Common-Source Phase Error of a Dual-Mixer Stability Analyzer

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The conceptual design of a dual-mixer stability analyzer for the Deep Space Network is described. A bound for the mean-square common-source phase error is given as a function of the power spectrum of the offset source and the response of the lowpass mixing filter. A derivation of the result is given in two appendices.

I. Introduction

As part of the Frequency and Time Subsystem Monitor and Control Upgrade Task, development of a frequency standards analyzer (FSA) is under way. One analyzer is to be installed at the Signal Processing Center of each Deep Space Communications Complex to monitor the stability of up to eight frequency sources. The current short-term sensitivity goal is an Allan deviation of $3 \times 10^{-15}/\tau$ ($1 \le \tau \le 100$ s) for a comparison of any two sources at 100 MHz. One needs a noise floor at least this low to measure the stability of the compensated sapphire oscillators (CSOs) [1], which are being installed in support of the Cassini mission. The purpose of this article is to calculate two components of the noise floor in terms of quantities that are known or measurable.

Single-mixer stability analyzers have been used at JPL and the Deep Space Network for a long time (see [2], for example). In a single-mixer analyzer, each pair of sources to be compared requires an offset generator, a mixer, and a zero crosser to provide a low-frequency square-wave beat note, typically at 1 Hz, whose zero crossings are time tagged and processed into phase residuals for the pair. In one time-tagging method, the readings of a free-running counter are latched. In another method, the "picket fence," an interval timer and an auxiliary reference signal are used to emulate an event timer [3]. In all these systems, the beat period is the same as the sample period of the phase residuals.

For the FSA, a dual-mixer design (Fig. 1) has been proposed[2] to take advantage of a new commercial multichannel event timer. A dual-mixer analyzer has a single reference source, or transfer oscillator, with its own offset generator. The resulting offset source is mixed against each of the sources under test to provide beat notes whose zero crossings yield phase residuals of each test source against the offset source. Any two of these phase-residual time series can be subtracted in software to give phase residuals of the two test sources against each other, with the phase deviations of the offset source canceling

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[1] Tracking Systems and Applications Section.

[2] C. A. Greenhall, "Conceptual Design of a Dual-Mixer Measurement System Based on a Timestamp Counter," JPL internal document, Jet Propulsion Laboratory, Pasadena, California, April 1, 1999.

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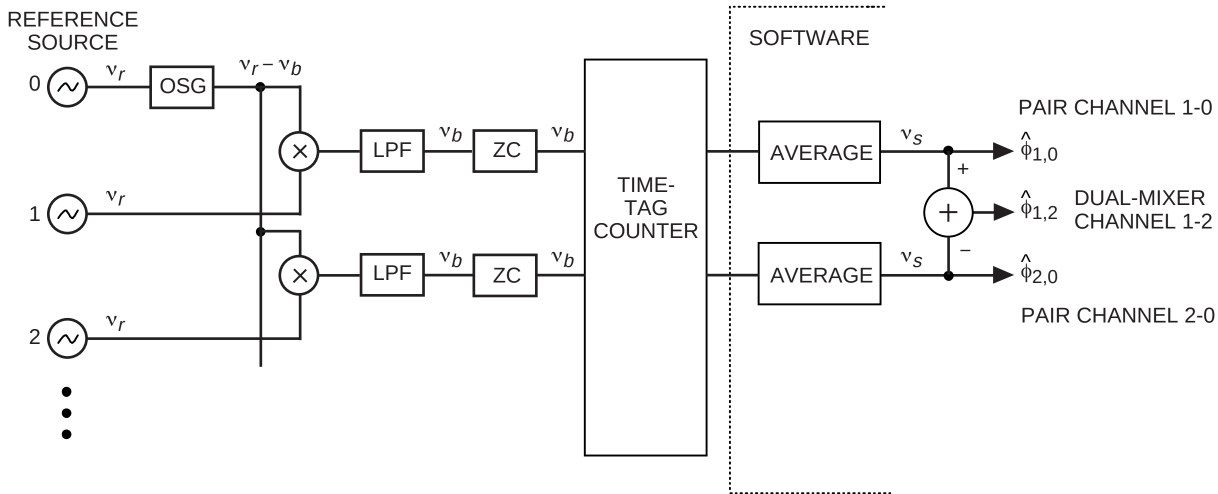

Fig. 1. The dual-mixer stability analyzer. Each source under test (1,2,...) is mixed with the offset reference source 0. Averaged phase residuals from channels *i*-0 and *j*-0 are subtracted to give phase residuals for dual-mixer chanel *i*-*j*. (OSG = offset generator; LPF = lowpass filter; ZC = zero crosser.)

to some degree. Although there can be many test sources, each with its own mixer, the term "dual mixer" is used because each pair comparison requires two mixers. Nevertheless, the hardware configuration of a dual-mixer instrument is simpler than that of a single-mixer instrument that has to monitor the same set of sources against each other.

