Stochastic Methods in Electrostatics: Applications to Biological and Physical Science

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Outline of the Talk

First-Passage Algorithms

- Walk on Spheres (WOS)
- Greens Function First Passage (GFFP)
- Simulation-Tabulation (S-T)
- Walk on Subdomains (biochemistry)
- Walk on the Boundary

Applications

- Materials Science
- Biochemistry

Last-Passage Algorithms

Conclusions and Future Work

Stochastic Methods for Partial Differential Equations (PDEs)

Examples for Solving Elliptic PDEs (Path Integrals)

- Exterior Laplace problems and electrostatics
- Electrical capacitance
- Charge density

Advantages of Stochastic Algorithms (Curse of Dimensionality)

- Can avoid complex discrete objects
- Can deal with complicated geometries/interfaces
- Can often cope with singular solutions

Brownian Motion and the Diffusion/Laplace Equations

Cauchy problem for the diffusion equation:

$$u_t = \frac{1}{2}\Delta u \tag{1}$$

$$u(x,0) = f(x) \tag{2}$$

in 1-D:

$$u(x,t) = \int_{-\infty}^{\infty} \omega(x-y,t) f(y) dy \tag{3}$$

where

$$\omega(x-y,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} \tag{4}$$

Brownian Motion and the Diffusion/Laplace Equations

$$u(x,t) = \mathbb{E}_x[f(X^x(t))] \tag{5}$$

- $X^x(t)$: a Brownian motion which has $\omega(x-y,t)$ as the transition probability of going from x to y in time t
- $\mathbb{E}_x[.]$: an expectation *w.r.t.* Brownian motion

$$\mathbb{E}_x[f(X^x(t))] = \int_{-\infty}^{\infty} \omega(x-y,t) f(y) dy \tag{6}$$

The First Passage (FP) Probability is the Green's Function

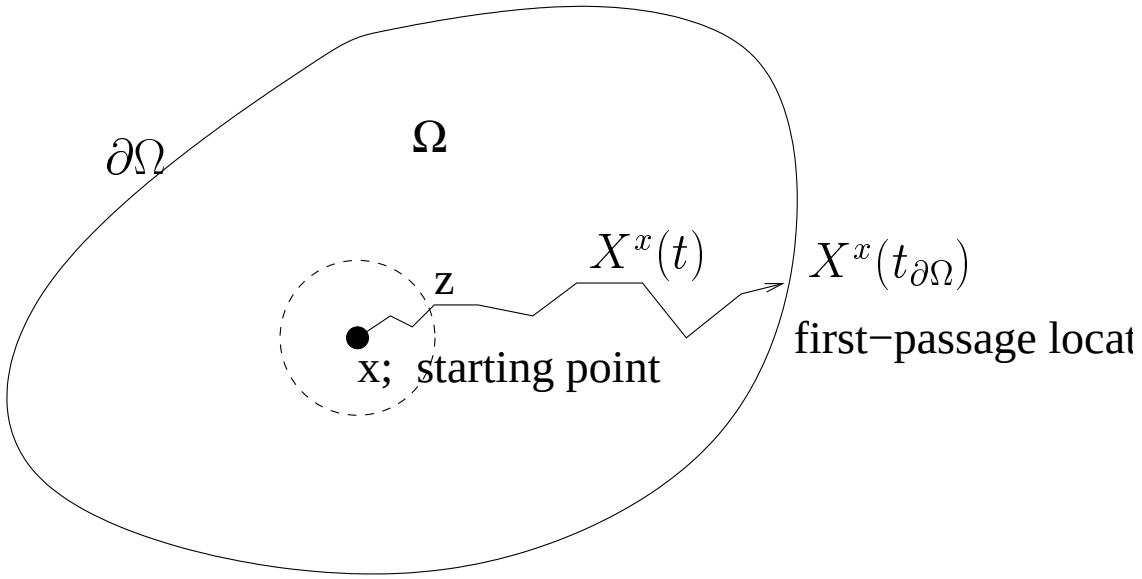
A related elliptic boundary value problem (Dirichlet problem):

$$\begin{aligned} \Delta u(\mathbf{x}) &= 0, \quad \mathbf{x} \in \Omega \\ u(\mathbf{x}) &= f(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega \end{aligned} \tag{7}$$

- Distribution of z is uniform on the sphere
- Mean of the values of $u(z)$ over the sphere is $u(x)$
- $u(x)$ has mean-value property and harmonic
- Also, $u(x)$ satisfies the boundary condition

$$u(\mathbf{x}) = \mathbb{E}_x[f(X^{\mathbf{x}}(t_{\partial\Omega}))] \tag{8}$$

The First Passage (FP) Probability is the Green's Function



The First Passage (FP) Probability is the Green's Function (Cont.)

Reinterpreting as an average of the boundary values

$$u(\mathbf{x}) = \int_{\partial\Omega} p(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) dy \tag{9}$$

Another representation in terms of an integral over the boundary

$$u(\mathbf{x}) = \int_{\partial\Omega} \frac{\partial g(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}} f(\mathbf{y}) dy \tag{10}$$

$g(\mathbf{x}, \mathbf{y})$ – Green's function of the Dirichlet problem in Ω

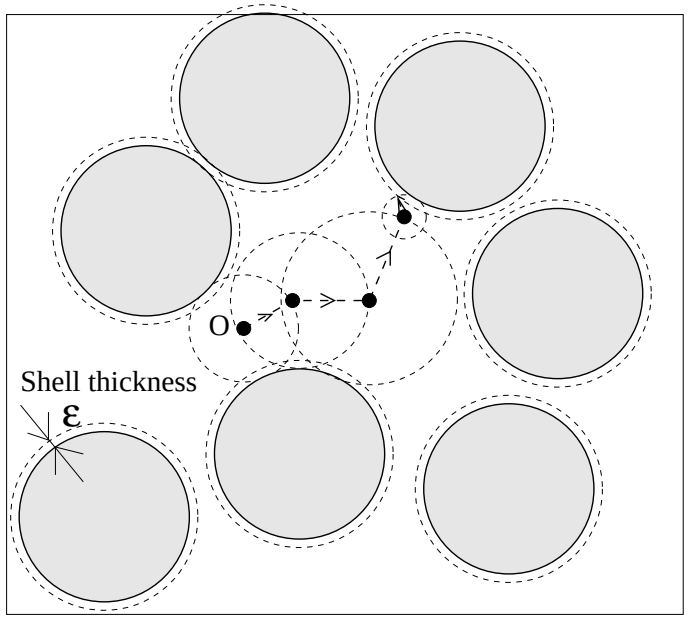
$$\Longrightarrow p(\mathbf{x}, \mathbf{y}) = \frac{\partial g(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}} \tag{11}$$

