Structural Characterizations of Computational (In)Tractability

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No notes

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Overview:

1 Sample of my work: Computational Complexity and Phase Transitions

2 Proposed work with PARC: structural parameters for "small-world networks"
– Definitions and discussions.
– Preliminary result: routing in small-world networks

3 Possible sources for structural parameters
– geometric embeddings of graphs
– tree-like decompositions

Can you predict whether a class of instances of a combinatorial problem is “easy/hard” ?

Computational Complexity: P/NP-complete.
Criticism: Pessimistic, worst-case theory.

Can one say something better than “hard in the worst case” ?

Example: 3-SAT:

Clause: $x_1 \vee \overline{x_3} \vee \overline{x_7}$.
Formula: Conjunction of clauses.
To decide: Is the formula satisfiable ?
NP-complete, hence “hard” in the worst case.

Random model: $c = \#clauses/\#variables$.

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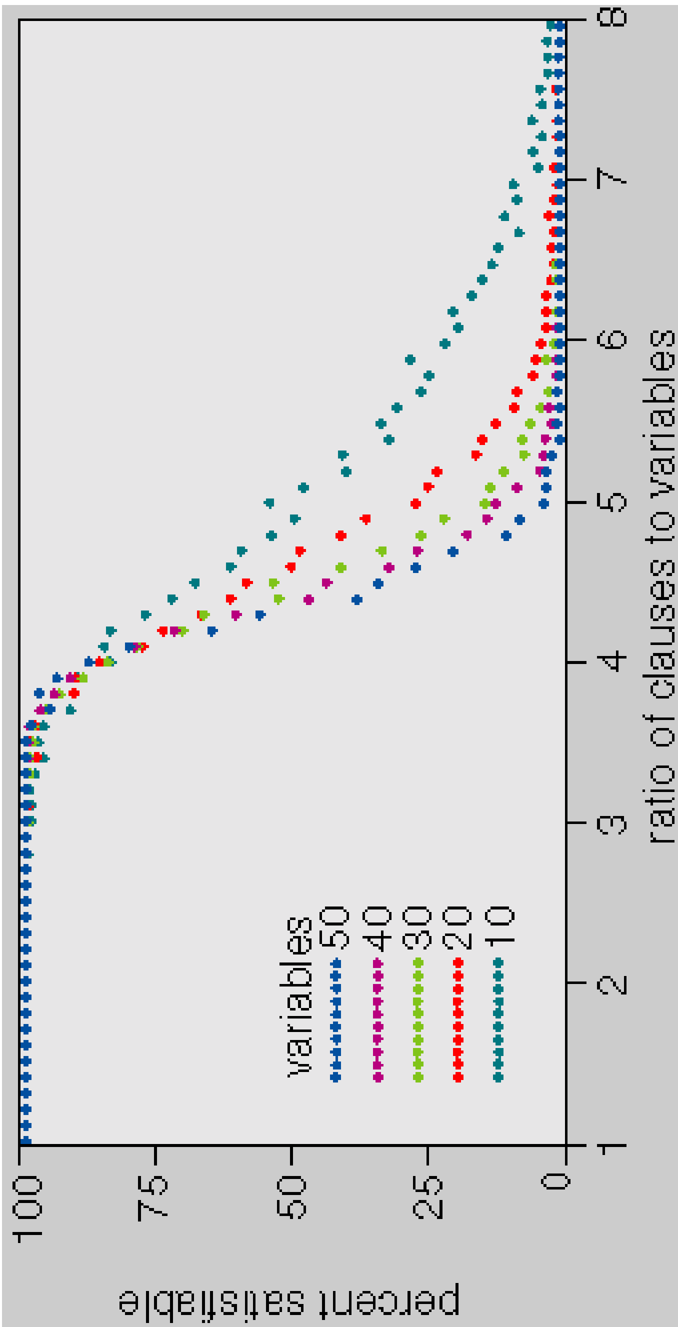

