Whole Language: Lessons for Math Teachers *

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Abstract

Two teachers with much experience in teaching language arts ask themselves: If mathematics were taught in the same spirit as we teach language skills, what would our math lessons be like? What would our students do? What would they learn? The teachers choose a problem from graph theory involving dominating sets, and use this as the central focus of a four-day lesson. The problem is easy to grasp, but only with great luck will the best solution leap off the page. This provides an opportunity for the students to explore the problem in their own way, and for the teachers to be amazed and excited by what occurs.

Introduction

Over the last 20 years a revolution has taken place in language arts teaching. The notion that reading and writing are learned by acquiring an ordered set of skills has been supplanted by the view that children employ the same mechanisms to learn to read and write that they used to learn to talk. Theory and research in language acquisition not only support this view, but have provided the foundation for teaching language skills (reading, writing, speaking) as part of a developmental process that occurs in a language-rich environment. The role of the teacher is to provide this linguistically stimulating environment and to observe and guide each child's language development. This approach to language arts teaching has come to be known as "Whole Language" and has ramifications across curriculum at all grade levels.

 Clearly, the author is endebted to Prudy Heimsch. First she arranged her and her students' schedules to accommodate this activity; then her insight and expertise were crucial for interpreting the classroom dynamics and guiding decisions about what to do next so that this was a positive experience for the students. (An earlier version of this essay appeared in: Swenson, Carl, ed. (1990). *Connections* Seattle: Washington State Mathematics Council. Subsequently, the material was unabashedly borrowed without citation by Marilyn Burns in her column in *Instructor* magazine, October, 1993.)

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In collaboration with Prudence Heimsch, Second Grade Teacher at McDonald Elementary School in Moscow, Idaho, I have been thinking about learning in mathematics as language learning. We have been seeking ways to adapt our knowledge and experience as language teachers to the teaching and learning of mathematics in the elementary classroom.

As language teachers we welcome young learners into the community of literate people so that they can find the same joy and satisfaction that we have known as readers and writers. It is natural, then to ask, how does one participate in a community of mathematicians, and what joys and satisfactions result from doing so?

Squinting over the wall that divides mathematics from language arts, we perceived the work of mathematicians to be recognizing and describing patterns, making abstractions from the patterns, and often, but not always, drawing conclusions about the natural world that are based on what has been discovered. Drawing upon a common vocabulary, notation, traditions, and body of knowledge, mathematicians communicate with one another about their perceptions and insights. The language they use is mathematics. We felt that children who are invited to use that language to communicate in meaningful ways about their perceptions of the world would blossom as young mathematicians, just as they blossom as readers and writers when the full communicative power of written language is made available to them.

For four consecutive days, I came to Prudy's second grade classroom and we taught math the way either one of us might have taught language arts. We wanted to set up a mathematically stimulating environment, define mathematics broadly, and invite the children to "do mathematics" by puzzling, reasoning, and communicating with us and with each other both orally and in writing. My mission was to find a problem that the students could readily understand, but definitely not solve on sight–a problem with multiple paths to a solution, and perhaps many promising blind alleys as well. Our focus was not to be on the solution to the problem so much as the reasoning and strategies the children used to seek a solution. We were prepared to stimulate and encourage the students rather than direct them. We hoped that they would find the magic of their reasoning power and problem solving abilities to be more interesting than the actual solution of the problem itself.

We planned to structure the class time (about 45 minutes) to resemble the reading and writing workshop time blocks with which the students were familiar. A short presentation to the whole group would begin the class period. This would provide instruction, cohesion, and direction. As much time as possible would be alloted to the students to simply work and think, pursuing directions that they chose.

I chose a problem from graph theory for several reasons. First, it would be unlikely that any student would have encountered the problem in another context. I also wanted an activity that didn't rely directly on numbers, in hopes that it would liberate students from notions that they already have about mathematics. Discrete mathematics, and graph theory in particular, are a source of many interesting problems that young children can grasp. Graphs are easily drawn, many of their properties are accessible and interesting in their own right, and graphs lend themselves to representation of real-world objects.