'Walk on Spheres' (WOS) and Green's Function First Passage (GFFP) Algorithms

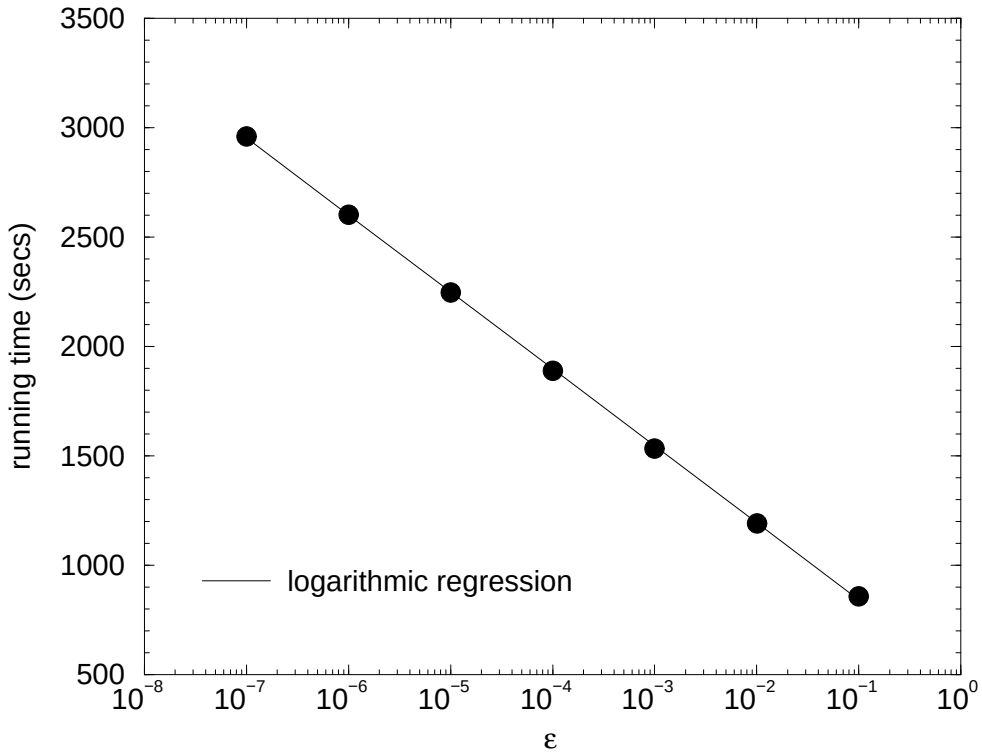
- Green's function is known
  $\Longrightarrow$ direct simulation of exit points and computation of the solution through averaging boundary values
- Green's function is unknown
  $\Longrightarrow$ simulation of exit points from standard subdomains of Ω, e.g. spheres
  $\Longrightarrow$ Markov chain of 'Walk on Spheres' (or GFFP algorithm)
  $\{\mathbf{x_0} = \mathbf{x}, \mathbf{x_1}, \ldots\}$
  $\mathbf{x_i} \to \partial\Omega$ and hits ε-shell is $N = O(\ln(\varepsilon))$ steps
  $\mathbf{x_N}$ simulates exit point from Ω with $O(\varepsilon)$ accuracy

'Walk on Spheres' (WOS) and Green's Function First Passage (GFFP) Algorithms

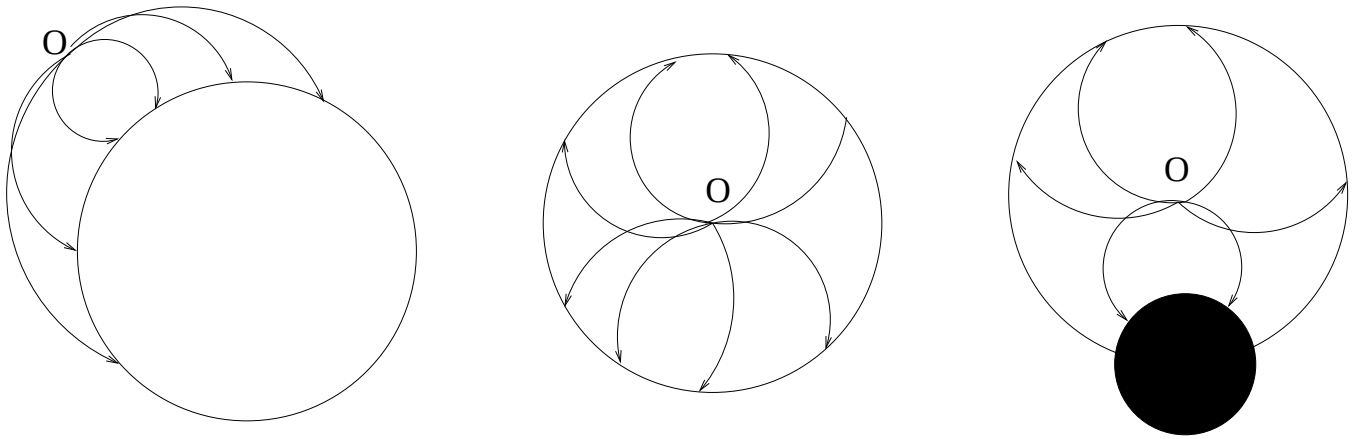
WOS:



Timing of the 'Walk on Spheres' Algorithm

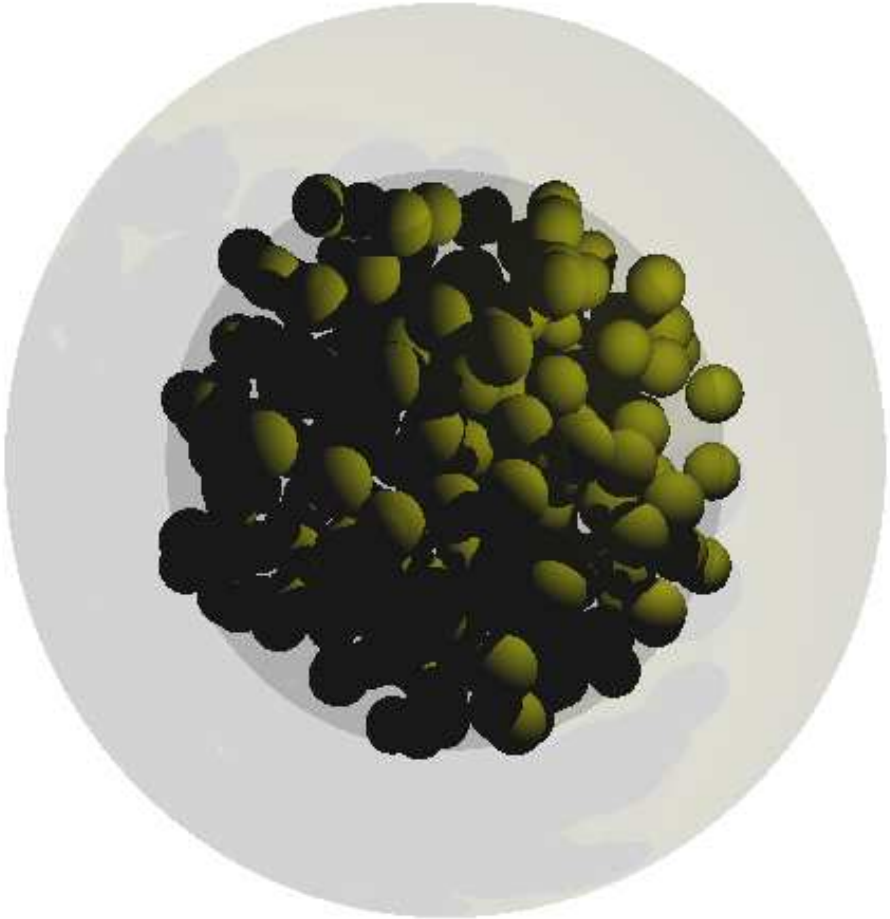


Various Laplacian Green's Functions: GFFP



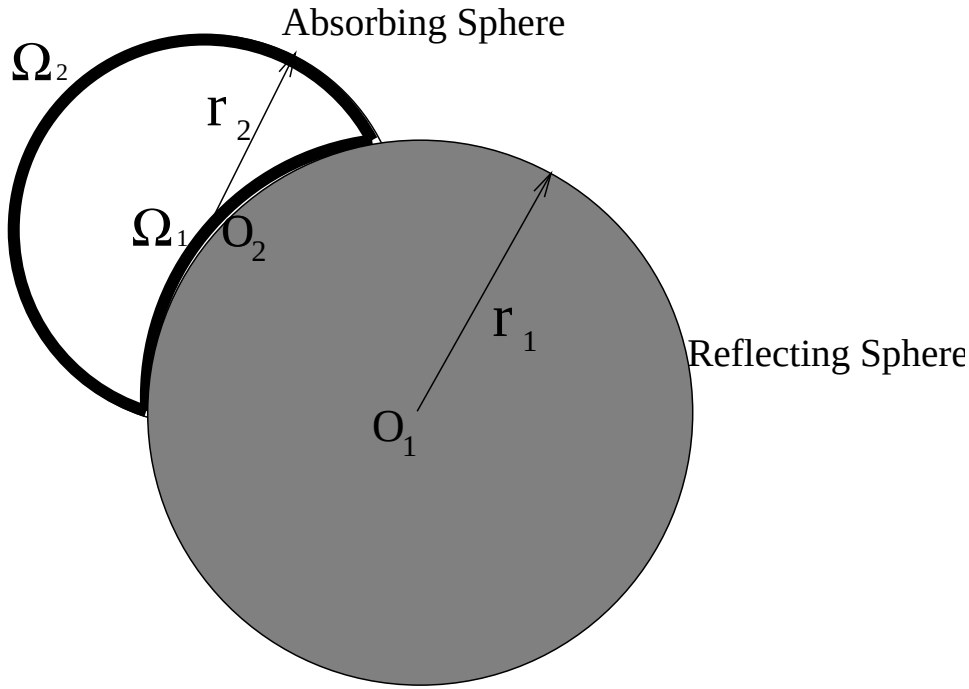
(a) Putting back (b) Void space (c) Intersecting

Geometry for Permeability Computations

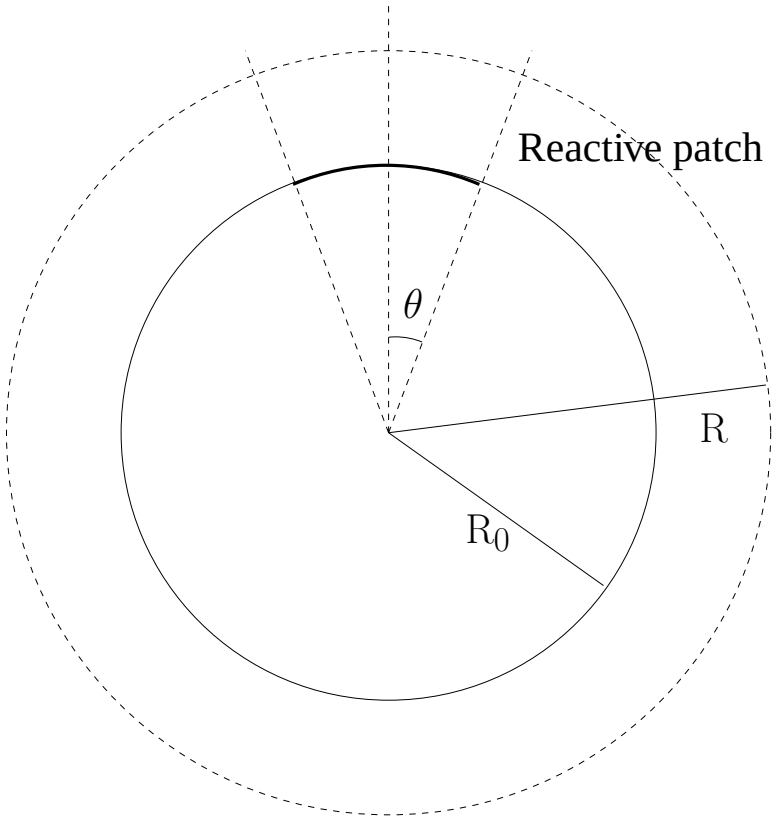


The Simulation-Tabulation (S-T) Method for Generalization

- Green's function for the non-intersected surface of a sphere located on the surface of a reflecting sphere

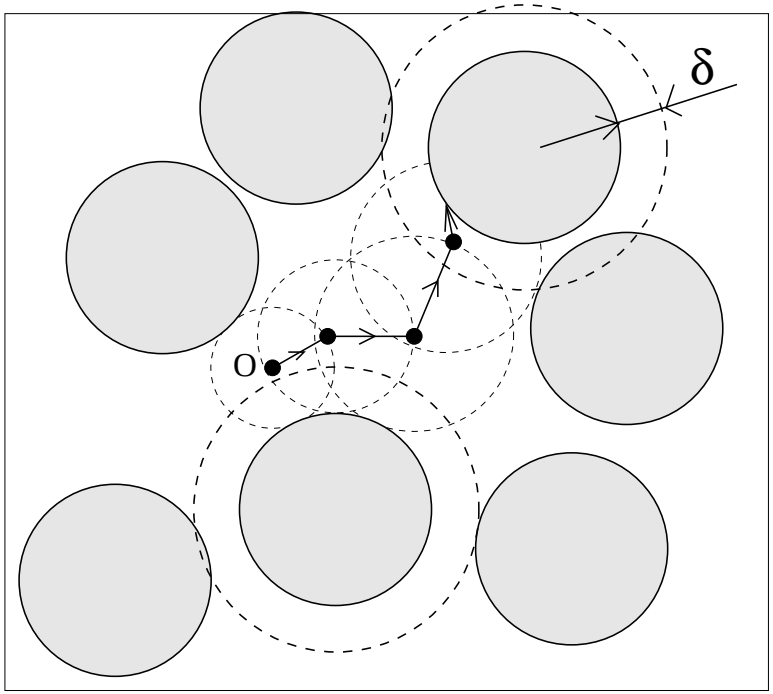


Example: Solc-Stockmayer Model without Potential



Another S-T Application: Mean Trapping Rate

In a domain of nonoverlapping spherical traps :



Biological Electrostatics: Motivation

Electrostatics are Extremely Important in Quantitative Biochemistry:

- Ligand binding
- Protein-protein interactions
- Protein-nucleic acid interactions
- Prediction from primary structure information

In Vivo Electrostatics Must Include the Solvent

- Explicit solvent model: individual water/ions computed, often with Molecular Dynamics
- Implicit solvent models: continuum model of water and dissolved ions used, Poisson-Boltzmann equation

Molecular Electrostatics Problem

Implicit solvent model

- Poisson equation for the electrostatic potential, ϕ:

$$-\nabla\epsilon(x)\nabla\phi(x) = 4\pi\rho(x) \ , \ x \in \mathbb{R}^3$$

  dielectric permittivity, ϵ, and charge density, ρ, are position-dependent

- molecule Ω – a compact cavity in $\mathbb{R}^3$ with low $\epsilon = \epsilon_i$
- surrounded by solvent with larger $\epsilon = \epsilon_e$
- point charges, q_m, at x_m inside molecule
- Boltzmann distribution of mobile ions in solvent

Molecular Electrostatics Problem (Cont.)