If the test sources are independent, then the beat notes of a dual-mixer analyzer are not coherent; thus, the phase residuals of two test sources, i and j, against the offset source 0 are time tagged at different sets of points on the time axis (the zero crossings themselves). In previous dual-mixer designs [4], the phase residuals for the pairs i-0 and j-0 are interpolated to a fixed set of time points before being subtracted. As a consequence, the synthesized i-j phase residuals are contaminated by short-term offset-source noise, here called common-source phase error. In the FSA design, the beat period, τ_b, is much shorter (0.01 s) than the desired sample period, τ_s (0.5 s, to achieve a measurement bandwidth of 1 Hz). An average i-0 phase residual for each successive interval of length τ_s is computed by a method called integrated interpolation (Fig. 2). Because the phase residuals for all i-0 channels are averaged over the same set of τ_s intervals, fluctuations in offset-source phase on time scales of order τ_s and greater are effectively canceled when the average phase residuals of two of these channels are subtracted. In fact,

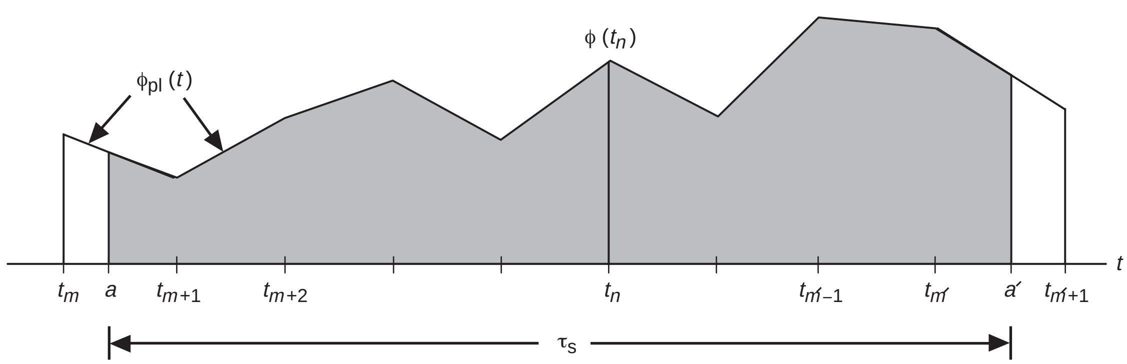

Fig. 2. Integrated interpolation of phase residuals. A beat-note zero crossing at t_n gives a phase residual $\phi(t_n)$. The average phase residual over the fixed interval $[a, a']$ of length τ_s is defined as the mean of the interpolated function $\phi_{pl}(t)$ over $[a, a']$.

the upper bound derived here for the mean-square common-source phase error depends on the sideband power of the offset source at integer multiples of $\nu_b = 1/\tau_b = 100$ Hz from the carrier, in bands of width $1/\tau_s = 2$ Hz.

The sideband noise is separated into two components. The first consists of spurs at multiples of ν_b from the carrier; these spurs come from the current design of the 100-Hz offset generator, which is based on two single-sideband mixer circuits in series.[3] The second is random broadband noise, coming mainly from the voltage-controlled oscillator (VCO) of the cleanup loop used to attenuate the spurs. The FSA design is intended to take advantage of the rapid decrease in spectral density of this noise as frequency offset increases from 1 Hz to 100 Hz.

II. Results

Let ν_r be the frequency of the sources under test (100 MHz). The offset source, with carrier power P_0, runs at frequency $\nu_0 = \nu_r - \nu_b$, where ν_b is the designed beat frequency (100 Hz). Let it have spurs with power $P_0 a_m^2$ at frequencies $\nu_0 + m\nu_b$, where m is a positive or negative integer, in addition to random modulation noise with a smooth single-sideband power density $S_{\text{ssb}}(\nu)$, relative to the carrier power, at offset frequency ν (positive or negative) from ν_0. We do not assume symmetry of sidebands or independence of phase and amplitude modulations. We do assume that the amplitude of the total noise is much less than that of the carrier, so that we can use linearized sideband analysis.

Test sources 1 and 2 at frequency ν_r are mixed with the offset source at frequency ν_0, thus producing beat notes 1 and 2 at nominal frequency ν_b. To reject the sum frequency and higher sidebands, we filter each beat note through a lowpass filter, with transfer function $G(\nu)$, having a bandwidth of perhaps $3\nu_b$. (From this narrow point of view, it would be better to use a bandpass filter with $G(0) = 0$; see Eqs. (1) and (2). Considerations of simplicity and stability probably will rule out this option.) The zero crossings of the beat notes are captured by the event timer and converted to phase residuals by the simple formula of Eq. (B-1). For a given interval of length $\tau_s = 1/\nu_s$ (default 0.5 s), we apply integrated interpolation (Fig. 2) to the phase residuals of each channel (about 50 per interval) to obtain average phase residuals $\hat{\phi}_{\text{1-0}}$, $\hat{\phi}_{\text{2-0}}$, representing the phase differences between the test sources and the offset source in a measurement bandwidth $(1/2)\nu_s$ (default 1 Hz). Then $\hat{\phi}_{\text{1-2}} = \hat{\phi}_{\text{1-0}} - \hat{\phi}_{\text{2-0}}$ represents the dual-mixer phase residual for source 1 − source 2. These phases are collected for a sequence of adjacent τ_s intervals and reduced to stability measures such as Allan deviation.

Assuming that the two beat notes are not coherent, we estimate the contribution of the offset-source noise to the error in $\hat{\phi}_{\text{1-2}}$. The result given here is an approximate upper bound for the long-term mean-square phase error $\sigma_\phi^2 = \left\langle E\hat{\phi}_{\text{1-2}}^2 \right\rangle$, where $\langle\rangle$ indicates an average over time, and E an average over the randomness of the noise. For any signed integer m, write $G_m = |G(m\nu_b)|$, the frequency response of the lowpass mixing filter at frequency $m\nu_b$. Under reasonable assumptions about the random noise, we find that

$$\sigma_\phi^2 \leq \frac{1}{G_1^2} \sum_{m \neq 0} \left\{ \begin{array}{l} G_{m-1}^2 \left[a_m^2 + \nu_s S_{\text{ssb}}(m\nu_b) \right] \\ + G_{m-1} G_{-m-1} \left[a_m a_{-m} + \nu_s \sqrt{S_{\text{ssb}}(m\nu_b) S_{\text{ssb}}(-m\nu_b)} \right] \end{array} \right\} \tag{1}$$

which gives a bound for σ_ϕ^2 in terms of the lowpass filter response, the spur powers a_m^2, and the broadband noise powers $\nu_s S_{\text{ssb}}(m\nu_b)$ in bands of width ν_s around the harmonics of the beat frequency.

