Chiral phase transition in large N QCD with overlap fermions

in collaboration with Herbert Neuberger

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Domain Wall Fermions at Ten years

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Domain Wall Fermions and Overlap Formalism 1992-1998

- Chiral fermions on the lattice using a mass defect: ***D. Kaplan: hep-lat/9206013***
- Anomalous currents flowing into an extra dimension: ***C.G. Callan and J.A. Harvey, Nucl. Phys. B250 (1985) 427.***
- Infinite number of Pauli-Villars regulators for the standard model: ***S.A. Frolov and A.A. Slavnov, May 1992 preprint, Phys. Lett. B309 (1993) 344.***
- Infinite number of Wilson fermions can be used to realize one chiral fermion: ***R. Narayanan and H. Neuberger, hep-lat/9212019***
- Integrating out infinite number of fermions – Overlap formalism: ***R. Narayanan and H. Neuberger, hep-lat/9307006***

Domain Wall Fermions and Overlap Formalism 1992-1998

- Schwinger model – Test of the overlap formalism: ***R. Narayanan, H. Neuberger and P. Vranas, hep-lat/9503013***

- Can the results of the Schwinger model be reproduced using domain wall fermions (finite number of Wilson fermions)?: ***P. Vranas, hep-lat/9608078; hep-lat/9705023***

- Domain wall fermions with ten wilson fermions (length of the extra direction) are enough to get the expected behavior in the chiral limit: ***T. Blum and A. Soni: hep-lat/9611030***

- There is an explicit fermion operator for vector-like theories in the overlap formalism: ***H. Neuberger: hep-lat/9707022***

- Spectrum of the hermitian Wilson-Dirac operator: ***R. Edwards, U. Heller and R. Narayanan, hep-lat/9802016***

Can the number of Wilson fermions be kept finite in practice?

- If we view the domain wall fermion operator as a $d+1$ dimensional operator, the physical fermions are bound to the two walls and a free field analysis shows that there is only an exponential overlap between the two chiral modes.

- ***Preliminary investigation of Schwinger model: P. Vranas, hep-lat/9608078***

  - *Because the overlap of the left and right components is exponentially small in the free case, one would expect that very small explicit breaking of chiral symmetry can be obtained for a relatively small L_s. If this is the case, then the method will be very valuable in studies involving spontaneous chiral symmetry breaking.*

  - *.... This indicates that DWF correctly reproduce topological effects already at $L_s = 10 - 12$. Furthermore, the overlap result in the zero topological sector .... is also reached at $L_s = 10 - 12$.*

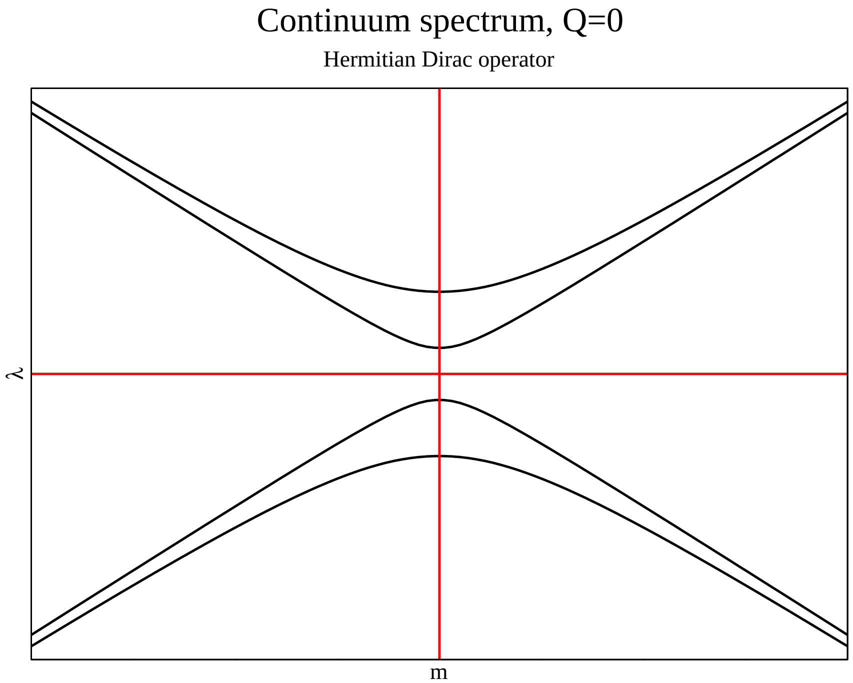
Can the number of Wilson fermions be kept finite in practice?