Day one:: An unexpected frenzy

I began by presenting an overview for the students that was deliberately open-ended and vague:

1. I explained that mathematics is a very broad field, and that mathematicians think about many interesting things in addition to numbers. During the math time this week, we would be thinking together as mathematicians in some areas of mathematics that I hoped they would find interesting.

2. Over the course of the four days, each child would be making a mathematics book of their very own. In it they would record their mathematical ideas, draw mathematical pictures, and do any other mathematical work that they decided to put in it. Materials were available for making a cover, for pages, etc.

3. For today, I would tell them a mathematical story. First they should listen carefully to the story. Then they could either work on solving the problem in the story, or put together the pages of their book.

The Story: Where do the fire stations go?

Each child received a handout, an "unusual map of a town." (See Figure **??** The lines on this map represent streets and the dots, street corners. The map doesn't show exactly where the houses are, we don't really need to know that. But we do know that houses are located at corners and there is at least one house at each corner.

Right now, there are no fire stations in this town. The people are very concerned about the fire danger, and the mayor has decided that they will build some fire stations. To make sure that everyone is very safe, the mayor has said that if a house catches on fire, the fire trucks should not have to drive more than one street to get to a house to put the fire out. When the fire stations are all built, every house will be either on the same corner as a fire station, or on a corner that is only one street away from a fire station.

The tricky thing is this: Fire stations are expensive to build, and the townspeople don't want to spend all the town's money on this project unless they have to. In fact, they hope to have some money left over to build some swimming pools, too. What is the fewest number of fire stations they can build to meet the requirements, and where should they be located?

What we didn't tell the students

In the language of graph theory, the map that we passed out is a *graph*: a set V of vertices, and a set E of edges that connect the pairs of elements of V. The solution to the problem—the locations of the fewest possible fire stations that meet the criteria–is the *minimum dominating*