This result is only a slight improvement over a simpler result that can be derived from Eq. (1) by applying Schwarz's inequality in the form

[3] G. Stevens, personal communication, Jet Propulsion Laboratory, Pasadena, California, 2000.

$$\sum x_m x_{-m} \leq \sqrt{\sum x_m^2 \sum x_{-m}^2} = \sum x_m^2$$

to the sum of the lower terms of Eq. (1). We obtain the slightly weaker estimate

$$\sigma_\phi^2 \leq \frac{2}{G_1^2} \sum_{m \neq 0} G_{m-1}^2 \left[a_m^2 + \nu_s S_{\mathrm{ssb}}\left(m\nu_b\right)\right] \tag{2}$$

For example, suppose that G is a single-pole lowpass filter with 3-dB bandwidth $3\nu_b$; then $G_m^2 = \left(1 + m^2/9\right)^{-1}$. Let[4]

$$\nu_s = 2 \text{ Hz}$$

$$S_{\mathrm{ssb}}\left(m\nu_b\right) \leq -160 \text{ dBc/Hz (at 5 MHz)} = -134 \text{ dBc/Hz (at 100 MHz)} = 4 \times 10^{-14} \text{ Hz}^{-1}$$

Using Maple,[5] we find that

$$\frac{2}{G_1^2} \sum_{m \neq 0} G_{m-1}^2 = 18.94$$

$$\frac{1}{G_1^2} \sum_{m \neq 0} \left[G_{m-1}^2 + G_{m-1} G_{-m-1}\right] = 18.67$$

Even though Eq. (1) is hardly an improvement over Eq. (2) in this case, we shall use it anyway. Considering only the broadband noise, we have

$$\sigma_\phi^2 \leq (18.67)\,(2)\left(4 \times 10^{-14}\right) = 1.49 \times 10^{-12} \text{ rad}^2$$

$$\sigma_\phi \leq 1.22 \times 10^{-6} \text{ rad} \tag{3}$$

which corresponds to a time deviation at $\nu_r = 100$ MHz of

$$\sigma_x = \frac{\sigma_\phi}{2\pi\nu_r} = 1.94 \times 10^{-15} \text{ s} \tag{4}$$

What does this mean for Allan deviation noise floor? The mean-square phase result gives no hint of the time dependence of the phase residual $\hat{\phi}_{1\text{-}2}$. If we assume that it is white, then

$$\sigma_y\left(\tau\right) = \frac{\sqrt{3}\sigma_x}{\tau} \leq \frac{3.36 \times 10^{-15} \text{ s}}{\tau} \tag{5}$$

[4] A. Kirk, personal communication on measurements of a commercial VCO, Jet Propulsion Laboratory, Pasadena, 2000.

[5] Copyright Waterloo Maple Inc.

To make the contribution of spurs to σ_ϕ^2 as small as that of the broadband noise, we could require spur power at offset $m\nu_b$ to be at most equal to the broadband noise power in a 2-Hz band about $m\nu_b$. This might be too much to ask. The author has carried out a close analysis[6] of the effect of one spur on one beat note, assuming that the offset and test sources are perfectly stable but the frequency of the test source differs from the nominal 100 MHz by much less than 1 Hz. This analysis shows that the time dependence of the phase disturbance depends on this frequency error, while the amplitude of the disturbance is compatible with Eqs. (1) and (2). Although $\sigma_y(\tau)$ would behave like σ_x/τ for large τ, it could be much less for small τ.

III. Conclusions

According to our results and noise measurements on a commercial VCO, the contribution of broadband offset-source noise to the Allan deviation noise floor of a dual-mixer channel is at most $3.4 \times 10^{-15}/\tau$. Combining this with the quantization effect of a 20-ns event timer, about $2 \times 10^{-15}/\tau$, gives an rss total of $4 \times 10^{-15}/\tau$, worse than the desired $3 \times 10^{-15}/\tau$. Nevertheless, a pair of CSOs could be measured with a lower noise floor by mixing them at 800 MHz in a single-mixer configuration; this is feasible because the 800-MHz synthesized output of a CSO can be offset by as much as several hundred hertz.

The effect of offset-generator spurs is more difficult to assess, both theoretically and experimentally. The analysis performed here gives only a bound for long-term mean-square phase error, on which no requirements have been placed. In a noise-floor test using the same oscillator for the offset source and for two test sources, the spurs would be aliased to a constant phase offset, which has no bearing on frequency stability. If at least one of the sources could be given a small ($\ll$1 Hz) additional frequency offset, then the effect of the spurs might be visible as a slow periodic modulation of the measured phase residuals.

References

[1] R. T. Wang and G. J. Dick, "Cryocooled Sapphire Oscillator with Ultrahigh Stability," *IEEE Trans. Instrum. Meas.*, vol. 48, no. 2, pp. 528–531, April 1999.

