University of Wisconsin - Madison

MADPH-95-885

September 1997

# Flux tubes in effective field theory

M.G. Olsson and Siniša Veseli[1]

*Department of Physics, University of Wisconsin, Madison,* WI 53706

Ken Williams

*Continuous Electron Beam Accelerator Facility*

*Newport News, VA 29606, USA*

*and*

*Physics Department, Hampton University, Hampton, VA 29668*

**Abstract**

Quark-antiquark bound states are examined in the long-range strong-coupling limit with the Wilson minimal area law assumed as input. Matrix element relations are established which in the effective theory obtain dynamical equations resembling the confining part of the relativistic flux-tube model.

[1]Present address: Theory Group, Fermilab, P.O. Box 500, Batavia, IL 60510

# 1 Introduction

The success of QCD as the correct theory of strong interactions has rested primarily upon perturbative calculations. The prediction of hadronic properties is inherently non-perturbative and has been much slower to develop. A promising approach has been the low quark velocity Wilson loop expansion [1]. The aim of this paper is to extend this method to arbitrary quark velocity.

The QCD Lagrangian density is a fundamental object in the theory of strongly interacting particles. It is given by the minimally transformed free Lagrangian (sum goes over quark flavors $f$)

$$\begin{aligned} p_\mu &\rightarrow p_\mu - gA_\mu \ , \qquad (1) \\ \mathcal{L}_{free} &\rightarrow \mathcal{L}_{QCD} = \sum_f \bar{\psi}_f (i \not{D} - m_f)\psi_f - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \ , \qquad (2) \end{aligned}$$

from which follow the Euler Lagrange equations

$$\begin{aligned} (i \not{D} - m_f)\psi_f &= 0 \ , \qquad (3) \\ D^\mu F_{\mu\nu} &= g \sum_f \bar{\psi}_f T \gamma_\nu \psi_f \ . \qquad (4) \end{aligned}$$

These are non-linear and amenable only as a perturbation expansion in small coupling $g$. Wilson's observation [2] over two decades ago has led to some of the most fruitful work in the non-perturbative regime. The simple picture offered in his area law for static quarks is compelling: contributions to the $q\bar{q}$ propagator fall off exponentially in the area swept out by lines of gauge invariant chromoelectric flux joining the constituents' world lines; in this way, widely separated paths are suppressed. Subsequent lattice simulations verify the same qualitative behavior for dynamical quarks.

Here we derive an effective Hamiltonian for a quark-antiquark system in the confinement region with Wilson's minimal area (MA) law [2] as input. In this arrangement, the gauge field's non-Abelian character is manifest through the minimal area law; complete non-Abelian expressions are therefore given only where useful, e.g. to clarify a point. We make the usual assumption that the minimal surface area is swept out by a straight line containing the two quarks.

We also make an additional ansatz. The vacuum expectation value of an observable $\vartheta$ weighted by the Wilson loop is taken to be nearly equal to the classical value of $\vartheta$,[2]

$$\langle\vartheta\rangle_W \approx \vartheta_{classical} \ . \tag{5}$$

This is exactly the case for Abelian gauge fields. In the effective theory the Wilson loop expectation values are transformed into first-quantized operators by symmetrization. The results are fully relativistic, and represent a generalization of the previous work [3] to include quarks with spin.[3]

## 2 Wilson loop expectation values

We begin by assuming that the MA analysis has been carried out and hence fermion and gauge fields are related. This effectively reduces the number of degrees of freedom in (2), and allows us to take the gauge fields specified by the MA relations to be "external" to the Lagrangian. In this way, the Euler-Lagrange non-linearities (3)-(4) will be hidden within the undetermined vacuum expectation values, which can be related to the corresponding quantities of the relativistic flux tube model [4].

From (2) the commuting observables are the usual

$$\begin{aligned} H &= \sum_i (\boldsymbol{\alpha}_i \cdot \boldsymbol{\pi}(r_i) + \beta m_i) + gA_0(r_1) - gA_0(r_2) \ , & (6) \\ \mathbf{P} &= \sum_i \mathbf{p}_i = \sum_i (\boldsymbol{\pi}(r_i)) + g\mathbf{A}(r_1) - g\mathbf{A}(r_2) \ , & (7) \end{aligned}$$

[2]The Wilson loop expectation value is defined in (12).