Explicit geometric models for solute molecule

- van der Waals surface:
  union of intersecting spheres (atoms): $\Omega = \bigcup_{m=1}^{M} B(x^{(m)}, r^{(m)})$
  point charges – at their centers, $x_m = x^{(m)}$
- contact and reentrant surface, Γ:
  $\partial\Omega$ smoothed by the probe molecule of the solute rolling on it
- ion-accessible surface $\partial\Omega'$:
  $\Omega' = \bigcup_{m=1}^{M} B(x^{(m)}, r^{(m)} + r_{ion})$;
  ion-exclusion layer between $\partial\Omega'$ and Γ

Molecular Electrostatics Problem (Cont.)

Mathematical model (one-surface geometry)

- Poisson equation for the electrostatic potential, ϕ, inside a molecule

$$-\epsilon_i \Delta \phi_i(x) = \sum_{m=1}^{M} 4\pi q_m \delta(x - x_m) \ , \ x \in \Omega$$

- linearized Poisson-Boltzmann equation outside, $x \in \mathbb{R}^3 \setminus \overline{\Omega}$:

$$\Delta \phi_e(x) - \kappa^2 \phi_e(x) = 0 \ ,$$

- Continuity condition on the boundary

$$\phi_i = \phi_e \ , \ \epsilon_i \frac{\partial \phi_i}{\partial n(y)} = \epsilon_e \frac{\partial \phi_e}{\partial n(y)} \ , \ y \in \Gamma \equiv \partial\Omega$$

Electrostatic Potential, Field and Energy (Linear Problem)

- Point values of the potential: $\phi(x) = \phi^{(0)}(x) + g(x)$
  Here, singular part of ϕ:

$$g(x) = \sum_{m=1}^{M} \frac{q_m}{\epsilon_i} \frac{1}{|x - x_m|}$$

- Free electrostatic energy of a molecule = linear combination of point values of the regular part of the electrostatic potential $\phi^{(0)}$:

$$E = \frac{1}{2} \sum_{m=1}^{M} \phi^{(0)}(x_m) q_m \ ,$$

- Point values of the electrostatic field: $\nabla\phi(x)$

Monte Carlo Estimates for Point Potential Values

Two different approaches to constructing Monte Carlo algorithms

1. Probabilistic representation for the solution

2. Classical potential theory

First approach

Laplace equation for the regular part of the potential inside Ω

$$\Delta \phi^{(0)} = 0$$

Probabilistic representation

$$\phi^{(0)}(x) = \mathbb{E}_x[\phi^{(0)}(x^*)]$$

x^* – exit point from Ω of Brownian motion starting at x

'Walk on Spheres' Algorithm

x – center of a sphere $\Rightarrow$ exit points are distributed isotropically. Ball $B(x, R)$ lies entirely in Ω. Strong Markov property of Brownian motion $\Rightarrow$ probabilistic representation holds valid for exit points.

Hence follows 'random walk on spheres' algorithm for general domains with regular boundary:

$$x^k = x^{k-1} + d(x^{k-1}) \times \omega^k \ , \quad k = 1, 2, \dots \ .$$

Here
$d(x^{k-1})$ – distance from x^{k-1} to the boundary
$\{\omega^k\}$ – sequence of independent unit isotropic vectors
x^k is exit point from the ball, $B(x^{k-1}, d(x^{k-1}))$, for Brownian motion starting at x^{k-1}

Green's Function First Passage Simulation

Other domains with known Green's function (G) $\Longleftrightarrow$ one-step simulation of exit points distributed on the boundary in accordance with $\partial G/\partial n$

For general domains:
Efficient way to simulate x^* – combination of 'walk in subdomains' approach and 'walk on spheres' algorithm

The whole domain, Ω, is represented as a union of intersecting subdomains:

$$\Omega = \bigcup_{m=1}^{M} \Omega_m$$

Simulate exit point separately in every Ω_m

Green's Function First Passage Simulation (cont.)

$x^0 = x, x^1, \ldots, x^N$ – Markov chain, every x^{i+1} is exit point from the corresponding subdomain for Brownian motion starting at x^i

For spherical subdomains, $B(x^i_m, R^i_m)$, exit points are distributed in accordance with the Poisson's kernel

$$\frac{|x^i - x^i_m|}{4\pi R^i_m} \frac{|x^i - x^i_m|^2 - R^i_m}{|x^i - y|^3}$$

$x^* = x^N$ is exit point of Brownian motion from Ω

Schwartz lemma $\Rightarrow$ Markov chain $\{x^i\}$ converges to x^* geometrically

'Walk on Spheres' and 'Walk in Subdomains' Algorithms

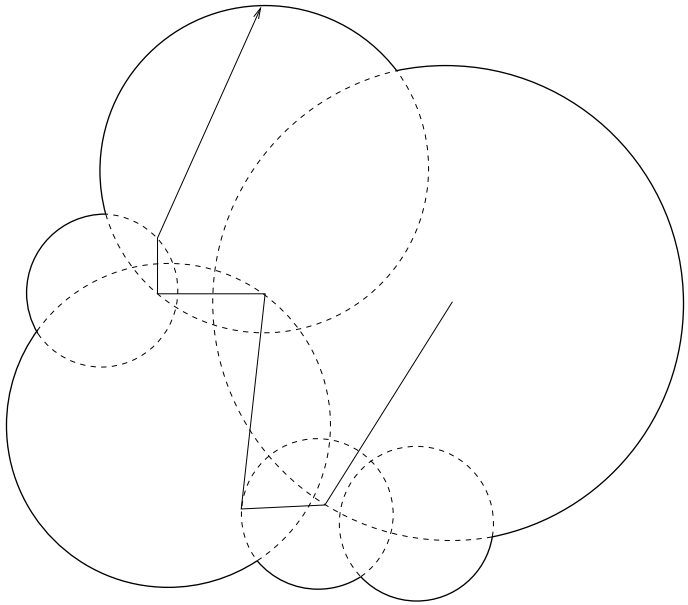


Figure 1: Walk in subdomains example.