[2] T. Y. Otoshi and M. M. Franco, "DSS 13 Frequency Stability Tests Performed During May 1985 Through March 1986," *The Telecommunications and Data Acquisition Progress Report 42-86, April–June 1986*, Jet Propulsion Laboratory, Pasadena, California, pp. 1–14, August 15, 1986.
http://tmo.jpl.nasa.gov/tmo/progress_report/42-86/86A.PDF

[3] C. A. Greenhall, "A Method for Using a Time Interval Counter to Measure Frequency Stability," *The Telecommunications and Data Acquisition Progress Report 42-90, April–June 1987*, Jet Propulsion Laboratory, Pasadena, California, pp. 149–156, August 15, 1987.
http://tmo.jpl.nasa.gov/tmo/progress_report/42-90/90S.PDF

[4] S. Stein, D. Glaze, J. Gray, D. Hilliard, D. Howe, and L. A. Erb, "Automated High-Accuracy Phase Measurement System," *IEEE Trans. Instrum. Meas.*, vol. 32, no. 1, pp. 227–231, March 1983.

[6] C. Greenhall, "Effect of OSG Spurs on Stability Analyzer," JPL internal document, Jet Propulsion Laboratory, Pasadena, California, July 12, 2000.

Appendix A
Derivation of Results

This derivation is not rigorous. We use a discrete-frequency sideband model, results from which will be applied to noise that has a continuous-spectrum component. Other approximations are made along the way. Without further comment, we shall write $\omega = 2\pi\nu$, $\omega_0 = 2\pi\nu_0$, and so forth.

Let the signal of the offset source be represented by

$$s_0(t) = \cos(\omega_0 t + \theta_0) + \sum_\nu a(\nu)\cos\big((\omega_0 + \omega)t + \theta_0 + \alpha(\nu)\big)$$

where the summation is taken over a discrete set of offset frequencies ν such that $0 < |\nu| < \nu_0$. We assume that $\sum_\nu |a(\nu)| \ll 1$. Let

$$\zeta(\nu) = a(\nu)e^{i\alpha(\nu)}, \quad z_0(t) = \sum_\nu \zeta(\nu)e^{i\omega t} \tag{A-1}$$

$$\epsilon_0 = \mathrm{Re}\ z_0(t), \quad \phi_0(t) = \mathrm{Im}\ z_0(t) \tag{A-2}$$

Then

$$s_0(t) = \mathrm{Re}\left\{e^{i(\omega_0 t + \theta_0)}\big[1 + z_0(t)\big]\right\} = \mathrm{Re}\left\{e^{i(\omega_0 t + \theta_0)}\big[1 + \epsilon_0(t) + i\phi_0(t)\big]\right\}$$

$$= \mathrm{Re}\left\{e^{i(\omega_0 t + \theta_0)}\big[1 + \epsilon_0(t)\big]e^{i\phi_0(t)}\right\} \quad \text{to 1st order}$$

$$= \big[1 + \epsilon_0(t)\big]\cos\big(\omega_0 t + \theta_0 + \phi_0(t)\big)$$

The complex-valued process $z_0(t)$ will be called the sideband process of $s_0(t)$. Relative to carrier power 1/2, the single-sideband power in the offset frequency band (ν_1, ν_2) is $\sum_{\nu_1<\nu<\nu_2} E\,|\zeta(\nu)|^2$, which is interpreted as $\int_{\nu_1}^{\nu_2} S_{\mathrm{ssb}}(\nu)\,d\nu$ plus the sum of the powers of any spurs in this band.

Let test signal 1 have a similar representation,

$$s_1(t) = \mathrm{Re}\left\{e^{i(\omega_r t + \theta_1)}\big[1 + z_1(t)\big]\right\}$$

where $|z_1| \ll 1$. Beat note 1 with double frequencies rejected can be obtained as follows:

$$b_1(t) = \frac{1}{2}\,\mathrm{Re}\left\{e^{i(\omega_r t + \theta_1)}\big[1 + z_1(t)\big]\overline{e^{i(\omega_0 t + \theta_0)}\big[1 + z_0(t)\big]}\right\}$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{e^{i(\omega_b t + \theta_1 - \theta_0)}\Big[1 + z_1(t) + \overline{z_0(t)}\Big]\right\} \quad \text{to 1st order}$$

Let $g(t)$ be the impulse response of the lowpass filter G. For any signal $s(t)$,

$$G\left(e^{i\omega_b \cdot} s\right)(t) = \int e^{i\omega_b (t-u)} s(t-u)\, g(u)\, du$$

$$= e^{i\omega_b t}\left[\left(e^{-i\omega_b \cdot} g\right) * s\right](t)$$

Accordingly (with the irrelevant factor 1/2 dropped),

$$Gb_1(t) = \operatorname{Re}\left\{e^{i(\theta_1-\theta_0)} G\left[e^{\omega_b \cdot}\left(1+z_1+\overline{z_0}\right)\right](t)\right\}$$

$$= \operatorname{Re}\left\{e^{i(\omega_b t+\theta_1-\theta_0)}\left(e^{-i\omega_b \cdot} g\right) * \left(1+z_1+\overline{z_0}\right)(t)\right\}$$

$$= \operatorname{Re}\left\{e^{i(\omega_b t+\theta_1-\theta_0)} G(\nu_b)\left[1+G_b\left(z_1+\overline{z_0}\right)(t)\right]\right\} \tag{A-3}$$

where G_b is the complex filter with impulse response and transfer function

$$g_b(t) = \frac{g(t)\, e^{-i\omega_b t}}{G(\nu_b)}, \quad G_b(\nu) = \frac{G(\nu+\nu_b)}{G(\nu_b)} \tag{A-4}$$

