LBL-46896

# Kinetic equation with exact charge conservation

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(September 28, 2000)

We formulate the kinetic master equation describing production of charged particles which are created or destroyed only in pairs due to conservation of their Abelian charge required by an $U(1)$ internal symmetry. Our equation applies to arbitrary particle multiplicities and reproduces the equilibrium results for both canonical (rare particles) and grand canonical (abundant particles) regimes. For canonical systems, the equilibrium multiplicity is much lower and the relaxation time is much shorter than the naive extrapolation from the grand canonical ensemble results. Implications for particle chemical equilibrations in heavy ion collisions are discussed.

## I. INTRODUCTION

Relativistic statistical thermodynamics has been long been used as a tool to describe particle production in heavy ion and in high energy particle collisions [1–3]. Recent analyses have shown that the statistical models can indeed give a satisfactory description of the multiplicities of most hadrons measured in A-A collisions at AGS and SPS energies [4,5]. However, the dynamics of particle equilibration, in particular chemical equilibration, is still not well understood [6–8]. In this letter we shall address the problem of chemical equilibration within the statistical kinetic approach.

Within this approach, particle production is commonly implemented using the grand canonical ensemble, where event-averaged multiplicities are controled by chemical potentials. In this description the net value of a given $U(1)$ charge (e.g., electric, baryon, strangeness, charm, etc.) fluctuates. These fluctuations can be neglected only if the particles carrying the charge in question are abundant. In this case the charge will be conserved on the average. In the opposite limit of rare particle production conservation laws must be implemented locally, event-by-event [1,2,9,10], i.e., canonical ensemble must be used.

The local conservation of quantum numbers in the canonical approach severely reduces the phase space available for particle production [1,2,5,9–12]. Recently it has been shown that the canonical statistical model provides a good description of particle yields measured in low energy heavy ion [13] and high energy hadron-nucleus, hadron-hadron and $e^+e^-$ reactions [5,14,15].

In view of these results it is of particular importance to formulate a kinetic theory for the time evolution of particle production in order to investigate the approach to the canonical chemical equilibrium. Obviously if particles are abundantly produced the equilibrium result for the particle multiplicity should coincide with the grand canonical one. For rare processes, however, the particle production is strongly correlated and the canonical equilibrium result is expected. The kinetic formulation for the production of strongly correlated particles was studied in the literature [16,17]. However, no complete solution has been obtained.

In this letter we consider the time evolution of the multiplicity of particles that carry the charges corresponding to an $U(1)$ internal symmetry. We formulate a kinetic master equation valid for arbitrary particle multiplicity. It reproduces the canonical equilibrium solution for rare particle production, and reduces to the standard grand canonical rate equation for abundant particle production.

## II. RATE EQUATION

In the standard formulation [16,18–20], the rate equation for a binary process $a_1a_2 \to b_1b_2$ with $a \neq b$ is described by the following population equation:

$$\frac{d\langle N_b\rangle}{d\tau} = \frac{G}{V}\langle N_{a_1}\rangle\langle N_{a_2}\rangle - \frac{L}{V}\langle N_{b_1}\rangle\langle N_{b_2}\rangle, \tag{1}$$

where $G \equiv \langle\sigma_G v\rangle$ and $L \equiv \langle\sigma_L v\rangle$ give the momentum-averaged cross sections for the gain process $a_1a_2 \to b_1b_2$ and the loss process $b_1b_2 \to a_1a_2$, respectively. $N_k$ represents the total number of particle $k$, and $V$ is the proper volume. A typical example of such processes is the kaon production/annihilation via $\pi^+\pi^- \leftrightarrow K^+K^-$. The above rate equation, however, cannot be applied to the situation where particle production is rare and is strongly correlated by exact charge conservation.

To account for the correlation between the production/annihilation of particles $b_1$ and $b_2$, let us define $P_{i,j}$ as the probability to find $i$ number of particle $b_1$ *and* $j$

number of particle $b_2$ in an event. We also denote as $P_i$ the probability to find $i$ number of particle $b$ in an event. The average number of particle $b$ per event is then defined as:

$$\langle N_b \rangle = \sum_{i=0}^{\infty} i P_i. \tag{2}$$

We can now write the following general rate equation for the average particle multiplicity:

$$\frac{d\langle N_{b_1}\rangle}{d\tau} = \frac{G}{V}\langle N_{a_1}\rangle\langle N_{a_2}\rangle - \frac{L}{V}\sum_{i,j} ij P_{i,j}. \tag{3}$$