[3]In [3] a relativistic QCD-string Lagrangian for spinless quarks has been deduced.

$$\mathbf{J} \;=\; \sum_i (\mathbf{r}_i \times \mathbf{p}_i + \mathbf{S}_i) \;, \tag{8}$$

where $\boldsymbol{\pi}$ is the mechanical-momentum operator, $r_i$ and $\mathbf{r}_i$ are the four and three vector coordinates of the $i^{th}$ quark, and $\mathbf{S}$ is the spin operator. The MA law [2],

$$i \ln\langle W(\mathcal{C})\rangle = a S_{MA} \;, \tag{9}$$

together with (6)-(8) define our bound state problem. In the above $a$ denotes the string tension.

We consider the Wilson loop of a straight slice out of the $q\bar{q}$ minimal world sheet from time $\tau'$,

$$\begin{aligned} \langle W(\mathcal{C})\rangle &= \frac{1}{3}\langle\, \mathrm{Tr}\, \mathrm{P} \exp(ig \oint_{\mathcal{C}} dy^\mu A_\mu(y))\rangle \\ &= \frac{1}{3}\langle\, \mathrm{Tr}\, \mathrm{P} \exp(-ig \int_{\tau_1'}^{\tau_{1f}} d\tau \, [A_0(r_1) - \mathbf{v}_1 \cdot \mathbf{A}(r_1)])\, U(r_2', r_1') \\ &\times \;\; \mathrm{P} \exp(-ig \int_{\tau_{2f}}^{\tau_2'} d\tau \, [A_0(r_2) - \mathbf{v}_2 \cdot \mathbf{A}(r_2)])\, U(r_{1f}, r_{2f})\, \rangle \;. \end{aligned} \tag{10}$$

The $U$'s here are straight-line path ordered exponentials, and the average is taken over gauge fields,

$$\langle \vartheta \rangle = \frac{\int \mathcal{D}A \exp(iS[A]) \vartheta[A]}{\int \mathcal{D}A \exp(iS[A])} \;, \tag{11}$$

with the Wilson loop expectation defined by

$$\langle \vartheta \rangle_W \equiv \langle W(\mathcal{C})\, \vartheta\rangle / \langle W(\mathcal{C})\rangle \;, \tag{12}$$

and where the pure gauge action $S[A]$ also includes a gauge fixing term. The classical path variables, $\mathbf{v}_i$, are related to the mechanical-momentum operator by (5) as

$$m_i \gamma_i \, \mathbf{v}_i \;\;\equiv\;\; \langle \boldsymbol{\pi}(r_i)\rangle_W \;. \tag{13}$$

where $m_i\gamma_i$ is the relativistic mass. The quark degrees of freedom have not yet entered the dynamics except as a means to specify the physical gauge field configurations. Our method consists in taking the Wilson loop expectation (12) of the observables (6)-(8). The gauge field part will be evaluated directly and the particle expectation value will be identified by (5) with quark mechanical quantities as in (13).

We parametrize the enclosed area in the Nambu-Goto form,

$$S = \int_{\tau'}^{\tau_f} d\tau \int_0^1 d\sigma [\dot{x}^2 x'^2 - (\dot{x} \cdot x')^2]^{1/2} \tag{14}$$

with $x_\mu = x_\mu(\tau, \sigma)$, $x_0 = \tau$, $\dot{x}_\mu = \frac{\partial x_\mu}{\partial \tau}$, and $x'_\mu = \frac{\partial x_\mu}{\partial \sigma}$. At the boundary, $\mathbf{x}(\tau, 1) = \mathbf{r}_1(\tau)$ and $\mathbf{x}(\tau, 0) = \mathbf{r}_2(\tau)$. In the usual straight-line and equal-times approximations, which we assume here, the minimum is given by

$$\mathbf{r}_t(\tau, \sigma) = \sigma \mathbf{r}_1(\tau) + (1 - \sigma) \mathbf{r}_2(\tau) \ , \tag{15}$$

so that

$$\begin{aligned} S_{MA} &= \int_{\tau'}^{\tau_f} d\tau |\mathbf{r}_1 - \mathbf{r}_2| \int_0^1 d\sigma [1 - (\sigma \mathbf{v}_{1\perp} + (1 - \sigma) \mathbf{v}_{2\perp})^2]^{1/2} \ , \\ &= \int_{\tau'}^{\tau_f} d\tau \, r \int_0^1 d\sigma [1 - \mathbf{v}_t^2]^{1/2} \ . \end{aligned} \tag{16}$$