To first order, the phase residual of filtered beat note $Gb_1(t)$ is given by

$$\phi_{1-0}(t) = \operatorname{Im}\left[G_b\left(z_1+\overline{z_0}\right)(t)\right]$$

The zero crossings of beat note 1 have approximate spacing $\tau_b = 1/\nu_b$, the nth zero crossing $t_{n,1}$ giving rise to a sample $\phi_{1\text{-}0}(t_{n,1})$ of phase residual according to Eq. (B-1). Let us approximate this situation by pretending that, locally in time, we have equally spaced samples $\phi_{1\text{-}0}((n-\lambda_1)\tau_b)$, where $0 < \lambda_1 < 1$. As time goes on, the fractional offset λ_1 changes slowly; we assume that it is essentially constant over any interval of length τ_s, the sample period. We shall simulate an average over many τ_s intervals by averaging over λ_1. Let $\tau_s = N\tau_b$ for an integer N, and let $T_j = j\tau_s + \tau_b$. Over an interval $[(j-1)\tau_s, j\tau_s]$, the average value of $\phi_{1\text{-}0}$ from the integrated-interpolation procedure is given by

$$\hat{\phi}_{1\text{-}0}(j\tau_s) = \left(B_{\lambda_1} H_{\lambda_1} \phi_{1-0}\right)(T_j) \tag{A-5}$$

where B_λ is a delay filter, $B_\lambda s(t) = s(t-\lambda\tau_b)$, and H_λ is a finite impulse response (FIR) filter that comes from the averaging procedure. It is given as a polynomial in the τ_s delay B_1 by

$$H_\lambda = \frac{1-B_1^N}{N(1-B_1)} F_\lambda(B_1), \quad F_\lambda(B_1) = \frac{1}{2}B_1 + \frac{1}{2}\left[B_1 + \lambda(1-B_1)\right]^2 \tag{A-6}$$

(see Appendix B).

Similarly, for beat note 2 with sideband process $z_2(t)$, we have

$$\left.\begin{aligned}\phi_{2\text{-}0}(t) &= \operatorname{Im}\big[G_b\left(z_2+\overline{z_0}\right)(t)\big] \\ \hat{\phi}_{2\text{-}0}(j\tau_s) &= \left(B_{\lambda_2}H_{\lambda_2}\phi_{2\text{-}0}\right)(T_j)\end{aligned}\right\} \tag{A-7}$$

Thus, keeping only the dependence on the sideband process z_0 of the offset source, we can represent one sample of the common-source phase error by

$$\begin{aligned}\hat{\phi}_{1\text{-}2}(j\tau_s) &= \hat{\phi}_{1\text{-}0}(j\tau_s)-\hat{\phi}_{2\text{-}0}(j\tau_s) \\ &= \operatorname{Im}\big[\left(B_{\lambda_1}H_{\lambda_1}-B_{\lambda_2}H_{\lambda_2}\right)G_b\overline{z_0}(T_j)\big]\end{aligned}$$

Applying the filters to $\overline{z_0(t)}=\sum_\nu \overline{\zeta(\nu)}e^{-i\omega t}$ term by term, we obtain

$$\hat{\phi}_{1\text{-}2}(j\tau_s)=\sum_\nu \operatorname{Im}\left[\Delta(-\nu;\lambda_1,\lambda_2)\,G_b(-\nu)\,\overline{\zeta(\nu)}e^{-i\omega T_j}\right] \tag{A-8}$$

where

$$\Delta(\nu;\lambda_1,\lambda_2)=\frac{1-e^{-iN\omega\tau_b}}{N\left(1-e^{-i\omega\tau_b}\right)}\big[e^{-i\lambda_1\omega\tau_b}F_{\lambda_1}\left(e^{-i\omega\tau_b}\right)-e^{-i\lambda_2\omega\tau_b}F_{\lambda_2}\left(e^{-i\omega\tau_b}\right)\big] \tag{A-9}$$

the transfer function of $B_{\lambda_1}H_{\lambda_1}-B_{\lambda_2}H_{\lambda_2}$.

At this point, let us split off from $z_0(t)$ a nonrandom portion due to spurs: assume that

$$\zeta(m\nu_b)=a_m e^{i\alpha_m}\quad \text{if } m\neq 0$$

where $a_m\geq 0$. The spur portion of $\hat{\phi}_{1\text{-}2}(j\tau_s)$ is given by

$$\hat{\phi}_{\text{spur}}(j\tau_s)=\sum_{m\neq 0}\operatorname{Im}\big[\Delta(-m\nu_b;\lambda_1,\lambda_2)\,G_b(-m\nu_b)\,a_m e^{-i\alpha_m}e^{-im\nu_b T_j}\big] \tag{A-10}$$

The last exponential is just 1, and $\Delta(-m\nu_b;\lambda_1,\lambda_2)=e^{i2\pi m\lambda_1}-e^{i2\pi m\lambda_2}$. Also write $G_b(-m\nu_b)\,a_m e^{-i\alpha_m}=|G_b(-m\nu_b)|\,a_m e^{-i\beta_m}$. Then Eq. (A-10) simplifies to

$$\begin{aligned}\hat{\phi}_{\text{spur}}(j\tau_s) &= \sum_{m\neq 0}\operatorname{Im}\big[\left(e^{i2\pi m\lambda_1}-e^{i2\pi m\lambda_2}\right)|G_b(-m\nu_b)|\,a_m e^{-i\beta_m}\big] \\ &= \sum_{m\neq 0}|G_b(-m\nu_b)|\,a_m\big[\sin(2\pi m\lambda_1-\beta_m)-\sin(2\pi m\lambda_2-\beta_m)\big]\end{aligned}$$