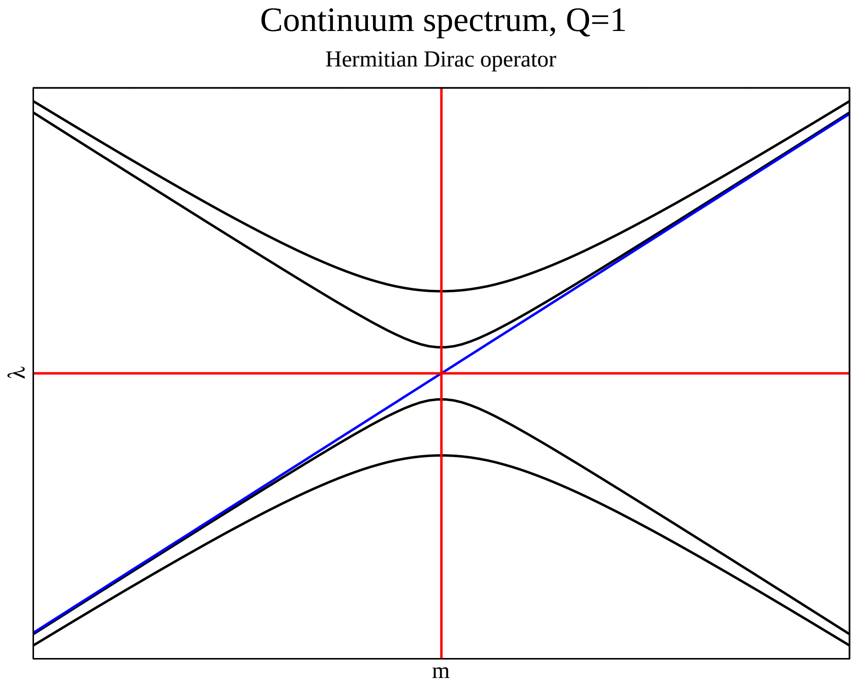
- ***QCD with domain wall fermions: T. Blum and A. Soni, hep-lat/9611030***
  - *In particular, we show that the pion mass and the four quark matrix element related to $K_0 - \bar{K}_0$ mixing have the expected behavior in the chiral limit, even on lattices with modest extent in the extra dimension, $N_s = 10$.*
  - *Finally, we re-emphasize that domain wall quarks at $N_s = 10$ retain the full chiral symmetry of continuum QCD to a remarkable degree, unlike either Wilson or Kogut-Susskind quarks.*

Can the number of Wilson fermions be kept finite in practice?

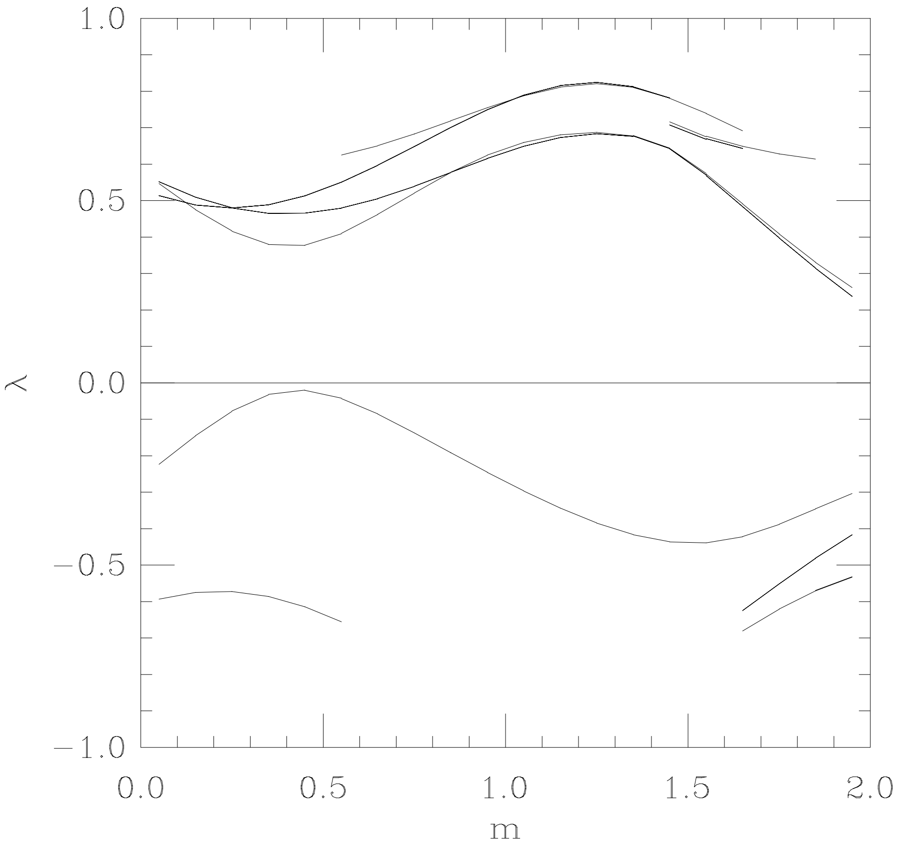
- ***Detailed investigation of the Schwinger model: P. Vranas, hep-lat/9705023***
  - *...... As can be seen from ..... the functional form of the fast decay is exponential. The slower decay is consistent with exponential but it turns out that it is also consistent with power law or exponential times power law behavior.*
  - *Using a simple model it was found that the first fast decay can be associated with restoration of chiral symmetry in the zero topological sector while the second slower decay can be associated with the regions of gauge field configuration space that connect the $q = 0$ and $q = \pm 1$ topological sectors.*
- Wilson fermions probe gauge field topology properly – **Good**
- Wilson fermions will feel topology changing configrations – **Potential problem**