Here $r \equiv |\mathbf{r}_1 - \mathbf{r}_2|$, and "$\perp$" is defined relative to the straight line. The quark velocities transverse to the minimal line are $\mathbf{v}_{i\perp} \equiv \mathbf{v}_i - (\mathbf{v}_i \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}}$, and the transverse velocity along the line is $\mathbf{v}_t = \sigma \mathbf{v}_{1\perp} + (1 - \sigma) \mathbf{v}_{2\perp}$. The functional derivative of (9) with respect to $\mathbf{v}_1$ for fixed endpoints and fixed time t

$$\frac{i}{\langle W(\mathcal{C}) \rangle} \frac{\delta \langle W(\mathcal{C}) \rangle}{\delta \mathbf{v}_1(t)} = \frac{\delta a S_{MA}}{\delta \mathbf{v}_1(t)} \tag{17}$$

obtains (see Appendix for details)

$$\langle g \mathbf{A}(r_1) \rangle_W = ar \int_0^1 \gamma_t \mathbf{v}_t \sigma d\sigma \, , \tag{18}$$

and similarly,

$$\langle g \mathbf{A}(r_2) \rangle_W = -ar \int_0^1 \gamma_t \mathbf{v}_t (1 - \sigma) d\sigma \ , \tag{19}$$

where $\gamma_t = (1 - \mathbf{v}_t^2)^{-1/2}$. These are the desired Wilson loop expectation values. It should be clear that the gauge fields quantized in this average carry spatial dependence in the radial coordinate only. By (7), (18) and (19), the total expectation value for the gauge field momentum is

$$\begin{aligned} \langle \mathbf{P} \rangle_W^{gauge} &= \langle g\mathbf{A}(r_1)\rangle_W - \langle g\mathbf{A}(r_2)\rangle_W \\ &= ar\int_0^1 \gamma_t \mathbf{v}_t d\sigma \ . \end{aligned} \tag{20}$$

The time component is obtained by differentiation of (9) with respect to $\tau'$,

$$\begin{aligned} \frac{d}{d\tau'} i \ln\langle W(\mathcal{C})\rangle &= \frac{i}{\langle W(\mathcal{C})\rangle} i \frac{1}{3} \Big\langle \operatorname{Tr} \mathrm{P} \exp\,(ig \oint_{\mathcal{C}} dx^\mu A_\mu(x)) \\ &\times \ g\{A_0(r_1) - A_0(r_2) - [\mathbf{v}_1 \cdot \mathbf{A}(r_1) - \mathbf{v}_2 \cdot \mathbf{A}(r_2)]\} \Big\rangle \\ &= -ar\int_0^1 (1 - \mathbf{v}_t^2)^{\frac{1}{2}} d\sigma \ , \end{aligned} \tag{21}$$

and hence

$$\begin{aligned} \langle gA_0(r_1)\rangle_W - \langle gA_0(r_2)\rangle_W &= \mathbf{v}_1 \cdot \langle g\mathbf{A}(r_1)\rangle_W - \mathbf{v}_2 \cdot \langle g\mathbf{A}(r_2)\rangle_W \\ &- \ ar\int_0^1 (1 - \mathbf{v}_t^2)^{\frac{1}{2}} d\sigma \ . \end{aligned} \tag{22}$$

Using (18) and (19) we then obtain

$$\langle gA_0(r_1)\rangle_W - \langle gA_0(r_2)\rangle_W = ar\int_0^1 \gamma_t d\sigma \ . \tag{23}$$

We are now ready to take Wilson loop expectation values of the observables (6-8) which in the effective theory are promoted again to noncommuting quantum operators requiring symmetrization. Using (23), Hamiltonian (6) becomes

$$\langle H \rangle_W = \Big[\sum_{i=1}^{2} (m_i \gamma_i \boldsymbol{\alpha}_i \cdot \mathbf{v}_i + \beta m_i) + ar\int_0^1 \gamma_t d\sigma\Big]_{sym} \ . \tag{24}$$

By (7) and (20) we have the total meson momentum

$$\langle \mathbf{P} \rangle_W = \Big[\sum_{i=1}^{2} (m_i \gamma_i \mathbf{v}_i) + ar\int_0^1 \gamma_t \mathbf{v}_t d\sigma\Big]_{sym} \ . \tag{25}$$

The angular momentum of the meson follows from (8),

$$\langle \mathbf{J} \rangle_W = \Big[\sum_{i=1}^{2}(m_i\gamma_i \mathbf{r}_i \times \mathbf{v}_i + \mathbf{S}_i) + ar\int_0^1 \gamma_t \mathbf{r}_t \times \mathbf{v}_t d\sigma\Big]_{sym} \ . \tag{26}$$