This depends on time $j\tau_s$ indirectly through λ_1 and λ_2, which are assumed to change slowly with j. We simulate the time average of $\hat{\phi}^2_{\text{spur}}$ by integrating over λ_1 and λ_2 as follows:

$$
\begin{aligned}
\left\langle \hat{\phi}^2_{\text{spur}} \right\rangle &= \int_0^1 \int_0^1 \hat{\phi}^2_{\text{spur}}\left(j\tau_s\right) d\lambda_1 d\lambda_2 \\
&= \sum_{m\neq 0} \sum_{m'\neq 0} \left|G_b\left(-m\nu_b\right) G_b\left(-m'\nu_b\right)\right| a_m a_{m'} I\left(m, m'\right)
\end{aligned}
\qquad \text{(A-11)}
$$

where

$$
\begin{aligned}
I\left(m, m'\right) = &\int_0^1 \int_0^1 \left[\sin\left(2\pi m\lambda_1 - \beta_m\right) - \sin\left(2\pi m\lambda_2 - \beta_m\right)\right] \\
&\times \left[\sin\left(2\pi m'\lambda_1 - \beta_{m'}\right) - \sin\left(2\pi m'\lambda_2 - \beta_{m'}\right)\right] d\lambda_1 d\lambda_2
\end{aligned}
$$

For any $f, g \in L^2\left(0,1\right)$,

$$
\int_0^1 \int_0^1 \left[f\left(x\right) - f\left(y\right)\right]\left[g\left(x\right) - g\left(y\right)\right] dx dy = 2\int_0^1 fg - 2\left(\int_0^1 f\right)\left(\int_0^1 g\right) \qquad \text{(A-12)}
$$

Consequently, since m and m' are both nonzero,

$$
\begin{aligned}
I\left(m, m'\right) &= 2\int_0^1 \sin\left(2\pi m\lambda - \beta_m\right)\sin\left(2\pi m'\lambda - \beta_{m'}\right) d\lambda \\
&= \delta_{m-m'} - \delta_{m+m'}\cos\left(\beta_m + \beta_{m'}\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left\langle \hat{\phi}^2_{\text{spur}} \right\rangle &= \sum_{m\neq 0} \left[\left|G_b\left(-m\nu_b\right)\right|^2 a_m^2 - \left|G_b\left(-m\nu_b\right) G_b\left(m\nu_b\right)\right| a_m a_{-m} \cos\left(\beta_m + \beta_{-m}\right)\right] \\
&\leq \sum_{m\neq 0} \left[\left|G_b\left(-m\nu_b\right)\right|^2 a_m^2 + \left|G_b\left(-m\nu_b\right) G_b\left(m\nu_b\right)\right| a_m a_{-m}\right]
\end{aligned}
\qquad \text{(A-13)}
$$

In view of Eq. (A-4), we have obtained the spur portion of Eq. (1). Additional knowledge about the phases of the spurs and of G would let us avoid estimating the cosine by -1 here.

Now we treat the random portion of the phase error, Eq. (A-8). For this purpose, let us agree that a sum over ν avoids the spurs. Abbreviating $\Delta\left(\nu; \lambda_1, \lambda_2\right)$ by $\Delta\left(\nu\right)$, we write

$$
\begin{aligned}
\hat{\phi}_{\text{rand}}\left(j\tau_s\right) &= \sum_{\nu} \operatorname{Im}\left[\Delta\left(-\nu\right) G_b\left(-\nu\right) \overline{\zeta\left(\nu\right)} e^{-i\omega T_j}\right] \\
&= \frac{1}{2i}\sum_{\nu}\left[\Delta\left(-\nu\right) G_b\left(-\nu\right) \overline{\zeta\left(\nu\right)} e^{-i\omega T_j} - \overline{\Delta\left(-\nu\right) G_b\left(-\nu\right)} \zeta\left(\nu\right) e^{i\omega T_j}\right]
\end{aligned}
\qquad \text{(A-14)}
$$

If the random variables $\zeta(\nu)$ satisfy the conditions

$$E\zeta(\nu) = 0 \quad \text{for all } \nu \tag{A-15}$$

$$E\zeta(\nu)\,\overline{\zeta(\nu')} = E\zeta(\nu)\,\zeta(-\nu') = 0 \quad \text{if } \nu \neq \nu' \tag{A-16}$$

then the random portion of the two-dimensional modulation process $[\epsilon_0(t), \phi_0(t)]$ [see Eq. (A-2)] is a mean-zero process with a stationary correlation matrix. Assume that Eqs. (A-15) and (A-16) hold. Then

$$E\hat{\phi}^2_{\text{rand}}(j\tau_s) = -\frac{1}{4}\sum_{\nu}\sum_{\nu'} E\left\{\begin{array}{l}\left[\Delta(-\nu)\,G_b(-\nu)\,\overline{\zeta(\nu)}e^{-i\omega T_j} - \overline{\Delta(-\nu)\,G_b(-\nu)}\zeta(\nu)\,e^{i\omega T_j}\right] \\ \times\left[\Delta(-\nu')\,G_b(-\nu')\,\overline{\zeta(\nu')}e^{-i\omega' T_j} - \overline{\Delta(-\nu')\,G_b(-\nu')}\zeta(\nu')\,e^{i\omega' T_j}\right]\end{array}\right\}$$

$$= \frac{1}{2}\sum_{\nu}|\Delta(\nu;\lambda_1,\lambda_2)|^2\left\{|G_b(-\nu)|^2 E|\zeta(\nu)|^2 - \operatorname{Re}\left[\overline{G_b(-\nu)\,G_b(\nu)}E\zeta(\nu)\,\zeta(-\nu)\right]\right\} \tag{A-17}$$