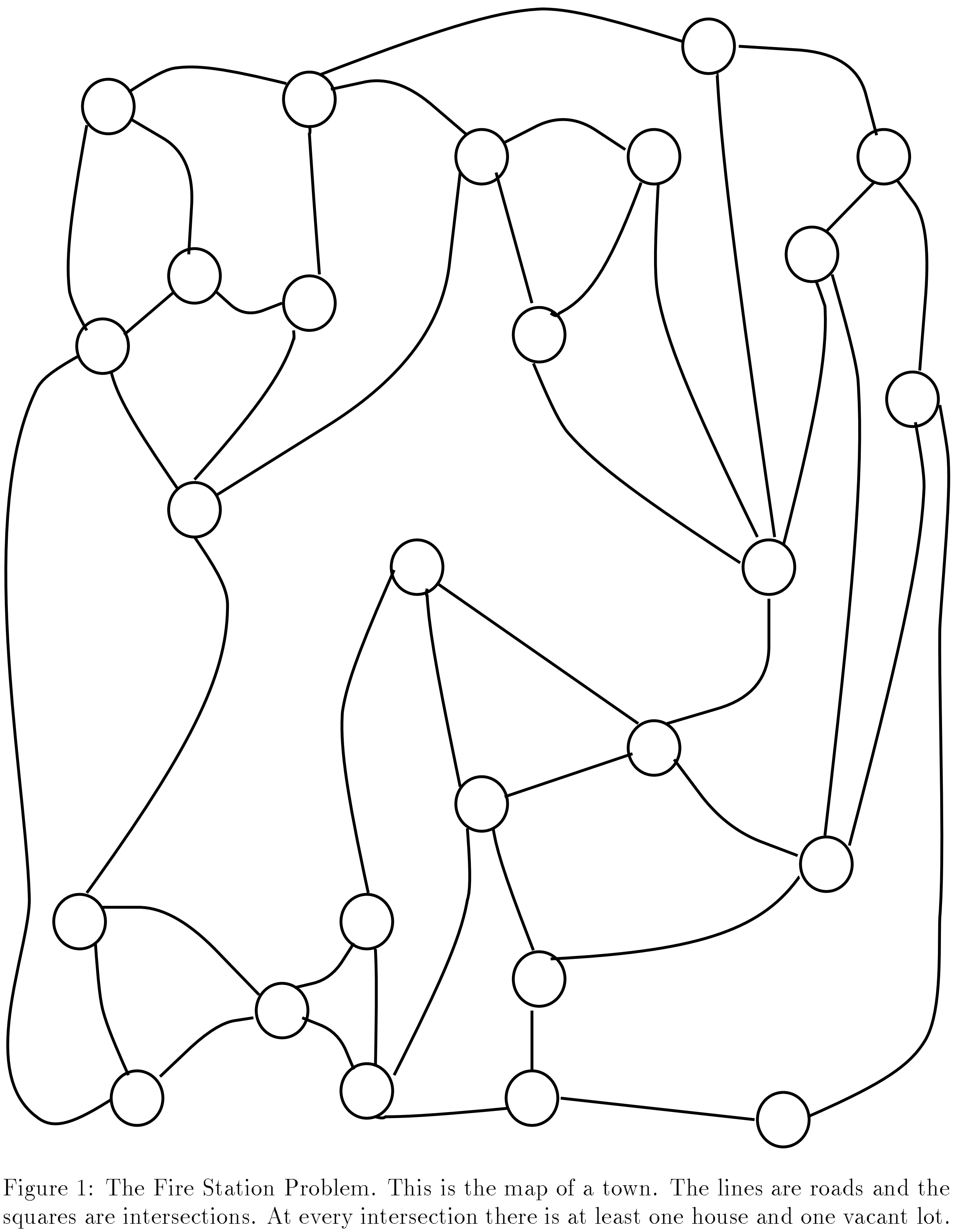

Figure 1: The Fire Station Problem. This is the map of a town. The lines are roads and the squares are intersections. At every intersection there is at least one house and one vacant lot. The town council has decided that when there is a fire, a fire truck shouldn't have to drive more than one street in order to get there. How many fire stations must the town build, and at which intersections should they be located?

set, or a subset S of V where every element of S is either an element of V or adjacent (connected by an edge) to an element of V. For the graph in Figure **??**, there are 6 elements in the minimum dominating set.

Although the only reliable algorithm for finding the miminum dominating set for any given graph is trial and error, there are many strategies for making informed guesses about how to proceed with a particular graph. It is a good idea at this point, before looking at the solution, for the reader to take some time with Figure **??** and work on the problem, focusing on the strategies that come to mind in seeking and modifying solutions. Then you will appreciate the the subtelties of the task we had laid before the students.

Tackling the problem

I told the students the story, then we passed out unifix cubes to each student and demonstrated how to use them to mark the locations where they thought fire stations should be built in the town.

The students attacked the problem with gusto, but not at all in the way I had expected. I struggled to remain true to our purpose of allowing the students to follow their own curiosity and reasoning, to describe and relate to the problem in their own ways, to play, to be creative, and to develop their own strategies and approaches. I passed judgement on myself: the task was mis-staged, ill-explained, and resulted in too much chaos. The classroom was in a frenzy. I felt uncomfortable about creating frenzies in school.

Many children built structures—fire stations—out of the unifix cubes. There was considerable discussion about what the graph best resembled: "It's a face! Here's the nose, this is the eyebrow..." "No! THIS is the nose, see..." One girl went quietly to her seat and colored the map.

A boy whose father is a fireman announced,"Moscow is a lot bigger than this town, and it only has two fire stations. That's plenty. I'll put them here, and here, and I've also drawn in where the swimming pools should go."

Simply understanding the problem was difficult for many of the students. Checking to see that all the houses were "covered" by a fire station was hard. It was easy to miss one. Students did not see the value of using unifix cubes to mark where the fire stations should go. They circled vertices, often in crayon or marker, not planning to be erasing. If they did not perceive problems with this approach, why should they use a tool to avoid the problems that *I* thought would occur?

