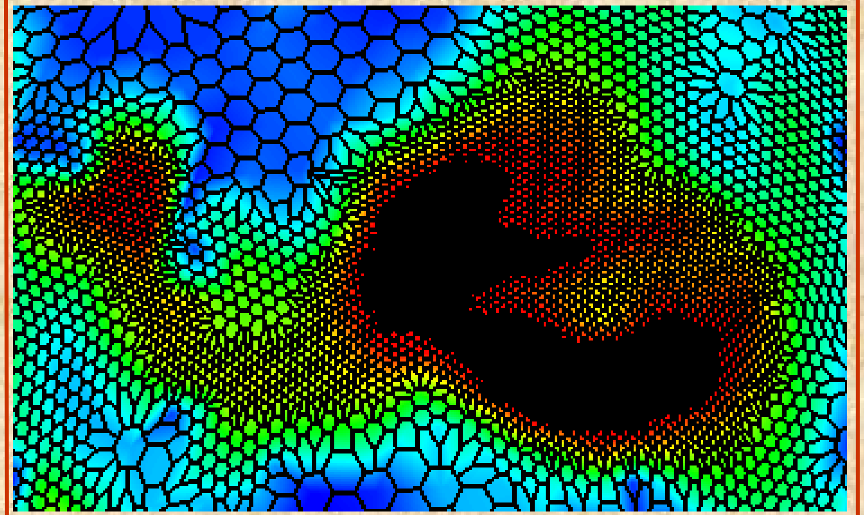
Orography field exhibiting high altitude regions of *Himalayas and Alps*



Quad adapted spherical mesh based on orographic field data



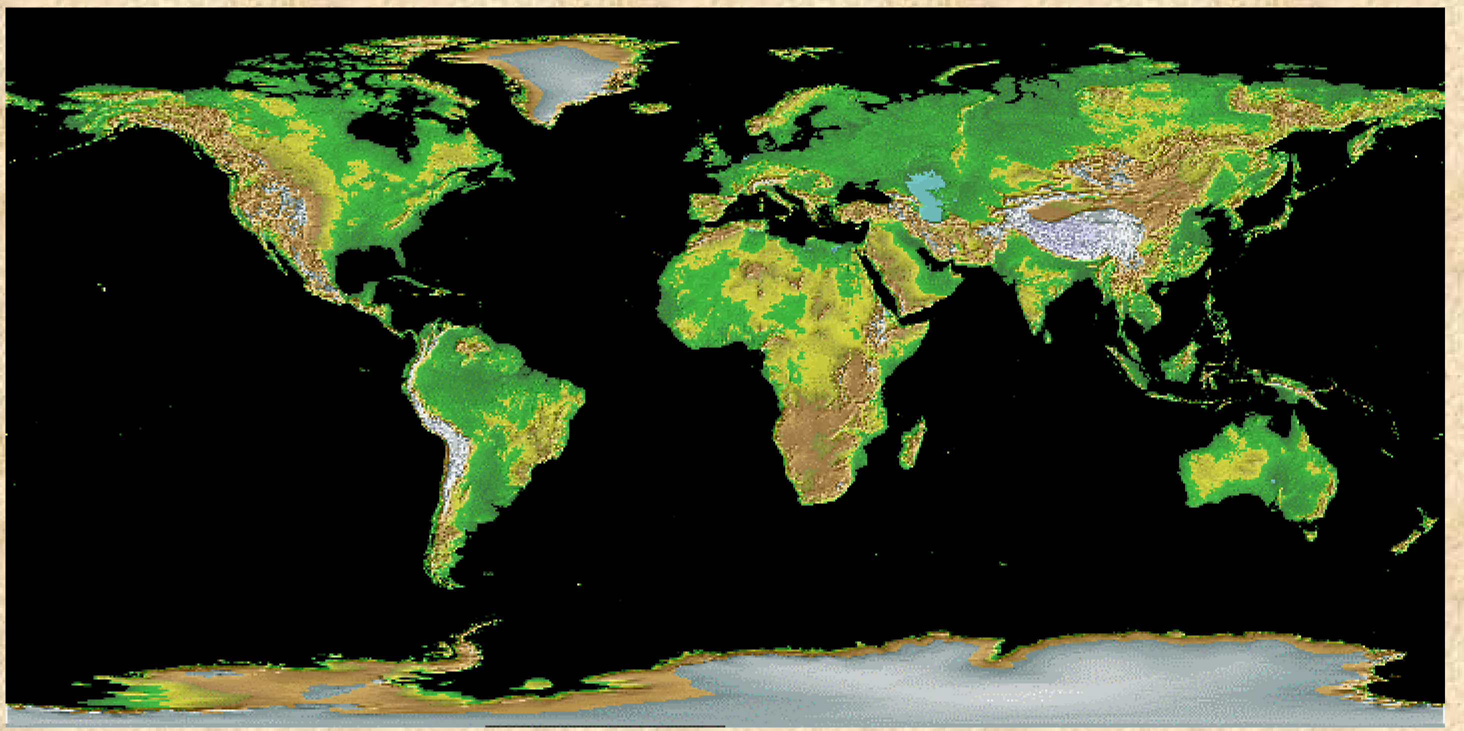
Unstructured adapted geodesic mesh based on orographic field data