We now average this over λ_1 and λ_2. From Appendix B,

$$\frac{1}{2}\int_0^1\int_0^1|\Delta(\nu;\lambda_1,\lambda_2)|^2\,d\lambda_1 d\lambda_2 =$$

$$\left[\frac{\sin(N\pi\nu\tau_b)}{N\sin(\pi\nu\tau_b)}\right]^2\left\{\frac{11}{20}+\frac{13}{30}\cos(2\pi\nu\tau_b)+\frac{1}{60}\cos(4\pi\nu\tau_b)-\left[\frac{\sin(\pi\nu\tau_b)}{\pi\nu\tau_b}\right]^6\right\} \tag{A-18}$$

The expression in braces equals δ_m when $\nu = m\nu_b$; therefore, for large N, Eq. (A-18) constitutes a comb with teeth at $m\nu_b$ having width and mass $\nu_b/N = \nu_s$, but missing the tooth at $\nu = 0$; see Fig. A-1. Letting

$$K(\nu) = \left[\frac{\sin(\pi\nu\tau_s)}{\pi\nu\tau_s}\right]^2$$

we have the large-N approximation

$$\frac{1}{2}\int_0^1\int_0^1|\Delta(\nu;\lambda_1,\lambda_2)|^2\,d\lambda_1 d\lambda_2 = \sum_{m\neq 0}K(\nu - m\nu_b)$$

Using this in Eq. (A-17), and assuming that $G(\nu)$ is essentially constant as far as $K(\nu - m\nu_b)$ is concerned, we have

$$\left\langle E\hat{\phi}^2_{\text{rand}}\right\rangle = \int_0^1\int_0^1 E\hat{\phi}^2_{\text{rand}}\,d\lambda_1 d\lambda_2$$

$$= \sum_{m\neq 0}\left\{\begin{array}{l}|G_b(-m\nu_b)|^2\sum_\nu K(\nu - m\nu_b)\,E|\zeta(\nu)|^2 \\ -\operatorname{Re}\left[\overline{G_b(-m\nu_b)\,G_b(m\nu_b)}\sum_\nu K(\nu - m\nu_b)\,E\zeta(\nu)\,\zeta(-\nu)\right]\end{array}\right\} \tag{A-19}$$

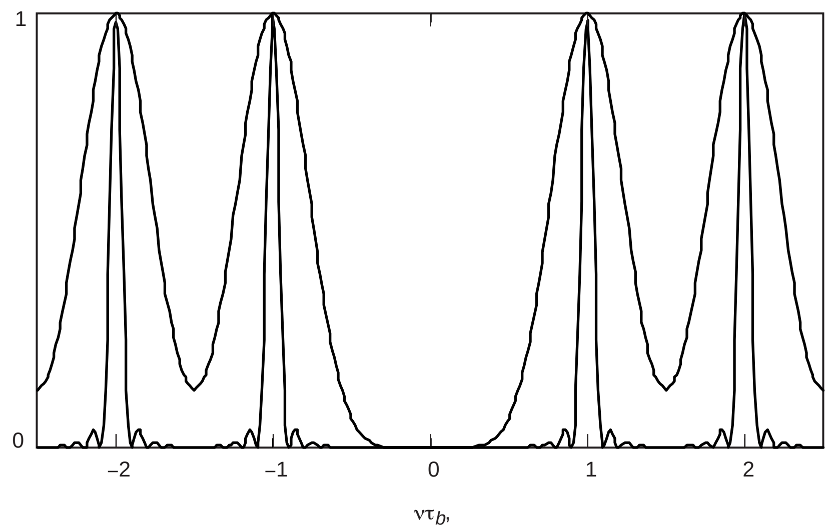

Fig. A-1. A comb with the central tooth missing. Equation (A-18) with *N* = 10 is plotted with and without the digital sinc^2 factor.

By Schwarz's inequality applied to the "integral" $\sum_\nu K(\nu - m\nu_b)\, E\,()$,

$$\left|\sum_\nu K(\nu - m\nu_b)\, E\zeta(\nu)\,\zeta(-\nu)\right| \le \left[\sum_\nu K(\nu - m\nu_b)\, E\left|\zeta(\nu)\right|^2 \sum_\nu K(\nu + m\nu_b)\, E\left|\zeta(\nu)\right|^2\right]^{1/2}$$

We interpret $\sum_\nu K(\nu - m\nu_b)\, E\left|\zeta(\nu)\right|^2$ as $\int K(\nu - m\nu_b)\, S_{\rm ssb}(\nu)\, d\nu$, which is approximately $\nu_s S_{\rm ssb}(m\nu_b)$ if $S_{\rm ssb}(\nu)$ is smooth. With this approximation,

$$\left\langle E\hat{\phi}^2_{\rm rand}\right\rangle \le \sum_{m\neq 0}\left[\begin{array}{l}\left|G_b(-m\nu_b)\right|^2 \nu_s S_{\rm ssb}(m\nu_b)\\ +\left|G_b(-m\nu_b)\, G_b(m\nu_b)\right|\sqrt{\nu_s S_{\rm ssb}(m\nu_b)\,\nu_s S_{\rm ssb}(-m\nu_b)}\end{array}\right]$$

This is the random contribution to the bound of Eq. (1) on σ^2_ϕ. Finally, $\sigma^2_\phi = \left\langle\hat{\phi}^2_{\rm spur}\right\rangle + \left\langle E\hat{\phi}^2_{\rm rand}\right\rangle$ because $E\zeta(\nu) = 0$.