An assumption prevaded the environment that there was a correct answer that could be found quickly, perhaps using a trick or a gimmick, and that only I could tell if a solution was a good one. I was mobbed with students showing me their papers. As I evaluated the solutions, even though I demonstrated how anyone could check to see if all the houses did have a fire station close enough, I seemed to confirm my authority as the only one who could tell if a solution was "right". While I tried to show one student how to evaluate the acceptability of a solution, a half dozen other little people were pulling at my elbows, poking my ribs, or pressing papers into my nose demanding, “Look at this!” “How’s mine?” “Here it is! I got it. See?” I was overwhelmed. The hysteria level seemed really high. Mine was going up too. It would have been so easy to silence them all and demand that they pay attention to a description of *my* thought process for approaching the problem.

In the confusion, I was moderately aware of Prudy, seated first in one clot of students, then another. Sometimes I heard her voice, “Well, let’s try. What happens when there is a fire at this house? Clang! Clang! Clang!” Siren sounds. Twenty minutes into the activity she bustled past me, her eyes wide, “This is fascinating. This is absolutely amazing. I just wish you knew these kids better.” She was off.

Eventually, about half the class arrived at solutions that met the criteria: every house located either on a corner with a fire station or exactly one street away from one. No one had a solution that used less than 13 fire stations, however. Hesitant to focus too much on “right” answers, I hadn’t uttered the magic number 6. When I said “As few as possible to students who had solutions using, perhaps, 15 fire stations, I could see my words washing past them with no comprehension whatsoever. Once they had done it with 15, it was inconceivable that there could be fewer. Every fire station was there for a reason. If they took any one of them off, some house(s) would be uncovered. I wanted to suggest, “Take one off and shift them around, see if you can get them covered with one less.” I had to remind myself that if I gave them strategies to try, they wouldn’t discover them, and that the purpose of the activity was for them to discover and discuss strategies, and *not* simply to solve the problem.

When the students began to declare themselves ”finished”, I gave them a journal page and told them to write how they figured it out. They were stymied. Prudy chimed in with some orienting instructions, “This is a writing assignment. So do what you always do when you write something: put your name and the date on the paper to start with. After that, just write what you did”

These children were experienced and fluent writers, yet they found writing in this new context difficult. It was hard for them to find the words to describe a process that was essentially spatial, much less describe the thoughts and motivations behind their actions. When students asked for help with what to write, we talked about what they had done.

One boy began writing because he had reached an impasse on the problem. He felt he had done a lot already, even though he hadn’t arrived at a solution. But when he put his pencil to the page, he was stuck. I asked him to explain how he got as far as he did. “Well, first I found the ones that. . . ” As his voice trailed off, the intensity in his eyes communicated that he knew exactly which “ones” he meant. After a long pause, he said, “Like this” and drew a vertex with many edges radiating from it. “When it looks like this, you know you should put one in the middle.” He continued to explain his strategy in this way, alternating demonstrative pronouns, pregnant pauses, and drawings. I encouraged him to continue writing words and pictures in whatever combination seemed best to convey his ideas.

“I got 14!” called a boy from across the room. A solution with 14 fire stations, I assumed, until he showed me. 14 houses “covered”, or protected by fire stations.. 14 out of a total of 26, so far using 4 fire stations. As he tried different configurations of fire stations, he explained their value in terms of "how many they got". He called these numbers out instantly and accurately; he did not seem to be counting. Was he so steeped in arithmetic that he turns all activities into arithmetic ones? Or do tasks like these provide students with opportunities to use and deepen their concept of number? The arithmetic reasoning that he was using can lead to a proof that for this particular graph the minimum dominating set must have at least 6 elements.

By the end of the class period, the students were finished—in the sense that they had lost interest, become weary, or been distracted. A few had assembled their books, some had decorated and colored the graphs, and many had gravitated to other areas of the classroom to build with unifix cubes and pattern blocks.

The luxury of two teachers debriefing

The students left for recess and we had 15 quick minutes to debrief. Where I saw "chaos" Prudy saw a combination of the excitement that comes naturally when second graders are stimulated to think and learn, and the discomfort that resulted from doing mathematics in ways that were new and unfamiliar. What had so fascinated her was the fact that the classroom had been turned topsy-turvy in terms of capabilities. A group of boys who loved math and plowed enthusiastically through math books, competing to see who could get the farthest, were visibly anxious and angry. They were the "best" in math: they did problems fast and did them right. In this context that was impossible. On the other hand, several students who do not do well in math and do not like it seemed extremely comfortable and enthused. They reveled in the chance to be assigned a task that relied on spatial reasoning; a few became leaders, explainers and helpers for the other students—a role not ordinarily theirs at math time.