Hybrid adapted geodesic mesh based on orographic field data

Test function (topography satellite data)

The Global Land One-km Base Elevation (GLOBE) Project (2-km) gridded resolution, quality-controlled global Digital Elevation Model (DEM). 2 GB data



http://www.ngdc.noaa.gov/mgg/topo/globe.html

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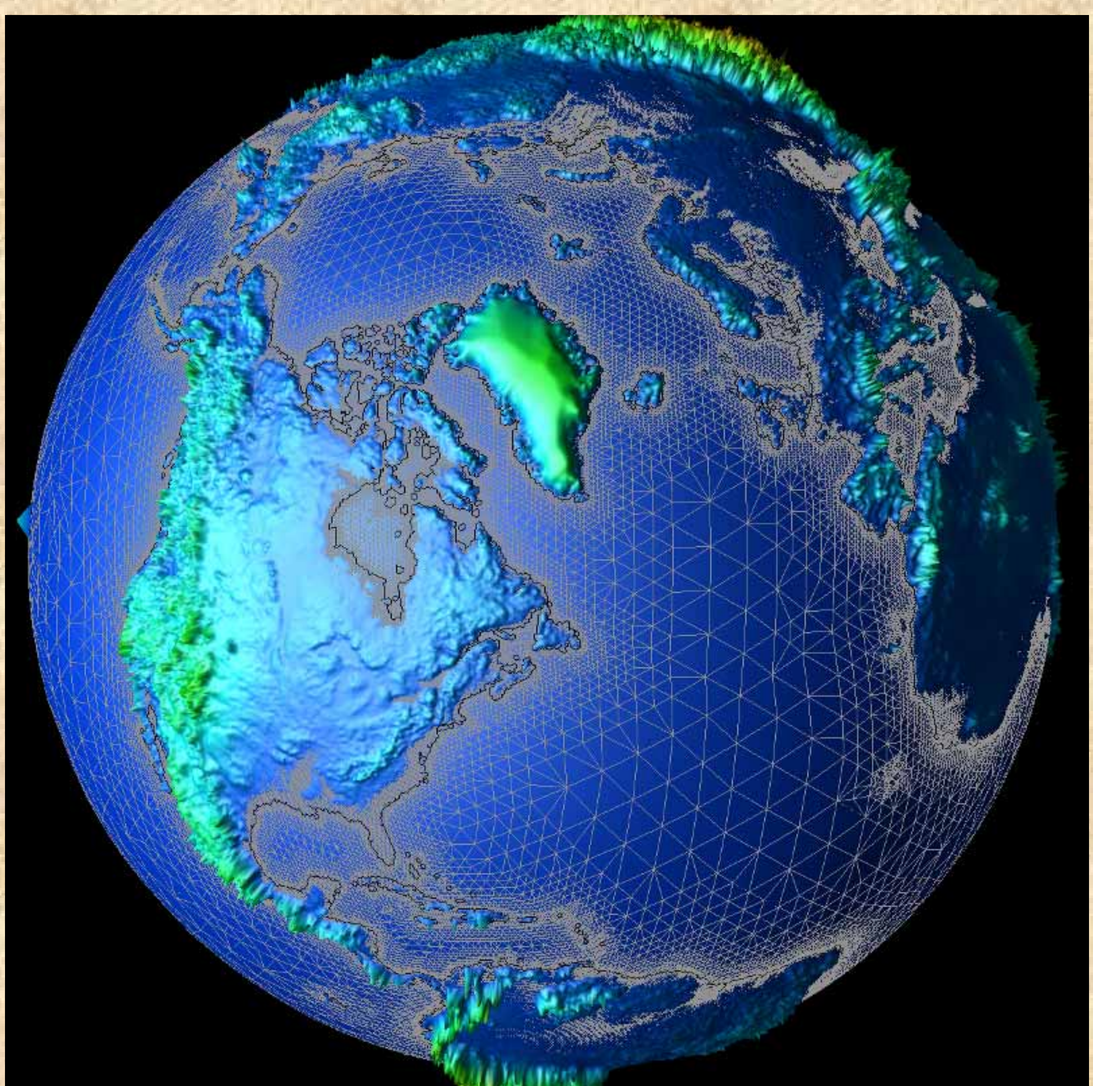
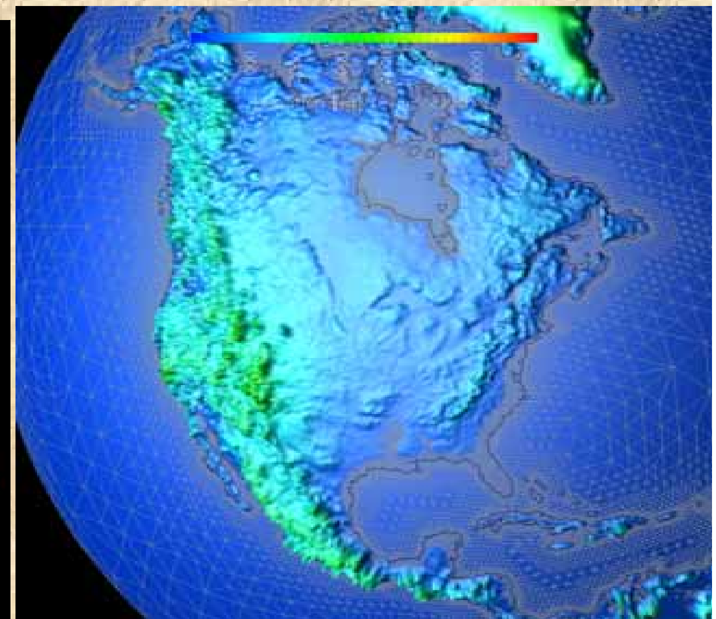
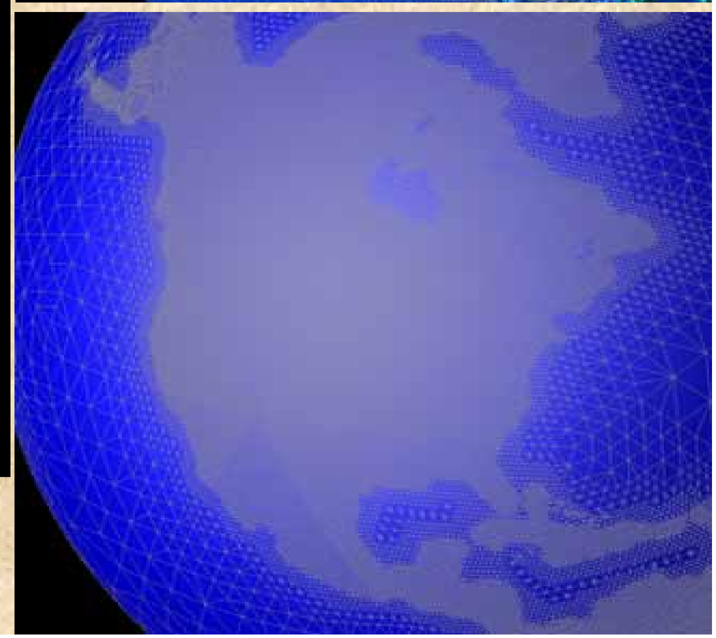
ADAPTIVE SHALLOW ATMOSPHERE SIMULATION







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Future Development and challenges

Meshing

- Develop time-dependent mesh tracking and alignment to follow and capture simulation vector field
- Implement on demand dynamic adaptive mesh coarsening and refinement
- Parallelization and coupling of meshing tools and libraries with simulation codes

Modeling

- Advance the implementation of the h-p Finite Element Discontinuous Galerkin Method
- Incorporate the energy equation in the climate modeler with species as moisture, humidity, etc.
- Benchmark simulation results using finite deference, finite volume, finite element, and spectral method on a variety of hybrid adapted meshes