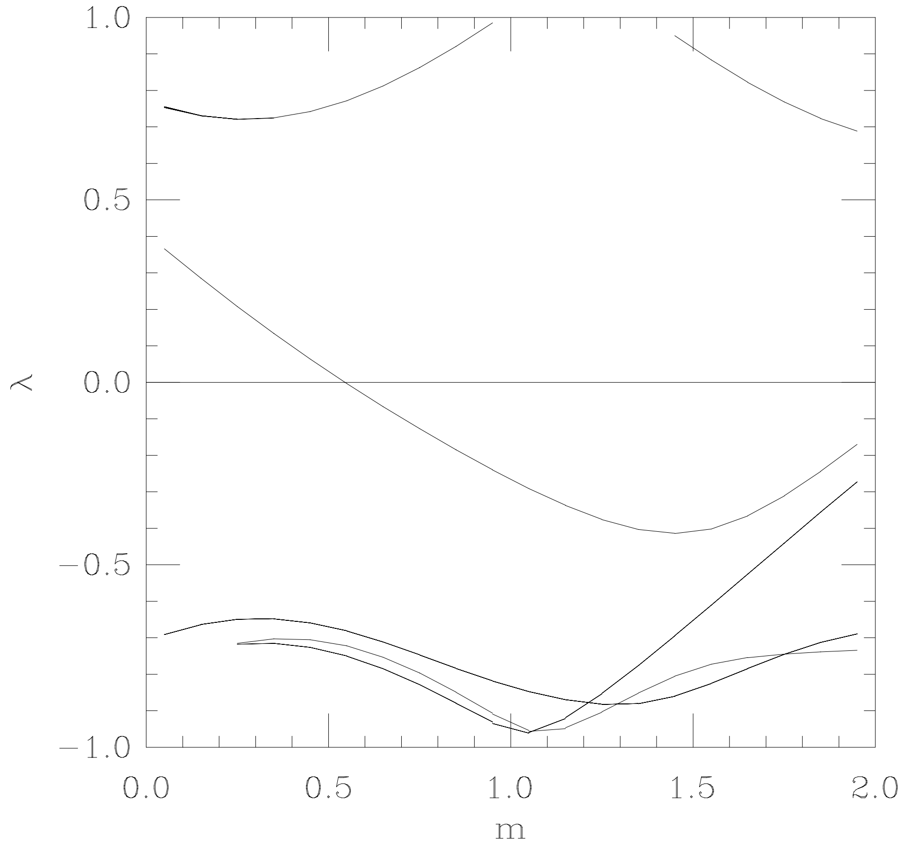
What the continuum fermion does:





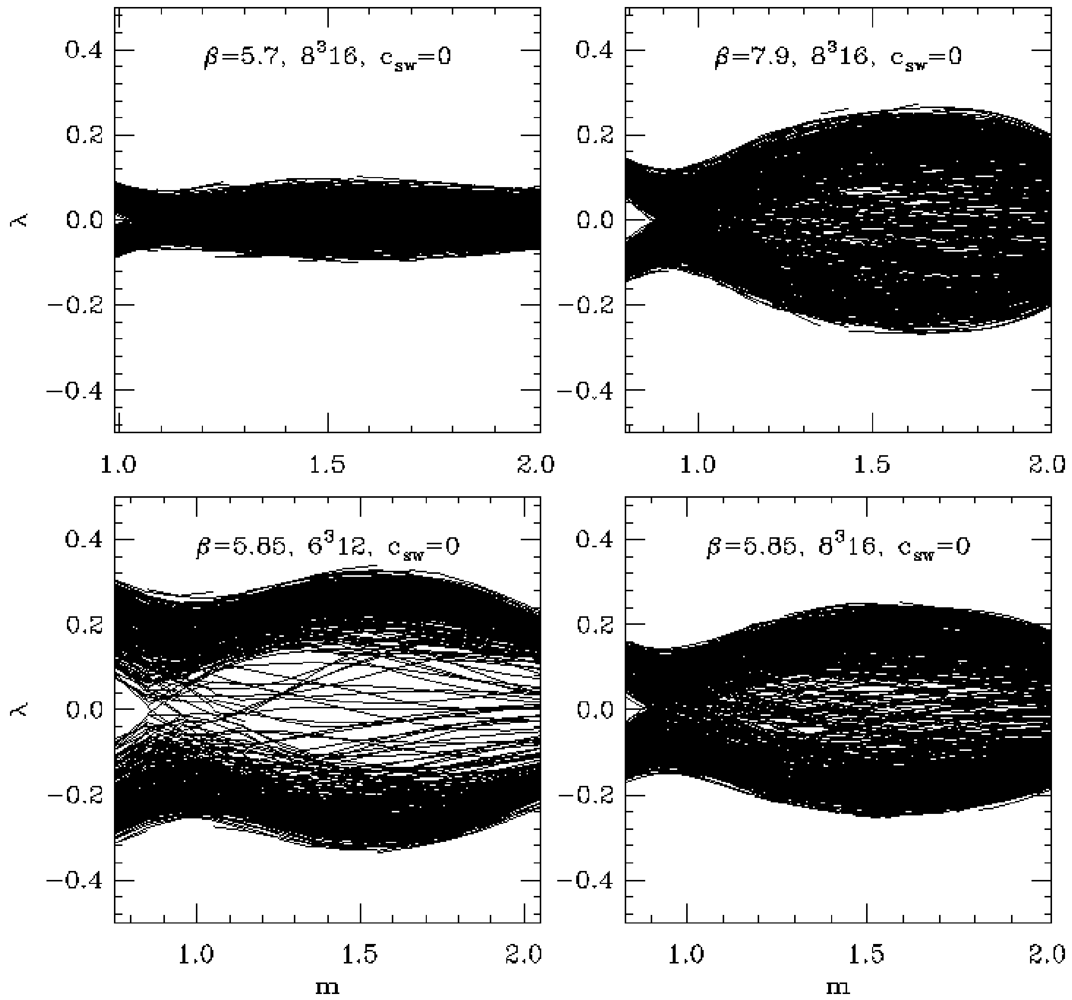
What Wilson fermion does in smooth gauge field backgrounds