Further, let particles $b_1$ and $b_2$ carry opposite units of a charge corresponding to an $U(1)$ internal symmetry (strangeness in the case of kaons). Then the $U(1)$ charge neutrality of the system gives $N \equiv N_{b_1} = N_{b_2}$. We have then,

$$\begin{aligned} P_{i,j} &= P_i\ \delta_{ij}, \\ \sum_{i,j} ij P_{i,j} &= \sum_i i^2 P_i \equiv \langle N^2\rangle = \langle N\rangle^2 + \langle \delta N^2\rangle, \end{aligned} \tag{4}$$

where $\langle \delta N^2\rangle$ represents the event-by-event fluctuation of the number of $b_1 b_2$ pairs. Note that we always consider $a_1$ and $a_2$ particles abundant (such as, e.g., pions) so that we can neglect the event-by-event fluctuations of their multiplicity and the change of their number due to the considered processes.

Following Eqs.(3-4), the general rate equation for the average number of $b_1 b_2$ pairs can be written as

$$\frac{d\langle N\rangle}{d\tau} = \frac{G}{V}\langle N_{a_1}\rangle\langle N_{a_2}\rangle - \frac{L}{V}\langle N^2\rangle. \tag{5}$$

For abundant production of $b_1 b_2$ pairs where $\langle N\rangle \gg 1$,

$$\langle N^2\rangle \approx \langle N\rangle^2, \tag{6}$$

and Eq.(5) obviously reduces to the standard form, i.e.,

$$\frac{d\langle N\rangle}{d\tau} \approx \frac{G}{V}\langle N_{a_1}\rangle\langle N_{a_2}\rangle - \frac{L}{V}\langle N\rangle^2. \tag{7}$$

However, for rare production of $b_1 b_2$ pairs where $\langle N\rangle \ll 1$, rate equations (1) and (7) are no longer valid. We have instead

$$\langle N^2\rangle \approx \langle N\rangle, \tag{8}$$

which reduces Eq.(5) to the following form:

$$\frac{d\langle N\rangle}{d\tau} \approx \frac{G}{V}\langle N_{a_1}\rangle\langle N_{a_2}\rangle - \frac{L}{V}\langle N\rangle. \tag{9}$$

Thus, in the limit $\langle N\rangle \ll 1$, the absorption term depends on the pair number only linearly, instead of quadratically for the limit $\langle N\rangle \gg 1$.

## III. EQUILIBRIUM MULTIPLICITY AND RELAXATION TIME

To illustrate the differences in the time evolution of particle abundances, we consider the two limiting cases, $\langle N\rangle \gg 1$ and $\langle N\rangle \ll 1$, and present their equilibrium values and relaxation times for the production of $b_1 b_2$ pairs. As an example, we consider a system at fixed temperature and volume and with no initial number of $b_1 b_2$ pairs, i.e., $\langle N\rangle(\tau = 0) = 0$.

In the limit when $\langle N\rangle \gg 1$, the standard Eq.(7) is valid and has the following well-known solution:

$$\langle N\rangle^{\mathrm{GC}}(\tau) = N_{\mathrm{eq}}^{\mathrm{GC}} \tanh\left(\tau/\tau_0^{\mathrm{GC}}\right), \tag{10}$$

where the equilibrium value for the number of $b_1 b_2$ pairs $N_{\mathrm{eq}}^{\mathrm{GC}}$ and the relaxation time constant $\tau_0^{\mathrm{GC}}$ are given by

$$N_{\mathrm{eq}}^{\mathrm{GC}} = \sqrt{\epsilon}, \qquad \tau_0^{\mathrm{GC}} = \frac{V}{L\sqrt{\epsilon}}, \tag{11}$$

respectively, with $\epsilon \equiv G\langle N_{a_1}\rangle\langle N_{a_2}\rangle/L$.

In the particular case where particle momentum distributions are thermal, the gain ($G$) and loss ($L$) terms just represent the thermal averages of the production and absorption cross sections with

$$\frac{G}{L} = \frac{d_{b_1}\alpha_{b_1}^2 K_2(\alpha_{b_1}) d_{b_2}\alpha_{b_2}^2 K_2(\alpha_{b_2})}{d_{a_1}\alpha_{a_1}^2 K_2(\alpha_{a_1}) d_{a_2}\alpha_{a_2}^2 K_2(\alpha_{a_2})}, \tag{12}$$

where $d_k$'s denote the degeneracy factors, and $\alpha_k \equiv m_k/T$. The equilibrium value for the number of $b_1 b_2$ pairs in Eq.(11) now reads as

$$N_{\mathrm{eq}}^{\mathrm{GC}} = \frac{d_{b_1}}{2\pi^2} V T^3 \alpha_{b_1}^2 K_2(\alpha_{b_1}), \tag{13}$$

Thus it is described by the Grand Canonical (GC) result with vanishing chemical potential due to our requirement of the (average) $U(1)$ charge neutrality of the system.