Appendix B
Integrated Interpolation Filter

Integrated interpolation is the method used in the proposed analyzer for associating an average phase to a fixed time interval $[a, a']$ (of length τ_s) from unequally spaced zero crossings t_n. Each t_n gives rise to a phase residual $\phi(t_n)$ according to the equation $2\pi\nu_b t_n + \phi(t_n) = 2\pi n$; thus,

$$\phi(t_n) = 2\pi(n - \nu_b t_n) \tag{B-1}$$

The piecewise linear function $\phi_{\mathrm{pl}}(t)$ is defined by linear interpolation between the points $(t_n, \phi(t_n))$, and the average phase is defined by

$$\hat{\phi} = \frac{1}{a' - a} \int_a^{a'} \phi_{\mathrm{pl}}(t)\, dt$$

(Fig. 2). Average phases for successive τ_s intervals can be produced by a simple real-time algorithm acting on the stream of measured t_n.

Here, we treat a local approximation in which $t_n = (n - \lambda)\tau_b$ for some $\lambda \in (0,1)$ and the averaging interval is $[0, N\tau_b]$. Evaluating the trapezoidal areas in Fig. 2, we find that

$$\begin{aligned}\hat{\phi} &= \frac{1}{N}\left[\begin{array}{l}\dfrac{(1-\lambda)^2}{2}\phi(t_0) + \left(1 - \dfrac{\lambda^2}{2}\right)\phi(t_1) + \sum_{n=2}^{N-1}\phi(t_n) \\ + \left(1 - \dfrac{(1-\lambda)^2}{2}\right)\phi(t_N) + \dfrac{\lambda^2}{2}\phi(t_{N+1})\end{array}\right] \\ &= (B_\lambda H_\lambda \phi)\big((N+1)\tau_b\big)\end{aligned} \tag{B-2}$$

Here, B_λ is a shift filter, $B_\lambda s(t) = s(t - \lambda\tau_b)$, and H_λ is a causal FIR filter with sample period τ_b and impulse response vector

$$\frac{1}{N}\left[\frac{\lambda^2}{2}, 1 - \frac{(1-\lambda)^2}{2}, \underbrace{1, \cdots, 1}_{N-2}, 1 - \frac{\lambda^2}{2}, \frac{(1-\lambda)^2}{2}\right]$$

A simple computation gives the expression in Eq. (A-6) for H_λ. In Eq. (B-2), ϕ is the actual underlying phase function, not the artificial interpolated function ϕ_{pl}.

We now look at properties of $B_\lambda H_\lambda$ in the frequency domain. Define transfer functions

$$B_\lambda(\nu) = e^{-i2\pi\nu\lambda\tau_b}$$

$$A_N(\nu) = \frac{1 - B_1^N(\nu)}{N\big(1 - B_1(\nu)\big)} = e^{-i\pi(N-1)\nu\tau_b}\frac{\sin(N\pi\nu\tau_b)}{N\sin(\pi\nu\tau_b)}$$

$$H_\lambda(\nu) = A_N(\nu)\, F_\lambda\big(B_1(\nu)\big)$$

Then

$$B_\lambda(\nu)\, H_\lambda(\nu) = A_N(\nu)\left\{1 - \frac{3}{2} i\omega\tau_b - \frac{5}{4}(\omega\tau_b)^2 + i(\omega\tau_b)^3\left[\frac{3}{4} + \frac{1}{12}\lambda(1-\lambda)(1-2\lambda)\right]\right\}$$

$$+\, O(\omega\tau_b)^4 \quad \text{as } \omega\tau_b \to 0$$

hence

$$\Delta(\nu;\lambda_1,\lambda_2) = B_{\lambda_1}(\nu)\, H_{\lambda_1}(\nu) - B_{\lambda_2}(\nu)\, H_{\lambda_2}(\nu) = O(\omega\tau_b)^3$$

for any λ_1, λ_2. Even though $A_N - B_\lambda H_\lambda$ has only one vanishing moment, the filter representing the local difference of average phases between two channels has three vanishing moments; this means that a dual-mixer channel 1-2 is not affected by a linear frequency drift common to both channels 1-0 and 2-0.

Finally, let us treat the integral, Eq. (A-18), of $1/2\,|\Delta|^2$ over λ_1 and λ_2. By Eq. (A-12),

$$\frac{1}{2}\int_0^1\int_0^1 |\Delta(\nu;\lambda_1,\lambda_2)|^2\, d\lambda_1 d\lambda_2 = |A_N(\nu)|^2\left[\int_0^1 |B_\lambda(\nu)\, F_\lambda(\nu)|^2\, d\lambda - \left|\int_0^1 B_\lambda(\nu)\, F_\lambda(\nu)\, d\lambda\right|^2\right]$$

Evaluating the integrals on the right with Maple, we obtain Eq. (A-18), the right side of which is $O(\omega\tau_b)^6$ as $\omega\tau_b \to 0$.