What we were able to share during this recess and on subsequent days demonstrated to both of us how *much* creative intellectual activity can go on within a single mathematics lesson. We related to each other dozens of anecdotes about what we observed students doing, about how they seemed to be thinking, and about what seemed to be happening in the classroom. Although our experiences were consistent with each other, seldom were they overlapping. The mere opportunity to share like this was a stimulating luxury—so seldom do two teachers have a chance to be together in the classroom and the time later to ask each other, "What did *you* see?"

Vocabulary had been a problem. We didn't want to flood the classroom with confusing terminology, and we wanted the students to be free to name the concepts that needed naming. They did, however, pick up on the words that we used. The verb "cover" became most ambiguous. Was a house "covered" when there was a unifix cube on its corner? In some contexts "covered" meant that a unifix cube was on an adjacent corner. Perhaps planning ahead to use the verb "dominate" would have avoided the ambiguity and led naturally to the term "dominating set". We had created a context in which the necessity for precision in mathematical language was obvious.

Most fascinating was the children's general inability to use words to describe what they had done. Did this indicate that an activity such as this one that is so spatial is inherently nonverbal? Or that we need to provide students with more vocabulary for describing spatial concepts and reasoning? What does their difficulty in speaking about mathematical processes tell us about their comprehension of what we tell them about mathematics? Had we provided a new context for communication, a context within which the students would become fluent with time and practice?

Day two: Quiet concentration.

To the degree that the previous day was frenetic, the second day of the activity was marked by quiet, careful thought.

We began by meeting in a circle and the children discussed what they had done the previous day. We laid a copy of the graph and some cubes on the floor, and they took turns showing what they had done. They posed questions and corrected each other, but most markedly, they listened to each other one at a time.

From the circle they went back to what they had worked on the previous day. The classroom was calm; people spoke in hushed voices. Students worked on the problem, assembled their books, and wrote. I had prepared some materials with more information: vocabulary and notation from graph theory, a chart of mathematical symbols, some other problems that involved graphs, a graph theory textbook. All of this was available for the students to browse, which some did. Some of this information resurfaced in their writing over the next few days.

Two boys, working independently, found the minimum solution with 6 vertices almost simultaneously. Both described a similar process: "I started out with 10. Then took away one so it was 9. (Demonstrates shifting fire stations to accommodate the one missing.) Then it was 8. And finally I looked and you could do it with 6. Neither one stair-stepped all the way down to 6. Did they abandon that process because they noticed something else? I don't know. They didn't say. They couldn't say? They got mad at me for asking. They were both absolutely and unwaveringly sure that the configuration they had both come upon was the only one that used 6 vertices. How do you know? It is, it just is.

Other students found the minimum solution, some working alone and some working together. Students freely shared the solution—it was not perceived as secret knowledge for the select group who discovered it. By the time the solution was found, however, many students were engaged in other activities, particularly writing, that pertained to the problem. The actual solution no longer seemed to be of paramount importance like it had been the day before.

Day Three: Refocusing and asking new questions

At the beginning of class, I used the overhead to show the students how I invented this problem. I knew that dominating sets were interesting, so I made one. First I drew 6 vertices that would be the vertices for the solution, then I drew edges coming out of the vertices as in Figure **??**. Demonstrating with unifix cubes, it is easy to see that the center vertices of each "star" are indeed the vertices of the dominating set. Then I drew in additional lines, the "camoflauge" vertices, and it was as if the apparent solution disappeared before our very eyes.

I talked about *representation*. A graph all by itself seemed kind of boring, so I made up a story to go along with it, pretending that it was a map of a town that had no fire stations, and so forth. I related how one student had already pointed out that this wasn't really a very good representation because the town ended up with far more fire stations than would have been necessary in the real world. In addition, some people may not think fire stations are all that interesting. The graph could represent anything, it didn't even have to be a town.