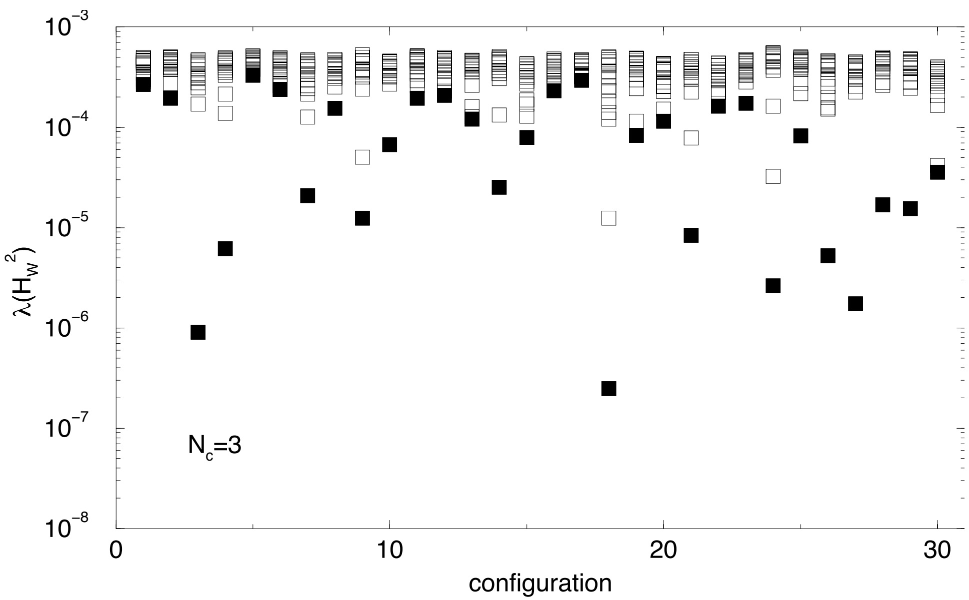
R. Narayanan, U. Heller and R. Edwards, hep-lat/9801015

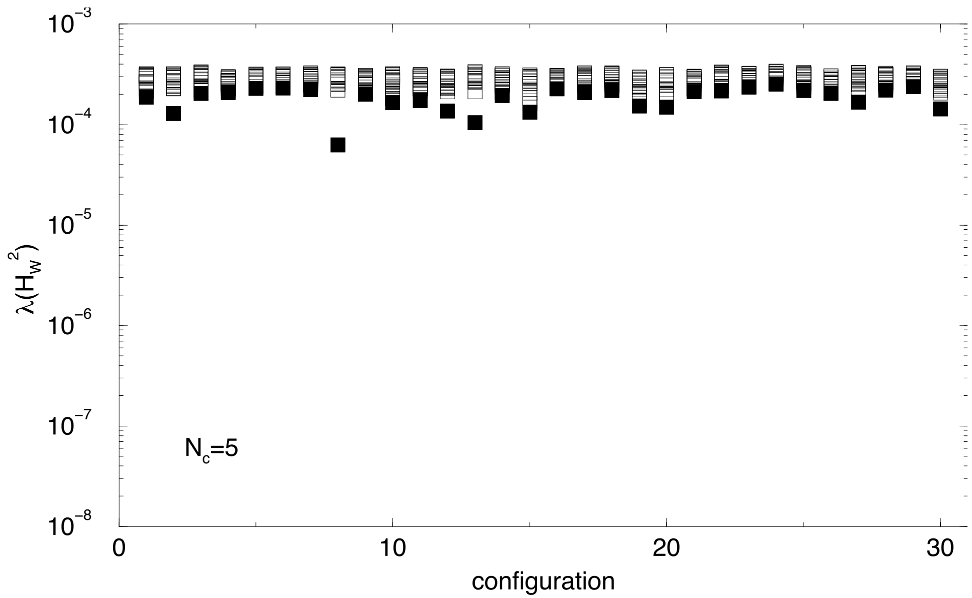
What Wilson fermion does in real lattice gauge field backgrounds



R. Narayanan, U. Heller and R. Edwards, hep-lat/9802016

What happens as one goes to larger N





N. Cundy, M. Teper and U. Wenger, hep-lat/0309011

A lattice phase transition at large N

- Let $P \in SU(N)$ denote the parallel transporter around a single plaquette.
- The eigenvalues $e^{i\theta_k}$, $k = 1, \cdots, N$ of P are gauge invariant and independent of the point where the loop is opened.
- Consider the quantity $\rho(\theta)d\theta$ which is the probability of finding an eigenvalue $e^{i\theta_k}$ in the range $\theta < \theta_k < \theta + d\theta$ for some k.
- $\rho(\theta)$ has no gap at lattice strong coupling and develops a gap around $\theta = \pi$ as the coupling gets weaker on the lattice in the large N limit.
- This is the bulk transition on the lattice and is a cross-over for finite N.
- It is the third order Gross-Witten transition in QCD$_2$ ***D. Gross and e. Witten, Phys. Rev. D21 (1980) 446***.
- This transition continues to be third order in $d = 3$ ***(F. Bursa and M. teper, hep-th/0511081)*** and it is first order in $d = 4$ ***(J. Kiskis, R. Narayanan and H. Neuberger, hep-lat/0203005)***.
- The weak coupling side of this transition separates lattice gauge fields into disconnected pieces. This prohibits topology changing configurations in a typical local update algorithm. Global changes in the gauge field are needed to change topology. Wilson fermions will not have small eigenvalues unlike the case in finite N ***(J. Kiskis, R. Narayanan and H. Neuberger, hep-lat/0203005)***.