In the opposite limit where $\langle N\rangle \ll 1$, the time evolution is described by Eq.(9), which has the following solution:

$$\langle N\rangle^{\mathrm{C}}(\tau) = N_{\mathrm{eq}}^{\mathrm{C}}\left(1 - e^{-\tau/\tau_0^{\mathrm{C}}}\right), \tag{14}$$

with the equilibrium value and relaxation time given by

$$N_{\mathrm{eq}}^{\mathrm{C}} = \epsilon, \qquad \tau_0^{\mathrm{C}} = \frac{V}{L}. \tag{15}$$

With thermal momentum distribution the equilibrium value of $b_1 b_2$ pair multiplicity has the following form:

$$N_{\mathrm{eq}}^{\mathrm{C}} = \left[\frac{d_{b_1}}{2\pi^2} V T^3 \alpha_{b_1}^2 K_2(\alpha_{b_1})\right] \cdot \left[\frac{d_{b_2}}{2\pi^2} V T^3 \alpha_{b_2}^2 K_2(\alpha_{b_2})\right]. \tag{16}$$

The above equation demonstrates the locality of the $U(1)$ charge conservation. With each particle $b_1$, a particle $b_2$ with the opposite charge is produced in the same event in order to conserve charge locally. This is the result expected from the Canonical (C) formulation of conservation laws [10,11].

We note that Eq.(16) is just the leading term in the expansion of the canonical result of multiplicity for particles carrying $U(1)$ charges. The general expression is known to have the following form [10,11]:

$$N_{\rm eq}^{\rm C} = N_{\rm eq}^{\rm GC} \frac{I_1(2N_{\rm eq}^{\rm GC})}{I_0(2N_{\rm eq}^{\rm GC})}, \tag{17}$$

where $N_{\rm eq}^{\rm GC}$ is given by Eq.(13) and $I_i$'s are the modified Bessel functions.

Comparing Eq.(11) and Eq.(15), we first find that, for $\langle N\rangle \ll 1$, the equilibrium multiplicity in the canonical formulation is much lower than what is expected from the grand canonical result as

$$N_{\rm eq}^{\rm C} = N_{\rm eq}^{\rm GC^2} \ll N_{\rm eq}^{\rm GC}. \tag{18}$$

This shows the importance of the canonical description of charge conservation when the multiplicity of charged particles is small. We also note that the volume dependence in the two cases is different. The particle density in the GC (abundant) limit is independent of $V$ whereas in the opposite canonical (rare) limit the density scales linearly with $V$.

Secondly, the relaxation time for a canonical system is far shorter than what is expected from the grand canonical result as

$$\tau_0^{\rm C} = \tau_0^{\rm GC} N_{\rm eq}^{\rm GC} \ll \tau_0^{\rm GC}. \tag{19}$$

We note from Eq.(5) that these two limits are essentially determined by the size of $\langle \delta N^2\rangle$, the event-by-event fluctuation of the number of $b_1 b_2$ pairs. The grand canonical results correspond to small fluctuations, i.e., $\langle \delta N^2\rangle/\langle N\rangle^2 << 1$; while the canonical description is necessary in the opposite limit.

## IV. MASTER EQUATION

In this section, we formulate the general evolution equation which is valid for arbitrary value of $\langle N\rangle$. It is a master equation for $P_n(\tau)$, the probability of finding $n$ pairs of $b_1 b_2$ at time $\tau$. This probability increases with time due to transitions from $n-1$ and $n+1$ states to the $n$ state, while it also decreases due to transitions from the $n$ state to $n-1$ and $n+1$ states. The transition probability $n \to n+1$ per unit time due to pair creation is $G\langle N_{a_1}\rangle\langle N_{a_2}\rangle/V$ and the transition probability $n \to n-1$ due to pair annihilation is $n^2 L/V$. Therefore, the master equation set has the form:

$$\begin{aligned}\frac{dP_n}{d\tau} &= \frac{G}{V}\langle N_{a_1}\rangle\langle N_{a_2}\rangle\,[P_{n-1} - P_n] \\ &- \frac{L}{V}\,\left[n^2 P_n - (n+1)^2 P_{n+1}\right],\end{aligned} \tag{20}$$