I invited them to make a puzzle like this one, beginning with a dominating set and building a larger graph, and then making up a story to go with it. Then they could show their puzzle to other people.

They dived into this task with great enthusiasm and confidence, even though it was not at all trivial. To come up with a story that is cohesive, they had to think of something that they liked that existed in a one-to-many relationship with something else that they liked, and then invent a narrative that poses the question "find the minimum dominating set" in the vocabulary of the story. Not all of their stories worked in that sense, but there wasn't really enough time to share them thoughtfully with a friend or edit and revise them after discovering inconsistencies or problems.

The students worked cheerfully and hard. They drew boldly with markers on big glossy pieces of paper. One girl drew an elaborate system of hotel swimming pools and walkways that connected them. Another made cats and their houses. One of the boys made an intricate representation involving configurations of chess pieces. Neither Prudy nor I understood what he explained, although it was quite clear to him. He raised an eyebrow of disappointment before impatiently abandoning us to our incomprehension.

Another student bounded up repeatedly to shove new pieces of paper under my nose and demand, "Here! What are my main dots?" I sat down to work on one, ostensibly to demonstrate that these problems aren't a snap to figure out, even if you do happen to be the teacher. I saw this as an opportunity (at last!) to think aloud and demonstrate my strategies for solving these problems. He sat excitedly next to me, watching. No sooner did I make a tentative selection of a single vertex, than he would peek at the back side of the page (where the solution dots were marked) and announce either yes or no. Subtract-and-shift, my beloved strategy, was not inside his reality. Nor was the possibility that it should take me a while to solve the problem—if I solved it at all. The idea was to stump me, he was succeeding, and he was joyous.