Fermions in large N QCD:

- There are N^2 gauge degrees of freedom but only N fermion degrees of freedom per fermion flavor if the fermions are in the fundamental representation

- There is no back reaction from the fermions in the 't Hooft limit: The number of colors, N goes to infinity at a fixed 't Hooft coupling, $\lambda = g^2 N$ with a finite number of fermion flavors in the fundamental representation.

- Fermions are naturally quenched in the 't Hooft limit.

- Fermions in the anti-symmetric or adjoint representation are not quenched and one will need dynamical simulations to numerically study supersymmetric QCD or QCD with symmetric/anti-symmetric fermions.

Phases of large N QCD:

- ***R. Narayanan and H. Neuberger, hep-lat/0303023; J. Kiskis, R. Narayanan and H. Neuberger, hep-lat/0308033***

- Large N gauge theory on a continuum l^4 torus has several phases denoted by Xc. X ranges from 0 to 4 and corresponds to the number of directions along which Polyakov loops are broken.

- There exists a critical size l_c that separates the 0c phase ($l > l_c$) from the 1c phase ($l < l_c$).

- Continuum reduction holds in the 0c phase and the theory does not depend on l if $l > l_c$. $l_c = 1/T_c$ and physics is independent of the temperature T for $T < T_c$. Chiral symmetry is broken in the 0c phase ***(R. Narayanan and H. Neuberger, hep-lat/0405025)***.

- The theory in the 1c phase behaves like finite temperature large N QCD in the deconfined phase. The deconfinement transition drives the chiral transition and chiral symmetry is restored in the 1c phase ***(R. Narayanan and H. Neuberger, hep-th/0605173)*** Numerical evidence indicates that the phase transition from 0c to 1c is first order ***(J. Kiskis, hep-lat/0507033)***.

Numerical computation of the chiral condensate

- Pick some L and choose $b < b_c(L)$ such that the theory is in the 0c (confined) phase
- No finite volume effects.
- Keep b close to $b_c(L)$ to minimize finite spacing effects.
- Use the overlap Dirac operator that respects exact chiral symmetry on the lattice. Let $A(\mu)$ denote the massive overlap Dirac operator with μ being the bare mass on the lattice.
- $$\Sigma = \lim_{\mu\to 0} \lim_{N\to\infty} \frac{1}{L^4 N} \langle \mathrm{Tr} A^{-1}(\mu) \rangle_{N,L}$$
- We are working at a fixed lattice volume.
- The first step is to the take the large N limit.
- The second step is to take the massless limit.
- Absence of finite volume effects means that Σ does not depend on L at the given gauge coupling.
- Continuum limit is obtained by increasing b and suitably changing L such that one is always in the 0c phase.

Quenched pathologies at finite N

- Quenched theory at finite N is an ill-defined field theory.
  - Anomalies are not taken into account.
  - Chiral condensate suffers from unphysical divergences: The chiral condensate $\bar{\Sigma}(L)$ defined in finite volume using chiral RMT diverges as L goes to infinity. ***P. Damgaard, hep-lat/0105010***.
- Such pathologies are suppressed by $\frac{1}{N}$ – fermions are naturally quenched in the large N limit.
- It is important to take the large N limit before one takes the large volume limit if one wants to work with the quenched theory on the lattice.

Chiral Random Matrix Theory:

- ***E.V. Shuryak and J.J.M. Verbaarschot, hep-th/9212088; J.J.M. Verbaarschot and T. Wettig, hep-ph/0003017.***
- Let $\pm i\lambda_i$ with $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_K$ be the eigenvalues of $A(0)$.
- Consider the scales variables $z_k = \lambda_k \Sigma N L^4$.
- Extensive work in the area of chiral RMT has shown that the probability distributions, $p(z_k)$, are universal functions as $L \to \infty$ at fixed N.
- Explicit formula can be found in ***P.H. Damgaard and S.M. Nishigaki, hep-th/0006111***.
- Chiral RMT can be derived from chiral effective Lagrangians. ***G. Akeman and P.H. Damgaard, hep-th/0311171***
- Chiral RMT should work even better for fixed L and N goes to infinity since we have N^2 degrees of freedom in the underlying field theory and we are asking for the behavior of only N observables.
- Compute the two lowest eigenvalues λ_1 and λ_2.
- Check if the ratio $r = \frac{\lambda_1}{\lambda_2}$ obeys the universal function dictated by chiral RMT.
- Find the common Σ that converts λ_1 and λ_2 into z_1 and z_2.