where $n = 0, 1, 2, 3, \ldots$. Multiplying the above equation by $n$ and summing over $n$, one recovers Eq.(5), the general rate equation for the time evolution of the average number of $b_1 b_2$ pairs. However, the master equation (20) contains much more information than the rate equation (5). It contains enough information to solve for the evolution of $\langle N\rangle$ (and all moments of $N$) for arbitrary $\langle N\rangle$. For example, one can also obtain the time evolution of particle fluctuations, $\langle N^2\rangle$, which are of physical importance for rarely produced particles.

We can convert the iterative equations (20) for $P_n$'s into a partial differential equation for the generating function

$$g(x,\tau) = \sum_{n=0}^{\infty} x^n P_n(\tau). \tag{21}$$

Multiplying Eq.(20) by $x^n$ and summing over $n$, we find

$$\frac{\partial g(x,\tau)}{\partial \tau} = \frac{L}{V}(1-x)\left(x g'' + g' - \epsilon g\right), \tag{22}$$

where $g' \equiv \partial g/\partial x$. Note that the $g(x=1,\tau)$ does not change with time, which is equivalent to the conservation of total probability evident in Eq.(20).

The equilibrium solution, $g_{\rm eq}(x)$, thus, obeys the following equation:

$$x g_{\rm eq}'' + g_{\rm eq}' - \epsilon g_{\rm eq} = 0. \tag{23}$$

By variable substitution ($x = y^2/(4\epsilon)$) this equation can be reduced to the Bessel equation. The solution regular at $x=0$ (since $g(0) = P_0 \le 1$) is given by

$$g_{\rm eq}(x) = \frac{1}{I_0(2\sqrt{\epsilon})} I_0(2\sqrt{\epsilon x}), \tag{24}$$

where the normalization is fixed by $g(1) = \sum P_n = 1$.

The equilibrium probability distribution $P_n$ can now be found from Eqs.(21,24) as

$$P_{n,{\rm eq}} = \frac{\epsilon^n}{I_0(2\sqrt{\epsilon})\ (n!)^2}, \tag{25}$$

and the value of the average number of $b_1 b_2$ pairs per event at equilibrium is given by

$$\langle N\rangle_{\rm eq} = g'(1) = \sqrt{\epsilon}\,\frac{I_1(2\sqrt{\epsilon})}{I_0(2\sqrt{\epsilon})}, \tag{26}$$

which obviously coincides with the expected result for particle multiplicity in the canonical ensemble given by Eq.(17) and also reduces to the results for the two limiting cases in Sec.III.

## V. CONCLUSIONS

We have formulated the kinetic master equation for strongly correlated production of particles, where the correlation is due to the local charge conservation required by an $U(1)$ internal symmetry. Our general rate equation is valid for arbitrary value of $\langle N\rangle$, thus it reduces to the grand canonical results for large $\langle N\rangle$ and to the canonical results for small $\langle N\rangle$. Therefore, our equation provides a generalization of the standard rate equation beyond the grand canonical limit. We have shown that, for rare particle production, the equilibrium multiplicity is much smaller and the relaxation time is much shorter than those expected from the standard rate equation. For abundant particle production, where the standard rate equation applies to first order, one can use the general rate equations to study finite-size corrections to the grand canonical results.

Our results could be of importance in the description and understanding of equilibration phenomena and equilibrium properties of partonic or hadronic matter created in heavy ion collisions. For example, it could provide insights into the equilibration time of strange particles produced at SIS, or open charm and charmonium productions at SPS and higher energies. It may also be meaningful for transport model studies of rare particle production, especially the perturbative procedure for rare processes [21], where the local charge conservation requires that the production probability should be assigned to the pair instead to each particle separately.

## ACKNOWLEDGMENT

We acknowledge stimulating discussions with M. Gyulassy, S. Jeon and L.D. McLerran. K.R. also acknowledges comments from C. Greiner, J. Knoll, and H. Feldmeier. M.S. and Z.L. are grateful to the nuclear theory group at LBNL for supporting the summer program where this work was started. This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of Nuclear Physics, the Office of Basic Energy Science, Division of Nuclear Science, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098, the Committee for Scientific Research (KBN-2P03B 03018), the National Science Foundation under Grant No. PHY-9870038, the Welch Foundation under Grant No. A-1358, and the Texas Advanced Research Program under Grant No. FY99-010366-0081.


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