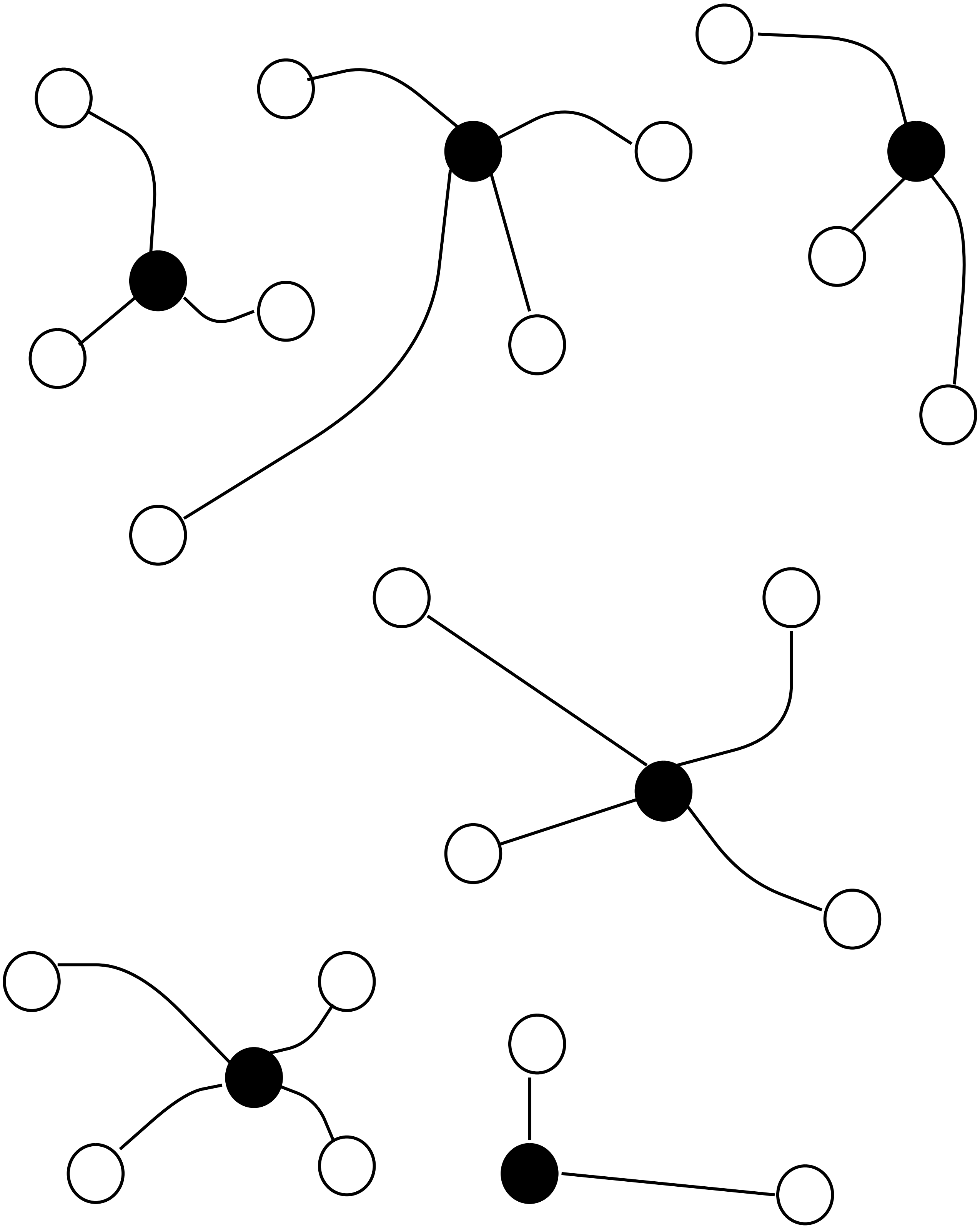

Figure 2: This is the drawing that was made before the map was produced. was produced. From this drawing, the solution to the Fire Station problem is obvious. Once the additional lines were added to produce the map, the solution was disguised and the problem became difficult to solve.

Day four: Mathematical delight

We left the last day wide open, simply inviting the students to do what they needed to do to feel finished. Some worked on their books, putting them all together, filling the pages with "mathematics": numbers, symbols, indications of correspondences, diagrams. No child lacked ideas for mathematical things to do, and the variety of approaches to their work was astounding. I received many mathematical pieces of art: beautifully decorated graphs, cryptic puzzles, and exciting scenes drawn entirely with numbers and mathematical symbols. The classroom was a happy, active place. Prudy described it as a celebration of the joy they had found in doing a new kind of mathematics for a few days. Unlike the first day, we had to suspend the activity, tell them to stop, and send them out for recess.

What if this hadn't been our last day? Had this been the regular writing workshop instead of an experimental mathematics workshop, the students would have continued working on projects that they chose, conferencing with one another for feedback to make sure their final products communicated their ideas, and publishing their results in handmade books, pamphlets, or newsletters.

As the "Whole Language" approach has evolved over the last 20 years, the reading and writing teachers have learned how to build a Language Arts curriculum that is organized around meaningful creative work by students. In this environment, students do indeed learn sound-letter correspondences, spelling, reading comprehension, vocabulary and other measurable "indicators" of language arts performance. The central question raised by our classroom experiment is not "Can an approach like this work for mathematics learning?" Rather it is, "What must we teachers learn about mathematics and children's mathematical development so that we can create the environments where full mathematical performance can take place, and how shall we teach our students to blossom mathematically in these environments?"

Conclusions

Everything that we did and saw served to convince us that our experiences in using the Whole Language approach to teaching reading and writing can and should be adapted to the teaching of mathematics. At present, these second graders work in reading and writing workshops. These large blocks of time allow them to devote long periods of concentration to reading and writing projects they have chosen. During the course of the school year they are taught how to engage in the activities of literate people during this time: not only reading, writing, and thinking, but also sharing with each other in meaningful and productive ways and producing polished final products. The prospect of expanding this or a similar environment to include learning in mathematics is exciting. Four days was enough time to see the potential, but not the fruits, of students acting authentically as members of a mathematical community—sharing ideas, pursuing thoughts, challenging each other's reasoning, choosing, inventing, and completing projects.