We now compute the above Wilson loop expectation values of the commuting observables (6)-(8) in the meson rest frame. We take our coordinate origin at the CM (which will be defined shortly). If $\boldsymbol{\omega}$ is the instantaneous angular velocity about the CM, then $\mathbf{v}_{i\perp} \equiv \boldsymbol{\omega} \times \mathbf{r}_i$, and

$$\begin{aligned} \mathbf{r}_t &= (\sigma|\mathbf{r}_1| - (1-\sigma)|\mathbf{r}_2|)\hat{\mathbf{r}}_1 = r_t \hat{\mathbf{r}}_1 \ , & (27) \\ \mathbf{v}_t &= \omega(\sigma|\mathbf{r}_1| - (1-\sigma)|\mathbf{r}_2|)\hat{\mathbf{v}}_{1\perp} = v_t \hat{\mathbf{v}}_{1\perp} \ . & (28) \end{aligned}$$

We observe by redefining the integration variable that for any function $F(v_t)$ we have ( $|\mathbf{r}_1| + |\mathbf{r}_2| \equiv r$)

$$r\int_0^1 F(v_t)d\sigma = \sum_{i=1}^{2} |\mathbf{r}_i| \int_0^1 F(\sigma v_{i\perp})d\sigma \ . \tag{29}$$

Using the above identity we may rewrite (24)-(26) in the CM frame as

$$\begin{aligned} \langle H \rangle_W &= \Big[\sum_{i=1}^{2}\Big(m_i\gamma_i \boldsymbol{\alpha}_i \cdot \mathbf{v}_i + \beta m_i + a|\mathbf{r}_i|\int_0^1 \gamma_{i\perp} d\sigma\Big)\Big]_{sym} \ , & (30) \\ \langle \mathbf{P} \rangle_W &= \Big[\sum_{i=1}^{2}\Big(m_i\gamma_i \mathbf{v}_i) + a|\mathbf{r}_i|\mathbf{v}_{i\perp}\int_0^1 \gamma_{i\perp}\sigma d\sigma\Big)\Big]_{sym} \equiv 0 \ , & (31) \\ \langle \mathbf{J} \rangle_W &= \Big[\sum_{i=1}^{2}\Big((m_i\gamma_i|\mathbf{r}_i|v_{i\perp} + a|\mathbf{r}_i|^2 v_{i\perp}\int_0^1 \gamma_{i\perp}\sigma^2 d\sigma)\hat{\mathbf{r}}_i \times \hat{\mathbf{v}}_{i\perp} + \mathbf{S}_i\Big)\Big]_{sym} \ . & (32) \end{aligned}$$

The condition (31) that the total linear momentum vanishes defines the CM system.

The first term in each of the above expressions has been identified as the quark's relativistic energy, linear momentum and angular momentum, respectively, by ansatz (5). The second term in each expression can be identified as the flux tube's rotational energy, linear momentum, and angular momentum, respectively [4].

# 3 Conclusion

In this paper we have demonstrated that under the same assumptions as in previous work [1, 3] the Wilson loop area law and the QCD Lagrangian yield a relativistically valid picture of dynamical confinement. We emphasize that we have not "solved" QCD, but that our ignorance can be distilled into the loop expectation values each of which turns out to be similar to the corresponding quantity of the mechanical relativistic flux tube model [4].

Our results represent an extension of the work of Brambilla and Prosperi [1] to quarks with arbitrary quark velocity. In addition, they also represent a generalization of the work of Dubin et al. [3] to include quarks with spin.

# A Appendix

Equation (18) follows from the direct application of the variation (17). We write from (16)

$$\begin{aligned} S_{MA} &= \int_{\tau'}^{\tau_f} d\tau |\mathbf{r}_1 - \mathbf{r}_2| \int_0^1 d\sigma [1 - (\sigma \mathbf{v}_{1\perp} + (1-\sigma)\mathbf{v}_{2\perp})^2]^{1/2} \ , \\ &\equiv \int_{\tau'}^{\tau_f} d\tau \, L \end{aligned} \tag{33}$$

express the Wilson loop in discrete form,

$$\begin{aligned} W(\mathcal{C}) &= \frac{1}{3} \mathrm{Tr}\ \mathrm{P} \prod_{n=0}^{N} \exp(ig\Delta t_n [A_0(r_{1n}) - \mathbf{v}_{1n} \cdot \mathbf{A}(r_{1n})]) U(r_2', r_1') \\ &\times \exp(-ig\Delta t_n [A_0(r_{2n}) - \mathbf{v}_{2n} \cdot \mathbf{A}(r_{2n})]) U(r_{1f}, r_{2f}) \ , \end{aligned} \tag{34}$$