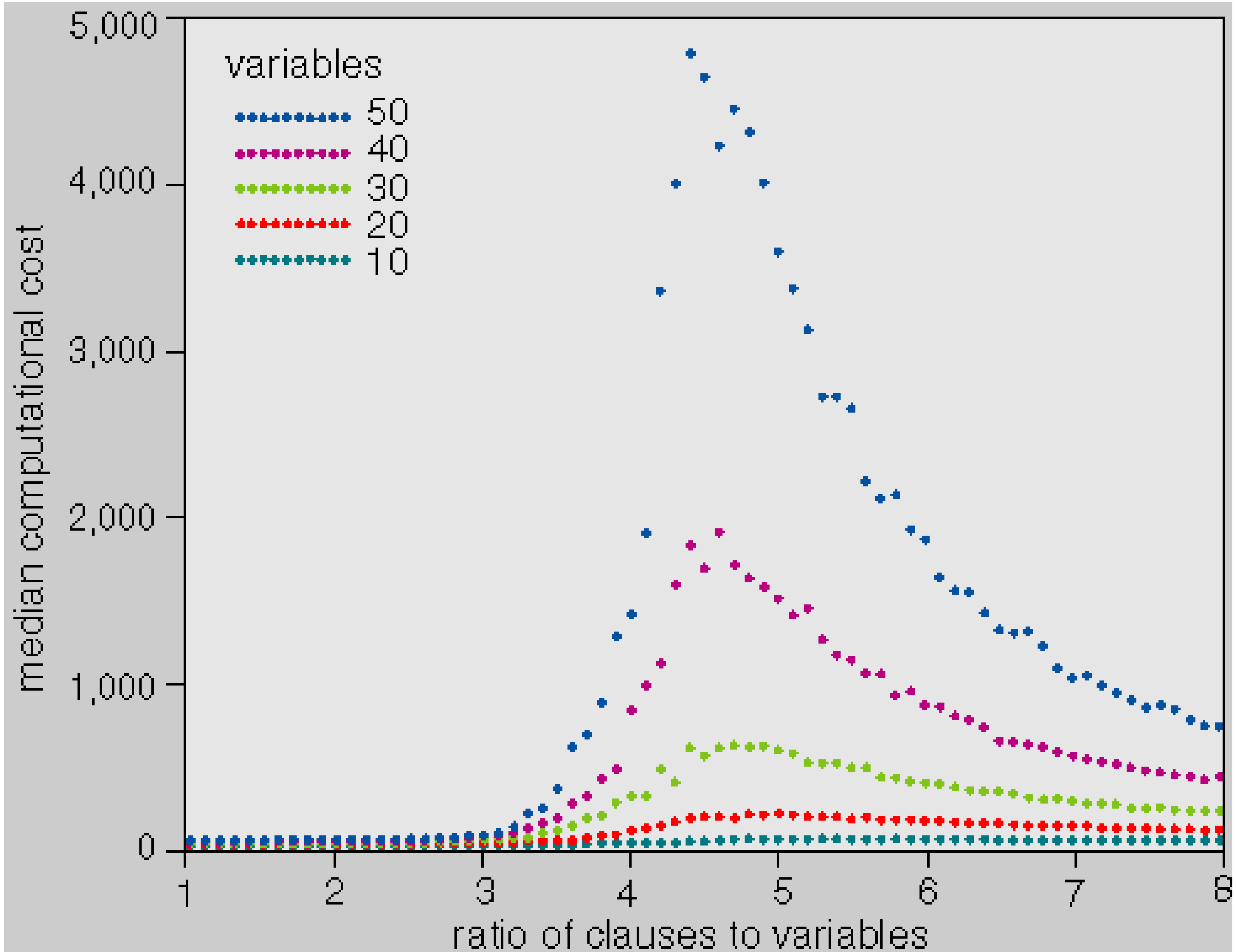
variables
50
40
30
20
10
median computational cost
5,000
4,000
3,000
2,000
1,000
0
1
2
3
4
5
6
7
8
ratio of clauses to variables

Survey up to 1996: Huberman, Hogg and Williams, "Artificial Intelligence" vol. 81

Cheeseman, Kanefsky and Taylor:
"The results reported above suggest the following conjecture: All NP-complete problems have at least one order parameter and the hard to solve problems are around a critical value [..]. This critical value (a phase transition) [...] The converse conjecture is P problems do not contain a phase transition".