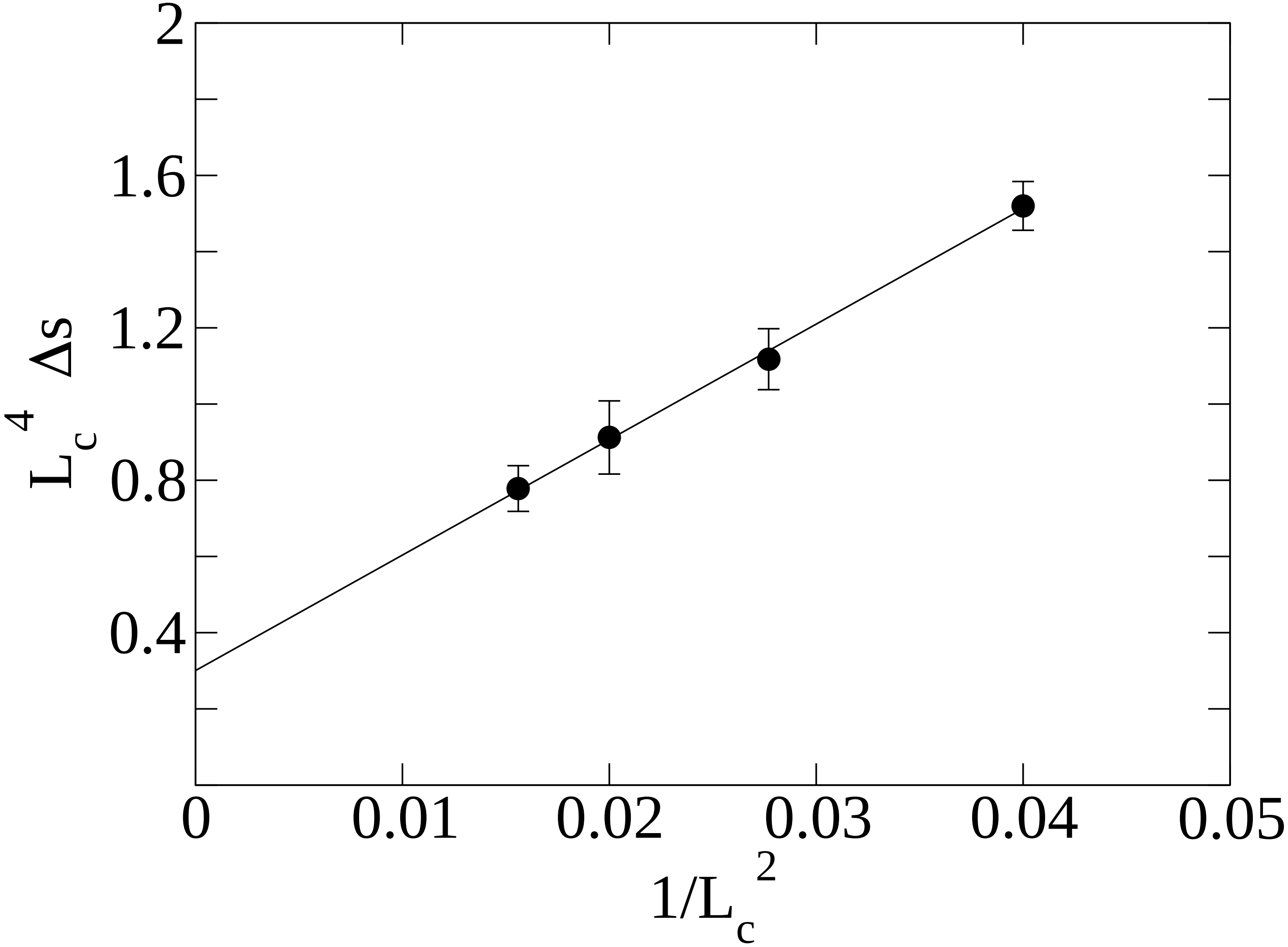
Main result for the chiral condensate

- If we use $T_c \approx 0.6\sqrt{\sigma} \approx 264\mathrm{MeV}$ ***(B. Lucini, M. Teper and U. Wenger, hep-lat/0307017)***, and if we use perturbative tadpole improved esimates for Z_S in the $\overline{MS}$ scheme, we get $\frac{1}{N}\langle\bar{\psi}\psi\rangle^{\overline{MS}}(2\mathrm{GeV}) \approx (174\mathrm{MeV})^3$

- Assuming $N = 3$ is large enough, we get $\langle\bar{\psi}\psi\rangle^{\overline{MS}}(2\mathrm{GeV}) \approx (251\mathrm{MeV})^3$for SU(3).

0c (confined) to 1c (deconfined) phase transition

- Polyakov loop in one of the four directions is spontaneously broken in the 1c phase ($l < l_c$ or $T > T_c$).
- There is a finite latent heat associated with the 0c to 1c transition ***(J. Kiskis,***