and take the path variations according to ($\delta \mathbf{r}_2 = 0$)

$$\begin{aligned} \delta(i \ln\langle W(\mathcal{C})\rangle) &= \frac{i}{\langle W(\mathcal{C})\rangle} \delta\langle W(\mathcal{C})\rangle \\ &= \frac{i}{\langle W(\mathcal{C})\rangle} \Big\langle \sum_{n=0}^{N} (\delta \mathbf{r}_{1n} \cdot \frac{\partial}{\partial \mathbf{r}_{1n}} + \delta \mathbf{v}_{1n} \cdot \frac{\partial}{\partial \mathbf{v}_{1n}}) W(\mathcal{C}) \Big\rangle \ , \end{aligned} \tag{35}$$

giving,

$$\frac{1}{\langle W(\mathcal{C})\rangle}\int_{\tau'}^{\tau_f} d\tau \frac{1}{3}\Big\langle \operatorname{Tr} \mathrm{P} \exp\left(ig\oint_{\mathcal{C}} dx^\mu A_\mu(x)\right)(\delta\mathbf{r}_1\cdot\frac{\partial}{\partial\mathbf{r}_1}+\delta\mathbf{v}_1\cdot\frac{\partial}{\partial\mathbf{v}_1})\, f(r_1,\mathbf{v}_1)\Big\rangle$$
$$= a\left(\delta\mathbf{r}_1\cdot\frac{\partial S_{MA}}{\partial\mathbf{r}_1}+\delta\mathbf{v}_1\cdot\frac{\partial S_{MA}}{\partial\mathbf{v}_1}\right)\ . \tag{36}$$

or

$$\frac{\delta(i\ln\langle W(\mathcal{C})\rangle)}{\delta\mathbf{v}_1} = \langle\frac{\partial}{\partial\mathbf{v}_1}f\,\rangle_W \tag{37}$$
$$= a\frac{\partial L}{\partial\mathbf{v}_1} \tag{38}$$

for $\mathbf{v}_1$ at fixed time. Here

$$f(r_1,\mathbf{v}_1) = -g[\mathbf{v}_1\cdot\mathbf{A}(r_1) - A_0(r_1)]\ . \tag{39}$$

From (38)

$$\langle g\mathbf{A}(r_1)\rangle_W = a|\mathbf{r}_1-\mathbf{r}_2|\int_0^1 \frac{(\sigma\mathbf{v}_{1\perp}+(1-\sigma)\mathbf{v}_{2\perp})}{[1-(\sigma\mathbf{v}_{1\perp}+(1-\sigma)\mathbf{v}_{2\perp})^2]^{1/2}}\sigma d\sigma\ , \tag{40}$$

which is equation (18).

## ACKNOWLEDGMENTS

We would like to thank N. Brambilla and G. Prosperi for reading the manuscript and making many helpful comments. This work was supported in part by the U.S. Department of Energy under Contract Nos. DE-FG02-95ER40896 and DE-AC05-84ER40150, the National Science Foundation under Grant No. HRD9154080, and in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation.

# References

[1] E. Eichten and F. Feinberg, Phys. Rev. D **23**, 2724 (1981); D. Gromes, Z. Phys. C **22**, 265 (1984); N. Brambilla, P. Consoli, and G. M. Prosperi, Phys. Rev. D **50**, 5878 (1994); N. Brambilla and G. M. Prosperi, *Wilson loops, $q\bar{q}$ and 3q potentials, Bethe-Salpeter equation*, Proceedings of the Conference *Quark Confinement and the Hadron Spectrum*, World Scientific, Singapore, 1995.

[2] K. Wilson, Phys. Rev. D **10**, 2445 (1974).

[3] A. Yu. Dubin, A. B. Kaidalov and Yu. A. Simonov, Phys. Lett. B **323**, 41 (1994).

[4] D. LaCourse and M. G. Olsson, Phys. Rev. D **39**, 2751 (1989); C. Olson, M. G. Olsson, and K. Williams, Phys. Rev. D **45**, 4307 (1992); M. G. Olsson and Ken Williams, Phys. Rev. D **48**, 417 (1993); M. G. Olsson and Siniša Veseli, Phys. Rev. D **51**, 3578 (1995); M. G. Olsson, Siniša Veseli, and Ken Williams, Phys. Rev. D **53**, 4006 (1996).