"fashionable nonsense".
Imprecise. What is an "order parameter" ?
"Canonical" property: monotonicity. With this interpretation: false (Erdős and Rényi).

Reason: Phase transitions are insensitive to changes on a set of instances "of measure zero". Worst-case complexity is not.

Any connection between computational complexity and phase transitions ? Reason: problems constructed in the above statement are "artificial". Maybe the situation is better for "natural" problems.

Similar: Computational Complexity: satisfiability problems are either in P or NP-complete (Schaefer,1978). Additional reason: all tractable cases from Schaefer's theorem have phase transitions that can be rigorously determined.

My work: classification of thresholds for the (clausal) subset of Schaefer's framework.

Clause template:

$C_{a,b} : \overline{x_1} \vee \ldots \overline{x_a} \vee x_{a+1} \vee \ldots \vee x_{a+b}$.

Examples:

$C_{a,0} = \overline{x_1} \vee \ldots \overline{x_a}$, $C_{0,b} = x_1 \vee \ldots \vee x_b$

S: finite set of clause templates.

SAT(S): allow clauses whose templates are in S.

Random model: m clauses, chosen uniformly at random, with repetition among those available.

Result: classification of satisfiability problems with a sharp/coarse threshold. Exact statement in the paper (15th I.E.E.E. Conference on Computational Complexity).

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Interpretation:

If $SAT(S)$ has a coarse threshold then the following "trivial" procedure works with probability $1 - o(1)$ everywhere outside the "critical region". Even in the critical region its success probability is (at least some) constant !

```
if 0^n or 1^n satisfy Φ
   then Φ is satisfiable
else
   declare Φ unsatisfiable.
```

Contrast: (new result) if $SAT(S)$ has a sharp threshold then All Davis-Putnam algorithms *provably* need exponential time on the average at the critical point.

Conclusion: In the case I study the lack of a phase transition *does* have algorithmic implications !

Caveats:

– the (combined) result is weaker than it might seem.
– the general case is more subtle
– ... however this is the first rigorous result that supports the existence of such connection. A more precise account is an exciting open question.

Milgrom (1957): sent letters between people in Nebraska and Massachusets and computed average number of hops.

Watts and Strogatz (Nature):power grid of Western U.S. is "locally dense" but is sparse, and has small diameter.

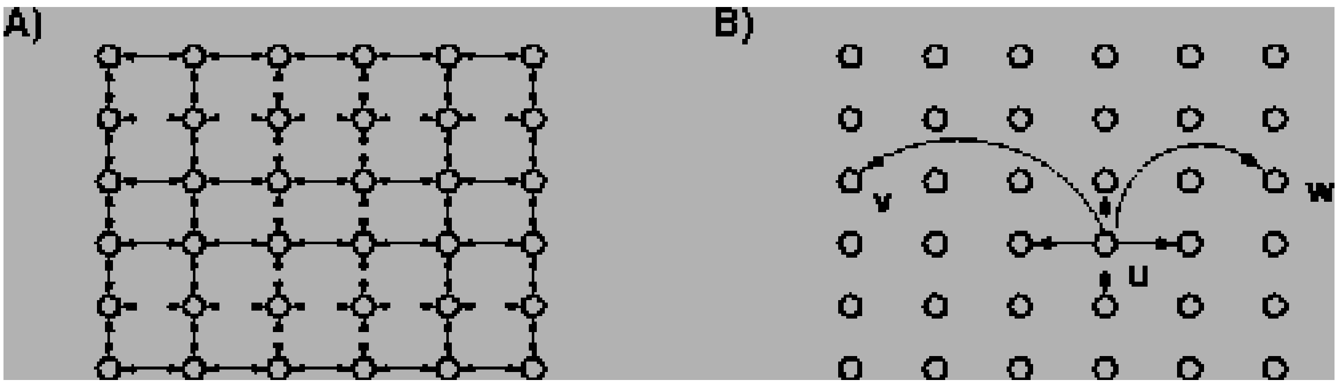
Flurry of work on "random graph models of small-world networks".
Web is "small-world" (Adamic, however see also Broder et. al.)