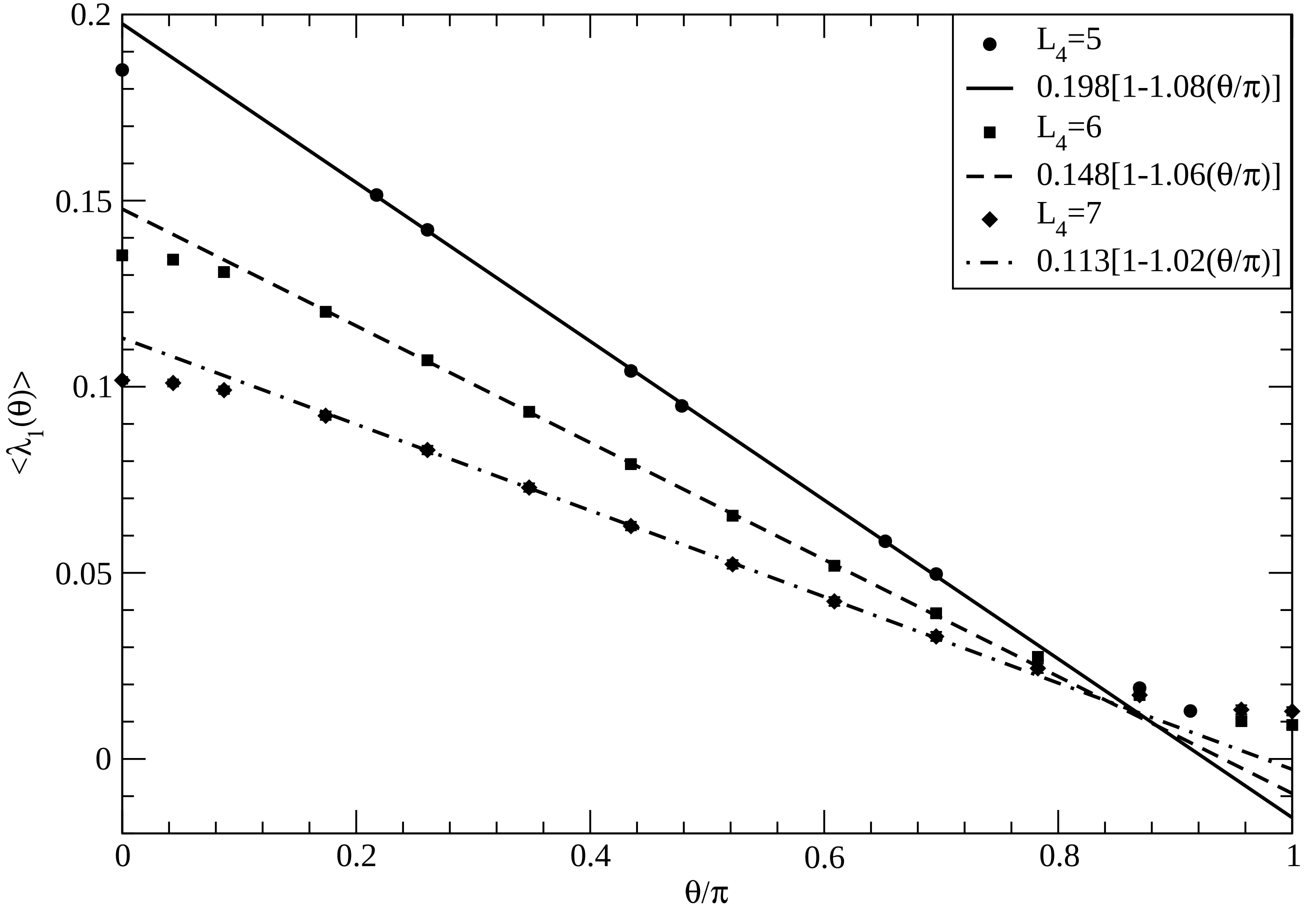


hep-lat/0507003)

Restoration of chiral symmetry in the 1c phase

- Fermions do matter in the 1c phase even in the 't Hooft limit.
- Fermion determinant will depend on the "momentum" that is force-fed in the broken direction.
- In other words, boundary conditions in the temperature direction matters.
- Let θ be the phase associated with the $U(1)$ that defines the boundary condition with respect to the phase of the Polyakov loop in the broken direction. Let $\theta = 0$ define anti-periodic boundary conditions.
- The fermion determinant will depend on θ and one hopes that fermions will pick $\theta = 0$.
- Consider the lowest eigenvalue of the overlap Dirac operator as a measure of the fermion determinant and look at this as a function of θ.
- The data shows a gap in the spectrum for all θ as long as $T > T_c$. This shows strong interaction in the color space.
- The gap is the biggest for $\theta = 0$.
- The gap is linear in θ indicating free-field like behavior and the effect of the interactions in color space is to lower the effective temperature.

Gap as a function of boundary condition



***R. Narayanan and H. Neuberger, hep-th/0605173*.**

Gap in the 1c phase

- Fermions are not quenched in the 1c phase but all we have to do is fix the boundary conditions to be anti-periodic with respect to the Polyakov loop.

- Fermions do not provide any other form of back reaction.

- Define the gap, G, to be the average of the lowest eigenvalue of the overlap Dirac operator.

- Work on a $L^3 \times L_4$ lattice for several couplings b such that they are all in the 1c phase.

- Use $L_c(b)$ to define a dimensionless gap, $g = GL_c(b)$, and a dimensionless temperature, $t = L_4 L_c(b)$.