Monte Carlo Estimate for Point Potential Value

On every step

$$\phi^{(0)}(x^i) = \mathbb{E}[\phi^{(0)}(x^{i+1})|x^i]$$

Hence

$$\phi^{(0)}(x) = \mathbb{E}\phi^{(0)}(x^*) \equiv \mathbb{E}[\phi(x^*) - g(x^*)]$$

Values of the electrostatic potential on the boundary, $\phi(x^*)$, are not known. We can use their Monte Carlo estimates instead

Monte Carlo Estimate for Boundary Potential Values

- *First approach*

  Discretization and randomization of the boundary condition ($y \in \Gamma$, $n = n(y)$ – normal vector);

$$\phi(y) = p_i \phi(y - hn) + p_e \phi(y + hn) + O(h^2)$$

$$\Longrightarrow$$

$$\phi(x_1^*) = \mathbb{E}(\phi(x_2^0 | x_1^*) + O(h^2)$$

  $x_2^0 = x_1^* - hn$ with probability p_i (reenter molecule)
  $x_2^0 = x_1^* + hn$ with probability $p_e = 1 - p_i$ (exit to solvent)
  $p_i = \dfrac{\epsilon_i}{\epsilon_i + \epsilon_e}$

Monte Carlo Estimate for Boundary Potential Values (cont.)

- *Second approach*

  Exact treatment of boundary conditions (mean-value theorem for boundary point, y, in the ball $B(y,a)$ with surface $S(y,a)$):

$$\begin{aligned}
\phi(y) &= \frac{\epsilon_e}{\epsilon_e+\epsilon_i}\int_{S_e(y,a)}\frac{1}{2\pi a^2}\frac{\kappa a}{\sinh(\kappa a)}\phi_e \\
&+ \frac{\epsilon_i}{\epsilon_e+\epsilon_i}\int_{S_i(y,a)}\frac{1}{2\pi a^2}\frac{\kappa a}{\sinh(\kappa a)}\phi_i \qquad (12)\\
&- \frac{(\epsilon_e-\epsilon_i)}{\epsilon_e+\epsilon_i}\int_{\Gamma\bigcap B(y,a)\setminus\{y\}}\frac{\cos\varphi_{yx}}{2\pi|y-x|^2}Q_{\kappa,a}\phi \\
&+ \frac{\epsilon_i}{\epsilon_e+\epsilon_i}\int_{B_i(y,a)}[-2\kappa^2\Phi_\kappa]\phi_i
\end{aligned}$$

Monte Carlo Estimate for Boundary Potential Values (cont.)

Here

φ_{yx} – angle between the normal $n(y)$ and $y - x$

$$\Phi_\kappa(x-y) = -\frac{1}{4\pi}\frac{\sinh(\kappa(a-|x-y|))}{|x-y|\sinh(\kappa a)}$$

– Green's function for the Poisson-Boltzmann equation in $B(y,a)$

$$Q_{\kappa,a}(|x-y|) = \frac{\sinh(\kappa(a-|x-y|)) + \kappa|x-y|\cosh(\kappa(a-|x-y|))}{\sinh(\kappa a)}$$

Monte Carlo Estimate for Boundary Potential Values (cont.)

Next

Randomization of approximation to (12), $y = x_1^*$, $x = x_2^0$:

$\phi(y) = \mathbb{E}\phi(x) + O(a/2R)^3$

Here

- with probability p_e exit to solvent:
  x is chosen isotropically on the surface of auxiliary sphere, $S_+(y, a)$, that lies above tangent plane; random walk survives with probability $\dfrac{\kappa a}{\sinh(\kappa a)}$

Monte Carlo Estimate for Boundary Potential Values (cont.)

- with probability p_i
  x is chosen isotropically in the solid angle below tangent plane;
  with probability $-2\kappa^2\Phi_\kappa$ it is sampled in $B_i(y,a)$ (reenter molecule);
  with the complementary probability x is sampled on the surface of auxiliary sphere, $S_-(y,a)$, that lies below tangent plane; x reenters molecule with conditional probability $1-a/2R$ and x exits to solvent with conditional probability $a/2R$

Higher order of approximation!

Monte Carlo Algorithm (cont.)

- x_2^0 *inside*

  Return to the boundary at x_2^*, the exit point of Brownian motion (Markov chain) starting at x_2^0, set

$$\phi(x_2^0) = \mathbb{E}(\phi(x_2^*) - g(x_2^*) + g(x_2^0)|x_2^0) \tag{13}$$

  Repeat the randomized treatment of the boundary condition at the point x_2^*

Monte Carlo Algorithm (cont.)

- x_2^0 *outside*

  'Walk on spheres' algorithm

  $x_2^{i+1} = x_2^i + \omega \times d_i, \qquad d_i =$ distance from x_2^i to $\partial\Omega$

  Terminates with probability $1 - \dfrac{\kappa d_i}{\sinh(\kappa d_i)}$ on every step, or when $d_{N_2} < \varepsilon$.

  x_2^* – the nearest to $x_2^{N_2}$ on the boundary

$$\phi(x_2^0) = \mathbb{E}(\phi(x_2^*)|x_2^0) + O(\varepsilon) \tag{14}$$

  Repeat the randomized treatment of the boundary condition at the point x_2^*

Molecular Electrostatics Problem (cont.)

In the exterior probability of terminating Markov chain depends linearly on the initial distance to the boundary, $d_0 \Rightarrow$
Mean number of returns to the boundary is $O(d_0)^{-1}$

- Finite-difference approximation of boundary conditions, $\varepsilon = h^2$
  Mean number of steps in the algorithm is $O(h^{-1}\log(h)\ f(\kappa))$,
  f is a decreasing function ($f(\kappa) = O(\log(\kappa))$ for small κ).
  Estimates for point values of the potential and free energy are $O(h)$-biased
- New treatment of boundary conditions provides $O(\overline{a})^2$-biased and more **efficient** Monte Carlo algorithm. Mean number of steps is $O((\overline{a})^{-1}\log(\overline{a})\ f(\kappa))$, $\overline{a} = a/2R$.

The same simulations give point values of the gradient and free electrostatic energy

Molecular Electrostatics Problem (cont.)

Mathematical model (with ion-exclusion layer)

- Poisson equation inside a molecule
- Laplace equation in the ion-exclusion layer:

$$-\Delta\phi_{lay}(x) = 0$$

- linearized Poisson-Boltzmann equation outside, $x \in \mathbb{R}^3 \setminus \overline{\Omega'}$:
- Continuity condition on the intermediate boundary, $\partial\Omega$:

$$\phi_i = \phi_{lay} \ , \ \epsilon_i \frac{\partial \phi_i}{\partial n(y)} = \epsilon_e \frac{\partial \phi_{lay}}{\partial n(y)}$$

- Continuity condition on the external boundary, $\partial\Omega'$:

$$\phi_{lay} = \phi_e \ , \ \frac{\partial \phi_{lay}}{\partial n(y)} = \frac{\partial \phi_e}{\partial n(y)}$$

Molecular Electrostatics Problem (Cont.)