This experiment raised many more questions than it answered. What behaviors and skills do we need to teach students so they can make the transition to learning mathematics in this context? To what extent is the discomfort, confusion and seeming chaos with which we began an integral part of the problem-solving process? What is the proper balance between instruction and discovery? What is the natural, comfortable role of specialized vocabulary and notation? What kind of discussion and communication would have resulted if the students had had time to share, review, discuss, edit and revise the dominating set puzzles they had made? If mathematics were taught always in this environment, how are we assured that the tasks and skills of the school math curriculum will be covered? How do you describe this approach in terms of curriculum? What are the techniques for assessment and evaluation?

We turn to the Curriculum and Evaluations Standards [NCTM] of the National Council of Teachers of Mathematics to see that our experiement fits with issues central to the national reform in mathematics teaching: development of mathematical power, encouraging mathematical reasoning and fostering mathematical communication. From our perspective as language and literacy teachers, classroom discourse is at the core of learning.

Like the NCTM Standards, the Whole Language Approach establishes goals and provides a framework, but does not dictate any specific methodology. It is only in the classroom that we really find out what works and what is truly valid. Just as the *Standards* seek to personalize learning, every teacher must develop a personal style for implementing them. We teachers are at the nexus of the reform in mathematics education. We are called upon to take risks, to observe, to learn and make mistakes, and to share with and inspire each other.

Afterthougts and other notes

The activity

I have done this activity with groups of second and fourth graders, high school students, and college students. At all these different levels, the problem is both accessible and challenging and lends itself to lots of discussion, experimentation and theorizing. Younger students can present or “teach” a problem like this to older students, which, I think, can go a long way towards building true mathematical communities in schools, as well as upsetting hard-to-shake notions about what constitutes “third grade work” and “mathematical ability”.

The choice of fire stations as the focus of this activity probably reflects the kinds of things adults (rather than children) find important. With other groups, I have used shopping malls, movie theaters, ice cream stands, convenience stores, and parks. My collaborator Mike Fellows of the University of Victoria did the activity with a group of children from the countryside near Lima, Peru, and at the teacher’s suggestion posed the question as “Where should we put the wells in this town?”

One particularly fascinating student

One of my most intriguing experiences with this activity was observing a fourth-grade boy. The students were using unifix cubes to locate, in this case, where the shopping malls should go. After working on the problem for a while, he made a huge construction out of unifix cubes, consisting of a shopping mall and a series of bridges that linked all the locations on the map to the single shopping mall. He kept calling me over to see what he was doing. Other students were working hard, helping each other, trying to solve the problem as I had posed it, and I felt they deserved my attention more than he did. I would pause by his construction when he asked me to, and tell him nicely-but-firmly that this wasn't exactly what I had in mind. He played like this for the whole class period.

None of the children solved the problem with anything close to the minimum number of shopping malls. The next day when I returned and we got out the materials again, this same boy produced a minimum solution in the first ten minutes of the class period. So...was he really "off task" the first day, or had he been thinking about the problem in his own way? Many times I have had the experience of suspecting that continued frontal assaults on a problem would be fruitless. Often, in such cases, I either set it aside or fiddle with the "irrelevant" details and wait for some kind of inspiration to sink in. Of course, it would have been great fun, if this young boy had articulted doing exactly that. He merely shrugged and said, "I just did it." His teacher described him as very bright and very angry, and said that this behavior was "typical'. I regret now that I hadn't tried to draw him out a little more when he was building his bridges. It seems that if there had been a way for him to examine and talk about his problem-solving strategies, everyone could have learned a lot from him.

Mathematical connections

This problem is an example of what is called a *one-way function*. If you construct an example (i.e., the graph or the map) in the manner described in the article, the solution is easy and obvious. However, if you do not have access to the information about the construction the problem becomes intractable for ever-larger maps. It is far easier to go *one way* (from the problem to the solution) than it is to go the other.

Prime factorization is another example of a one-way function. If I multiply two or more very large prime numbers together, and ask you to factor the result, you will need to work much harder to solve the problem than I did to produce it. It is fun and interesting to look for other examples of one-way functions around us, and to try to invent other problems that make use of them. One-way functions are fundamental in cryptography. I seldom find it hard to get students excited about secret codes.

References

[NCTM] *Curriculum and Evaluation Standards for School Mathematics*, National Council of Teachers of Mathematics, Reston, Virginia, 1989.