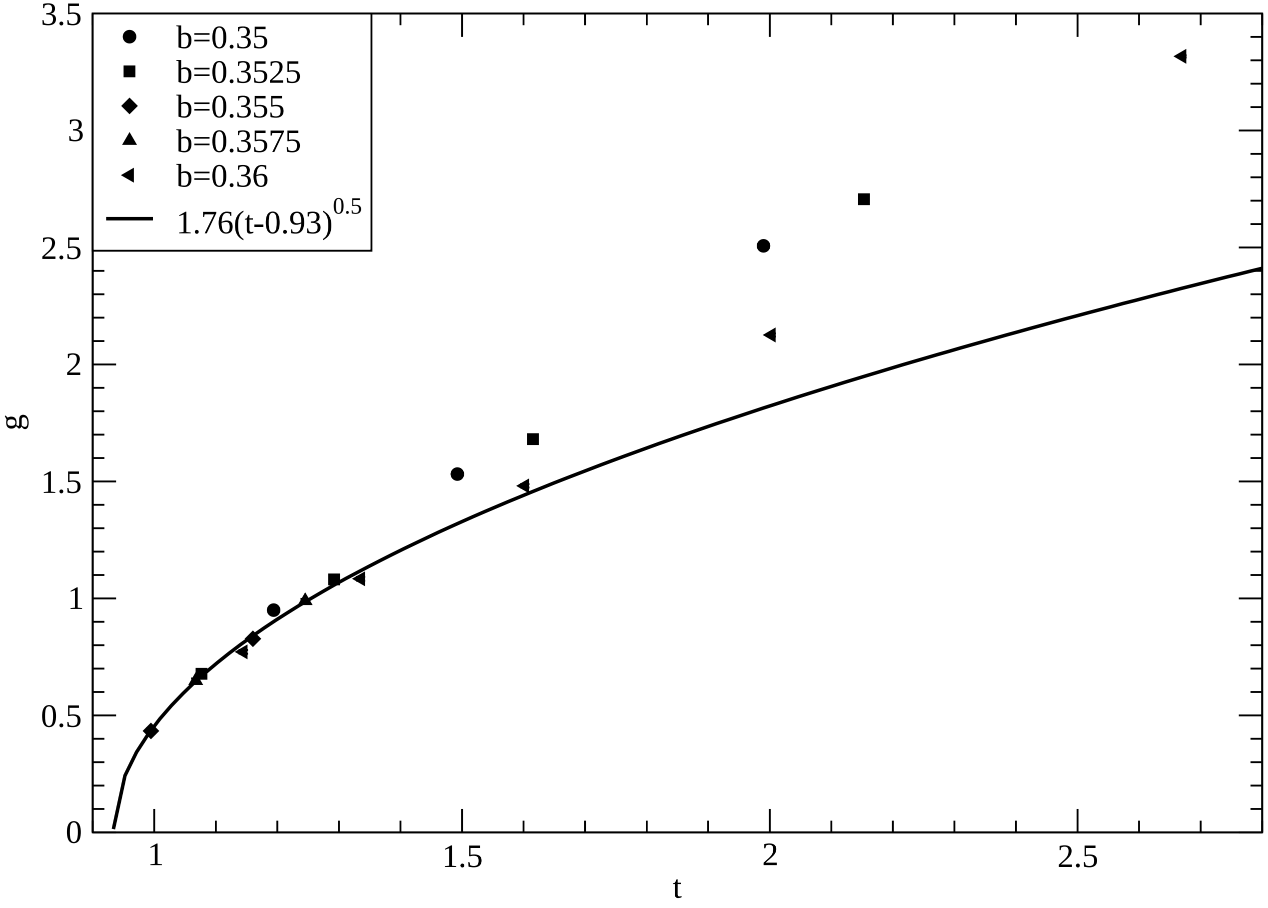
- A plot of g vs t shows that the data fall on a universal curve for small lattice spacing.

- The data fits $1.76\sqrt{t - 0.93}$ for $1 < t < 1.5$.

- There is clear numerical evidence for a first order phase transition in the fermionic sector.

- If we could supercool in the 1c phase below $t = 1$, we would find $T_c^{\mathrm{chiral}} < T_c^{\mathrm{deconfined}}$

Chiral symmetry restoration



R. Narayanan and H. Neuberger, hep-th/0605173.

Disagreement with standard random matrix models

- ***R. Narayanan and H. Neuberger, hep-lat/0612006***.
- We do not have a chiral Langrangian to motivate us toward a Random Matrix Model in the chirally symmetric phase.
- It is natural to write down a matrix model of the form $D = \begin{pmatrix} 0 & C + i\omega \\ -C^\dagger - i\omega & 0 \end{pmatrix}$ for the Dirac operator where ω is the lowest Matsubara frequency at a given temperature and C is a random comples matrix. ***J.J.M. Verbaarschot and T. Wettig, Ann. Rev. Nucl. Part. Sci. 50 (200) 343.***
- This model undergoes a phase transition at some ω_c and has a prediction for
$$c = \frac{\langle(\lambda_1 - \langle\lambda_1\rangle)(\lambda_2 - \langle\lambda_2\rangle)\rangle}{\sqrt{\langle(\lambda_1 - \langle\lambda_1\rangle)^2\rangle\langle(\lambda_2 - \langle\lambda_2\rangle)^2\rangle}}$$
in both phases. c goes from $\frac{1}{3}$ in the confined phase to $\frac{1}{2}$ in the deconfined phase.
- The lattice data does not agree with the predicition of this model. c is significantly above $\frac{1}{2}$ and close to 1 in the 1c deconfined phase. This is in contradiction to the agreement with the lattice data and the model in the 0c confined phase.

Two extra fluctating variables in the deconfined phase

- The lattice data, namely, the eigenvalues of the overlap Dirac operator, $i\lambda_j$; $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ are matched to random matrix model eigenvalues ξ_j; $0 < \xi_1 < \xi_2 < \xi_3 < \cdots$ by $\lambda_j = s\xi_j$ in the confined phase where $s = 1/VN\Sigma$ and Σ, the chiral condensate, is not a variable that fluctuates.

- In order to model the lattice data in the deconfined phase, we propose the following relation:

$$\lambda_j = \alpha' \xi_j + \beta'$$

  where α' and β' are two additional fluctuating variables. Let $\mu_j = \lambda_j - \lambda_1$ and $\eta_j = \xi_1 - \xi_j$.

- We have

$$\ln \mu_j \leftrightarrow \ln \eta_j + \ln(-\alpha')$$

  and this can be tested by looking for j independence of the LHS. The lattice data is in agreement.

- In order to look for fluctuations, we consider

$$\Delta_j = \ln \mu_j - < \ln \mu_j >, \ \delta_j = \ln \eta_j - < \ln \eta_j >, \delta = \ln(-\alpha') - < \ln(-\alpha') >$$

  and obtain

$$< \Delta_j \Delta_k > - < \delta_j \delta_k > = < \delta^2 > + < \delta(\delta_j + \delta_k) >$$

  The first term on the LHS is obtained from the data and the second term on the LHS from the simplest, unextended, RMM.

- If α' were a fixed parameter, $\delta \equiv 0$ and the LHS of the above equation should be zero. The data is not consistent with this scenario.

- If δ and δ_j are not correlated, the LHS should come out independent of j and k. The data is not consistent with this scenario.

- α' and β' are fluctuating random variables that are correlated to the rest of the random matrix model variables.