Mathematical model (nonlinear)

- nonlinear Poisson-Boltzmann equation outside (1-to-1 electrolyte):

$$\Delta\phi_e(x) - \kappa^2 \sinh \phi_e(x) = 0$$

Second order approximation to the non-linear term:

$$\Delta\phi_e(x) - \kappa^2\phi_e(x) = \frac{\kappa^2}{6}\phi_e^3(x)$$

Exit Point Probabilities

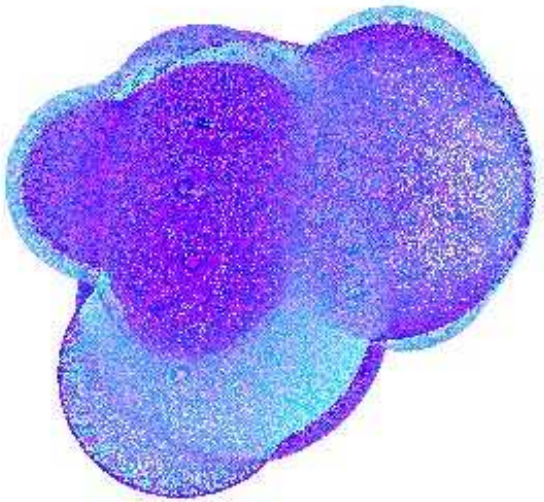


Figure 2: Exit points on the van der Waals surface for the first 12-atom cluster from the Barnase molecule.

Exit Point Probabilities (Cont.)

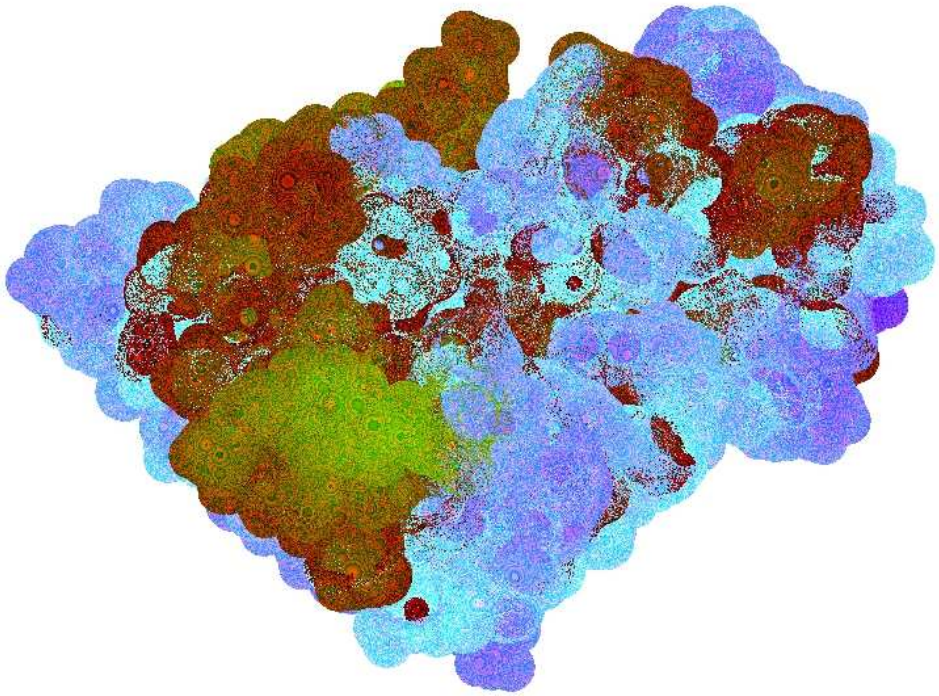


Figure 3: Exit points on the van der Waals surface for the entire Barnase molecule.

Capacitance of a Conductor G

$$C = -\frac{1}{4\pi} \int_\Gamma \frac{\partial u}{\partial n} \, ds \ ,$$

u – solution of the external Dirichlet problem for the Laplace equation

$$\Delta u(x) = 0 \ , \ x \in G_1 = \mathbb{R}^3 \setminus \overline{G} \ ,$$
$$u(y) = 1 \ , \ y \in \Gamma \ ,$$
$$\lim_{|x| \to \infty} u(x) = 0 \ .$$

By Green's formula

$$C = 4\pi R \overline{u}(R) = \mathbb{E}\left(4\pi u(R\omega)\right) = \mathbb{E}\left(4\pi \xi(R\omega)\right) ,$$

ξ – Monte Carlo estimate for u, ω – unit isotropic vector, $S(0, R)$ – sphere containing G.

Monte Carlo Estimates Based on Integral Equations

Consider $u(x)$ to be an integral functional of an integral equation solution:

$$u(x) = \int_Y h_x(y)\mu(y)d\sigma(y)$$

$$\mu(y) = \int_Y k(y,y')\mu(y')d\sigma(y') + f(y) \equiv K\mu(y) + f(y)$$

Example:

In the energy calculation, assume there are no charges outside the 'molecule' G. The non-singular part of the solution can be represented as a single-layer potential

$$u^{(0)}(x) = \int_\Gamma \frac{1}{2\pi}\frac{1}{|x-y|}\mu(y)d\sigma(y)$$

Integral Equations (Cont.)

Potential's density satisfies the integral equation

$$\mu(y) = -\lambda_0 \int_\Gamma \frac{1}{2\pi} \frac{\cos\varphi_{yy'}}{|y-y'|^2} \mu(y') d\sigma(y') + f(y)$$

Here $\lambda_0 = \dfrac{\epsilon_e - \epsilon_i}{\epsilon_e + \epsilon_i}$. The Neumann series

$$\sum_{i=0}^{\infty} (-\lambda_0 K)^i f$$

for this equation converges, but slowly.

Substitution of spectral parameter to speed up the convergence:

$$\mu = \sum_{i=0}^{n} l_i^{(n)} (-\lambda_0 K)^i f + O(q^{n+1})$$

Integral Equations (Cont.)

Monte Carlo Estimate

Markov chain of *random walk on the boundary*:

$p_0(y)$ – initial distribution density

$$p(y_i \to y_{i+1}) = \frac{1}{2\pi} \frac{\cos \varphi_{y_{i+1} y_i}}{|y_{i+1} - y_i|^2}$$

– transition density (uniform in the solid angle)

The estimate (biased, for a convex Γ)

$$u(x) = \mathbb{E}\left[\sum_{i=0}^{n} l_i^{(n)} (-\lambda_0)^i \frac{f(y_0)}{p_0(y_0)} h_x(y_i) \right]$$

Capacitance: Random Walk on the Boundary

Capacitance

$$C = \int_\Gamma \mu(y)\, d\sigma(y)$$

Charge distribution

$$\mu(y) = -\frac{1}{4\pi}\frac{\partial u}{\partial n}(y)$$

is the eigenfunction of the integral operator K:

$$\mu(y) = \int_\Gamma \frac{\cos\varphi_{yy'}}{2\pi|y-y'|^2}\mu(y')d\sigma(y')$$

Capacitance: Random Walk on the Boundary (Cont.)