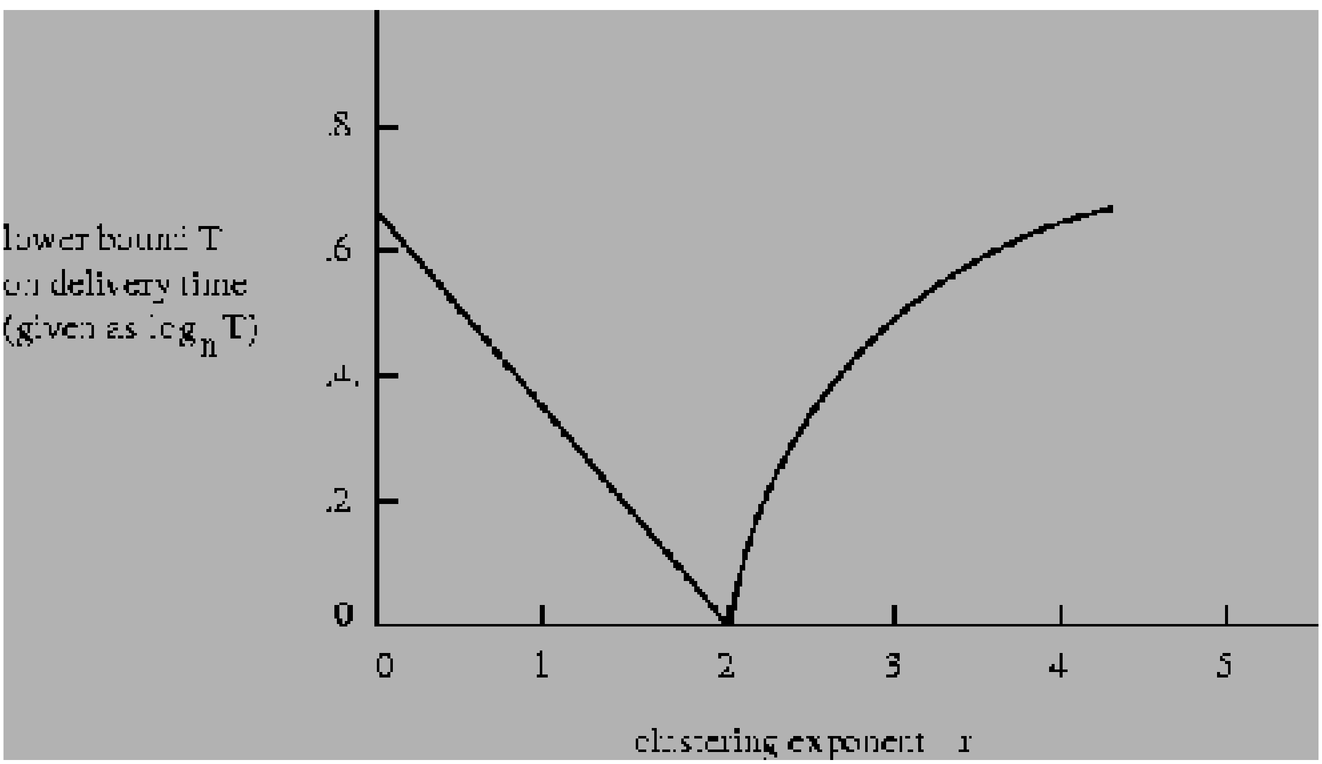
Ingredients: mixture of "ordered" and "random" structure.
What is "ordered" ?: not clear. Sometimes clique, sometimes lattice.