For a convex G, stationary distribution of isotropic random walk on boundary:

$$\pi_\infty = \frac{1}{C}\,\mu$$

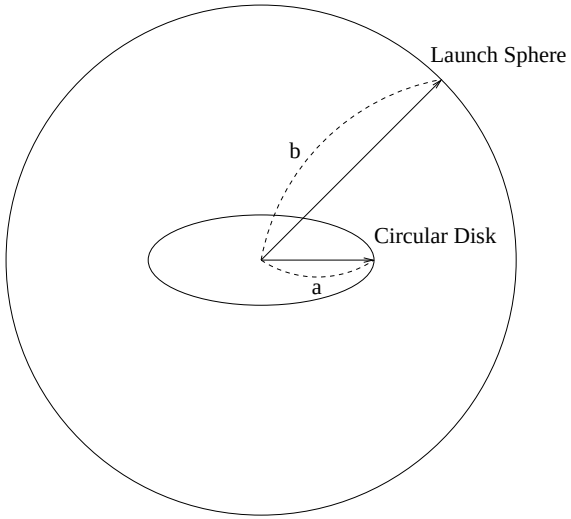
By the ergodic theorem

$$C = \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} v(y_n) \right)^{-1}$$

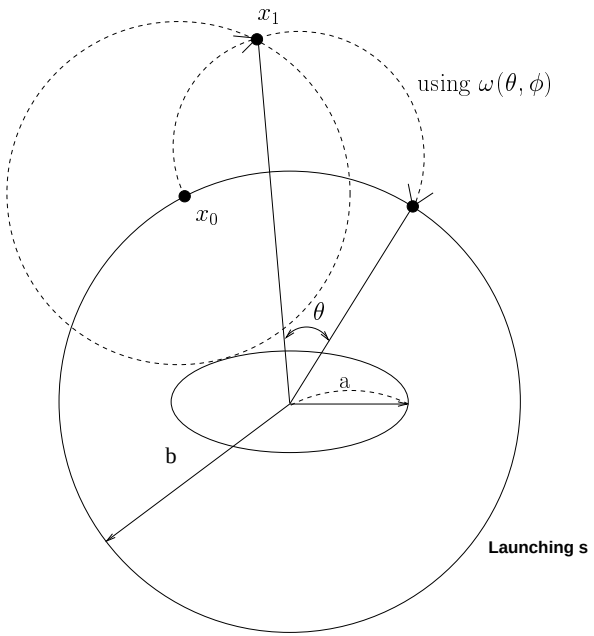
for $v(y) = \dfrac{1}{|x-y|}$ (arbitrary $x \in G$), since inside G the potential

$$\int_\Gamma \frac{1}{|x-y'|}\mu(y')d\sigma(y') = 1\ .$$

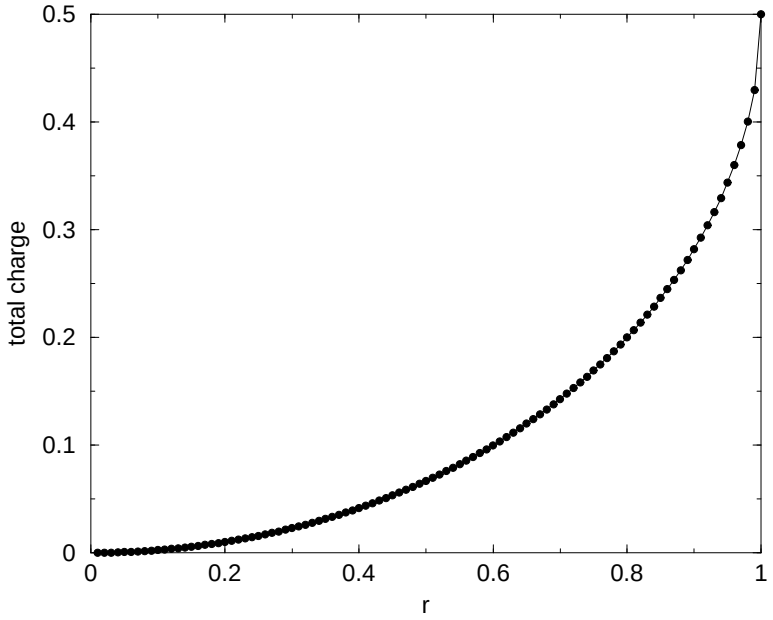
First-Passage Charge Density Calculation



First-Passage Methods

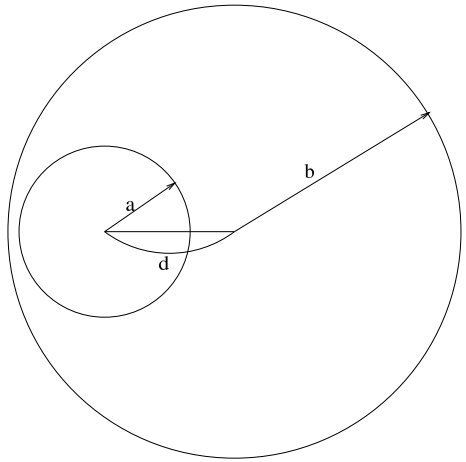


First-Passage Results: Cumulative Charge Distribution

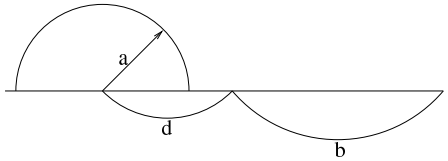


Charge Density on a Circular Disk via Last-Passage

From the top



From the side



Approach from the Outside

- $P(x)$: prob. of diffusing from ϵ above lower FP surface to ∞

$$P(x) = \int_{\partial\Omega_y} g(x,y,\epsilon)p(y,\infty)dS \tag{15}$$

$$\sigma(x) = -\frac{1}{4\pi}\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \phi(x) = \frac{1}{4\pi}\frac{d}{d\epsilon}\bigg|_{\epsilon=0} P(x) \tag{16}$$

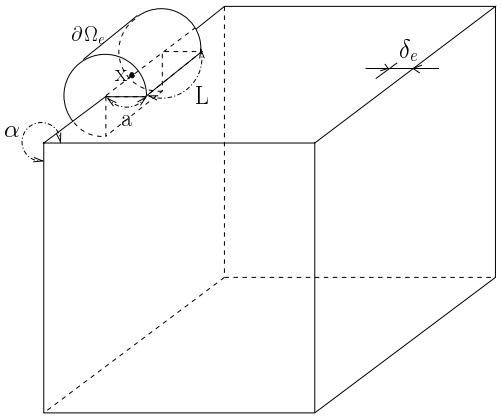
$$\sigma(x) = \frac{1}{4\pi}\int_{\partial\Omega_y} G(x,y)p(y,\infty)dS \tag{17}$$

where

$$G(x,y) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} g(x,y,\epsilon) \tag{18}$$

- $G(x,y)$ satisfies a point dipole problem

Unit Cube Edge Distribution



Unit Cube Edge Distribution (Cont.)

$$\sigma(x, \delta_e) = \delta_e^{\pi/\alpha - 1} \sigma_e(x) \tag{19}$$

- $\sigma(x, \delta_e)$: charge on a curve parallel to the edge separated by δ_e
- $\sigma_e(x)$: edge distribution
- α: angle between the two intersecting surfaces, here $\alpha = 3\pi/2$

$$\sigma_e(x) = \frac{1}{4\pi} \lim_{\delta_e \to 0} \delta_e^{1-\pi/\alpha} \int_{\partial\Omega_e} G(x, y) p(y, \infty) dS \tag{20}$$

- $\partial\Omega_e$: cylindrical surface that intersects the pair of absorbing surfaces meeting at angle α

Unit Cube Edge Distribution (Cont.)