Kleinberg (Nature, 2000): random graph models do *not* capture the *algorithmic* aspect of *efficient online search*.

Model: 2-d lattice with “long range connections”.
long-range connections: one (constantly many).
distribution: power-law, characterized by exponent r.

$$P(u \Rightarrow v) \sim D(u,v)^{-r}.$$


A)
B)
v
w
u


.8
.6
.4
.2
0
lower bound T
on delivery time
(given as $\log_n T$)
0
1
2
3
4
5
clustering exponent r

“Practical” motivation: routing in (ad-hoc) communication networks. Routing algorithms that use geographic information but no routing tables. GPSR (Hong and Karp, MOBICOM 2000), SORSRER (Barrett et al. LANL 2000).

Hot-potato routing: no packet is ever buffered. Everything is passed along.

Model: at each step, at each node independently, a new packet is injected with probability λ/n. Random destination.

Measure average time a packet stays in the network.
Grid: Broder, Frieze, Upfal (STOC 1996).

Result: $r = 2$ again $O(\log^2 n)$. Otherwise $n^{\Omega(1)}$.

Thesis: one needs to look for "structural" parameters that explain the *algorithmic (tractability)* properties of "small-world networks".

Analogy: random walk on graphs. Conductance $\Rightarrow$ rapid mixing.

Second analogy: Quasi-random graphs (Alon, Erdős, Spencer 1992) are *single* graphs whose properties mirror many of the properties of random graphs.

Possible directions: geometric embeddings and tree-like decompositions.

Geometric embedding: initially see a graph as a relational structure. It induces a graph distance d_G.

Embedding: mapping $\Phi : V(G) \to M$, (M, d) metric space such that $d_G(x, y) \le d_M(x, y)$.

Dilation of an embedding:

$$c_\Phi = \max_{x \neq y} \frac{d_M(x, y)}{d_G(x, y)}.$$

Lattice graphs: embedded into (R^2, l_1) with dilation 1. Optimal search (since geometric and graph distance coincide).

Heuristic: small dilation (+ other structural properties) $\Rightarrow$ geometric and graph distances correlate fairly well $\Rightarrow$ efficient search.

Kleinberg's result:

$0 < r < 2$ "small-world" (small diameter). Consequently high dilation.
$2 < r < \infty$ "the lattice takes over". Efficient (in the shortest path length) online search.

Conclusion: $r = 2$ is a phase transition in dilation.

General embedded graphs that support efficient search ?Some results, not yet satisfactory form.

Tree-decompositions:

Motivation: many hard problems are easy for trees. Tree-like decompositions: A.I. and Parameterized Complexity.

Treewidth k:

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Implicit constraints imply that "natural" graphs have small treewidth.

– Control-flow graphs arising from structured programming languages have treewidth ≤ 10 (Thorup).

– Dependency graphs arising from N.L.P. have bounded treewidth under a plausible cognitive model of language understanding (Kórnai and Tuza, 1992).

Why treewidth might be relevant to social networks ?

Model of citation: simple approximation. The references of a new paper are all “related”.

Semantic graph: metric embedding. All cited papers are “close” (clique).
Useful approximation: there is an upper bound on the number of references in a single paper.

Conclusion: the (semantic version of the) citation graph has bounded treewidth !

Other possible work: graph topology and the dynamics of multi-agent systems.
Related work: Peyton-Young (1999), Axtell (2000).

Pavlov rule for Iterated Prisoner's Dilemma:
– Graph G, labels from ± 1.
– choice: random pair.
– x_i and x_j replaced by $x_i x_j$.
– cooperation: unique stable state.

– Emergence of cooperation
– $O(nlogn)$ steps on cycles, exponential on complete graph (Dyer, Goldberg, Greenhill, Istrate, Jerrum 2000).
– polynomial on lattices.

What about small-world networks ? More importantly, why ?