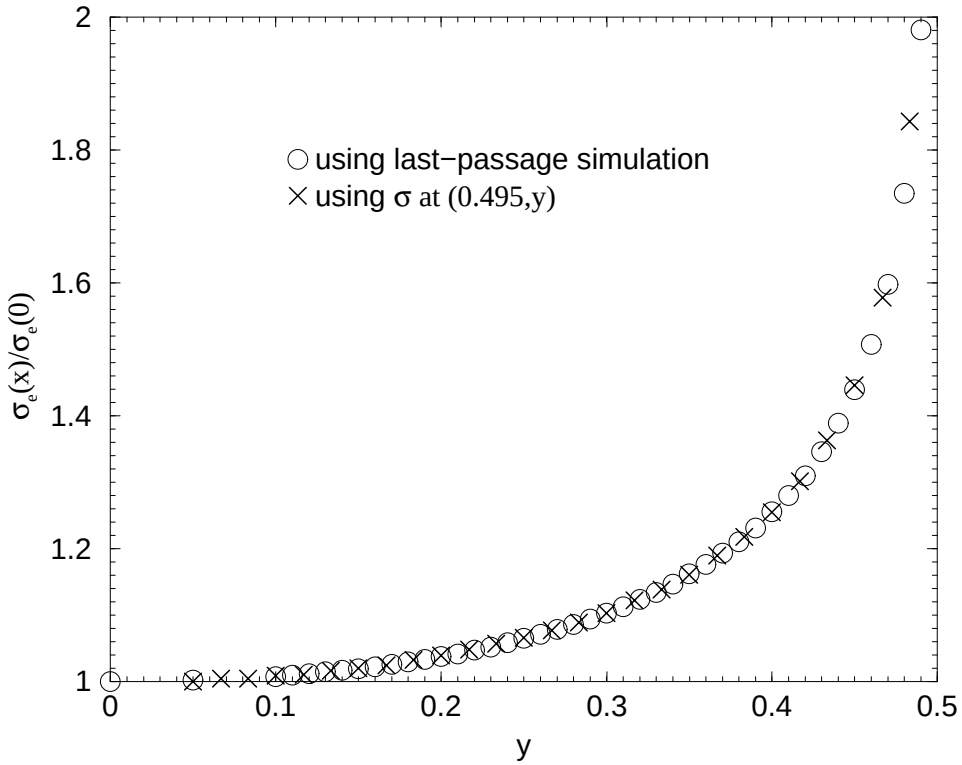


Figure 4: First- and last-passage edge computations

Unit Cube Edge Distribution (Cont.)

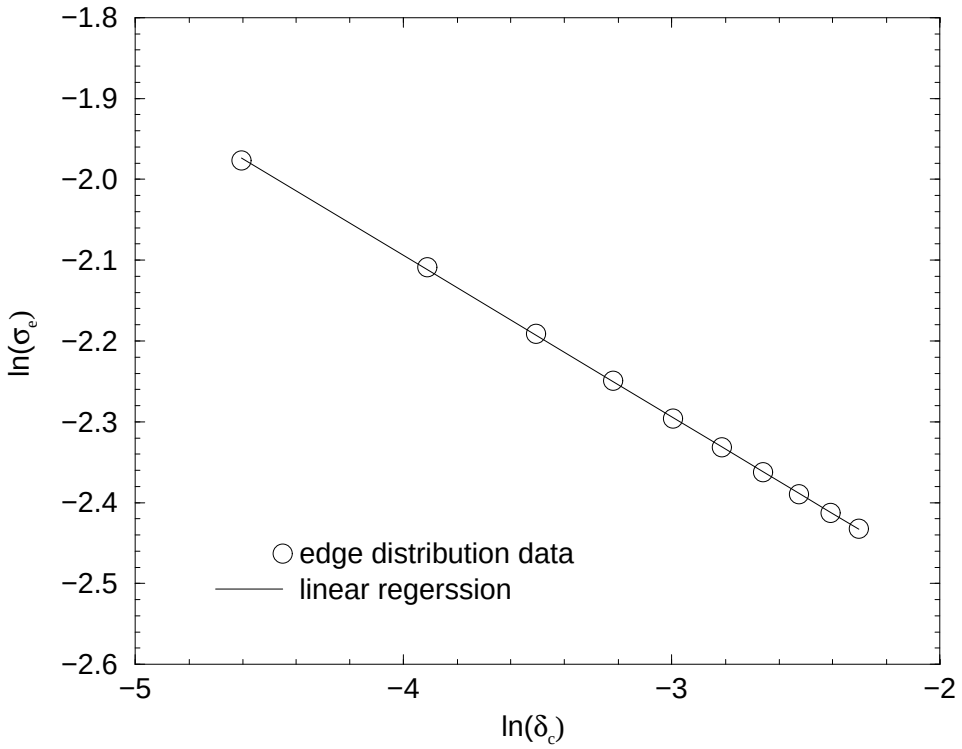


Figure 5: The slope, that is, the exponent of the edge distribution near the corner is approximately -0.20, that is, $\sigma_e \sim \delta_c^{-1/5}$

Conclusions and Future Work

Conclusions

- Stochastic algorithms are very effective in a wide range of partial differential equation and integral equation settings
- Efficiency comes from choosing among the appropriate variant: WOS, GFFP, S-T, 'Walk on the Boundary,' or 'Walk on Subdomains'
- Many applications can be addresses, here the examples are related through electrostatics

Conclusions and Future Work (Cont.)

Future Work

- Molecular Electrostatics
  - More complicated functionals of the solution
  - Derivatives (forces)
  - Nonlinear problem via branching processes and expansions
- Multiscale Monte Carlo

Local Bibliography

- M. Mascagni and N. A. Simonov (2004), "Monte Carlo Methods for Calculating Some Physical Properties of Large Molecules," *SIAM Journal on Scientific Computing*, **26(1)**: 339–357.
- N. A. Simonov and M. Mascagni (2004), "Random Walk Algorithms for Estimating Effective Properties of Digitized Porous Media," *Monte Carlo Methods and Applications*, **10**: 599–608.
- M. Mascagni and N. A. Simonov (2004), "The Random Walk on the Boundary Method for Calculating Capacitance," *Journal of Computational Physics*, **195(2)**: 465–473.
- C.-O. Hwang and M. Mascagni (2003), "Analysis and Comparison of Green's Function First-Passage Algorithms with "Walk on Spheres" Algorithms," *Mathematics and Computers in Simulation*, **63**: 605–613.
- J. A. Given, C.-O. Hwang and M. Mascagni (2002), "First- and last-passage Monte Carlo algorithms for the charge density distribution on a conducting surface," *Physical Review E*, **66, 056704**, 8 pages.
- (2001), "The Simulation-Tabulation Method for Classical Diffusion Monte Carlo," *Journal of Computational Physics*, **174**: 925–946.
- C.-O. Hwang, J. A. Given, and M. Mascagni (2000), "On the Rapid Calculation of Permeability for Porous Media Using Brownian Motion Paths," *Physics of Fluids*, **12**: 1699–1709.