IQI 04, Seminar 6

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- Distinguishable states of qubits.
- Overlaps.
- Bra-ket algebra.
- Measuring one of n qubits.
- Projective measurement.
- Rotation by state preparation.
- Eigenspace measurement.

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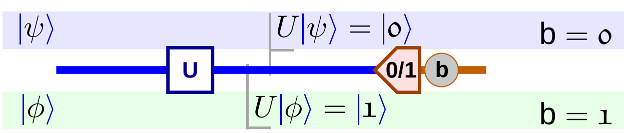
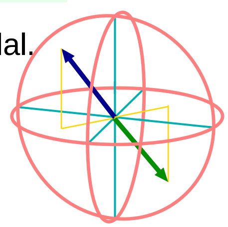
Distinguishable One-Qubit States

- $|\psi\rangle$ and $|\phi\rangle$ are distinguishable if for some unitary U, $U|\psi\rangle = |\mathbf{0}\rangle$ and $U|\phi\rangle = |\mathbf{1}\rangle$.



- Distinguishable states of one qubit are antipodal.

  ... by universality of rotations.



- Distinguishable states are orthogonal.

  ... by unitarity of U.

  Write $|\psi\rangle = \alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle$ and $|\phi\rangle = \gamma|\mathbf{0}\rangle + \delta|\mathbf{1}\rangle$.

$$|\psi\rangle \perp |\phi\rangle \quad \text{iff} \quad (\bar{\alpha}, \bar{\beta}) \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = 0$$

Bra-Ket Algebra

- Need to be able to "dagger" kets.

$$|\psi\rangle \perp |\phi\rangle \quad \text{iff} \quad (|\psi\rangle)^\dagger|\phi\rangle = (\bar{\alpha}, \bar{\beta}) \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = 0$$

  Write $\langle\psi| \doteq (|\psi\rangle)^\dagger$. Meaning derived from:

- Bra-ket rules.
  - $|\mathbf{0}\rangle^\dagger = \langle\mathbf{0}|, |\mathbf{1}\rangle^\dagger = \langle\mathbf{1}|$.
  - $(\alpha|\psi\rangle + \beta|\phi\rangle)^\dagger = \bar{\alpha}\langle\psi| + \bar{\beta}\langle\phi|$.
  - $\langle\mathfrak{a}||\mathfrak{b}\rangle \doteq \langle\mathfrak{a}|\mathfrak{b}\rangle = \delta_{\mathfrak{b}\mathfrak{a}}$.
  - Product distributes over addition.

  Example: $|\psi\rangle = \alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle, |\phi\rangle = \gamma|\mathbf{0}\rangle + \delta|\mathbf{1}\rangle$

$$\langle\psi|\phi\rangle = \langle\psi||\phi\rangle = (|\psi\rangle)^\dagger|\phi\rangle$$
$$= \bar{\alpha}\gamma \qquad + \bar{\beta}\delta$$

Bra-Ket Algebra

- Need to be able to "dagger" kets.

$$|\psi\rangle \perp |\phi\rangle \quad \text{iff} \quad (|\psi\rangle)^\dagger|\phi\rangle = (\bar{\alpha}, \bar{\beta}) \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = 0$$

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  - $(\alpha|\psi\rangle + \beta|\phi\rangle)^\dagger = \bar{\alpha}\langle\psi| + \bar{\beta}\langle\phi|$.
  - $\langle\mathfrak{a}||\mathfrak{b}\rangle \doteq \langle\mathfrak{a}|\mathfrak{b}\rangle = \delta_{\mathfrak{b}\mathfrak{a}}$.
  - Product distributes over addition.
  - For labeled systems: $|\psi\rangle_{\mathsf{A}}^\dagger = {}^{\mathsf{A}}\langle\psi|$.
  - Labeled expressions involving disjoint systems commute.

Ket-Bra Expressions

- Ket-Bra expressions are operators.

  Example: Apply $|0\rangle\langle 0| - |1\rangle\langle 1|$ to states.

$$\begin{aligned}
(|0\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle &= |0\rangle\langle 0||0\rangle - |1\rangle\langle 1||0\rangle \\
&= |0\rangle\underbrace{\langle 0|0\rangle}_{1} - |1\rangle\underbrace{\langle 1|0\rangle}_{0} \\
&= |0\rangle \\
(|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle &= |0\rangle\langle 0||1\rangle - |1\rangle\langle 1||1\rangle \\
&= |0\rangle\underbrace{\langle 0|1\rangle}_{0} - |1\rangle\underbrace{\langle 1|1\rangle}_{1} \\
&= -|1\rangle \\
(|0\rangle\langle 0| - |1\rangle\langle 1|)(\alpha|0\rangle + \beta|1\rangle) &= \alpha|0\rangle - \beta|1\rangle
\end{aligned}$$

Multiplication by $|0\rangle\langle 0| - |1\rangle\langle 1|$ acts as σ_z.

Ket-Bra Expressions

- Ket-Bra expressions are operators.
  - Multiplication by $|0\rangle\langle 0| - |1\rangle\langle 1|$ acts as σ_z.
  - Multiplication by $|0\rangle\langle 1| + |1\rangle\langle 0|$ acts as σ_x.

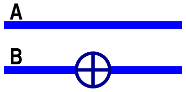
$$\begin{pmatrix} |0\rangle\langle 0| & |0\rangle\langle 1| \\ |1\rangle\langle 0| & |1\rangle\langle 1| \end{pmatrix}$$

$$\begin{aligned}
\sigma_x(\alpha|0\rangle + \beta|1\rangle) &= (|1\rangle\langle 0| + |0\rangle\langle 1|)(\alpha|0\rangle + \beta|1\rangle) \\
&= \alpha|1\rangle\langle 0||0\rangle + \beta|1\rangle\langle 0||1\rangle + \alpha|0\rangle\langle 1||0\rangle + \beta|0\rangle\langle 1||1\rangle \\
&= \alpha|1\rangle\underbrace{\langle 0|0\rangle}_{1} + \beta|1\rangle\underbrace{\langle 0|1\rangle}_{0} + \alpha|0\rangle\underbrace{\langle 1|0\rangle}_{0} + \beta|0\rangle\underbrace{\langle 1|1\rangle}_{1} \\
&= \alpha|1\rangle + \beta|0\rangle
\end{aligned}$$

Ket-Bra Expressions

- Ket-Bra expressions are operators.
  - Multiplication by $|0\rangle\langle 0| - |1\rangle\langle 1|$ acts as σ_z.
  - Multiplication by $|0\rangle\langle 1| + |1\rangle\langle 0|$ acts as σ_x.
  - To flip qubit B in system AB multiply by $\sigma_x{}^{(B)} = |0\rangle_B{}^B\langle 1| + |1\rangle_B{}^B\langle 0|$.

$$\begin{pmatrix} |0\rangle\langle 0| & |0\rangle\langle 1| \\ |1\rangle\langle 0| & |1\rangle\langle 1| \end{pmatrix}$$



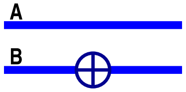
Ex.: $\sigma_{x(B)}\left(\frac{1}{\sqrt{2}}(|00\rangle_{AB} + i|11\rangle_{AB})\right)$

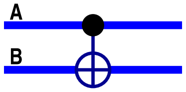
$$\begin{aligned}
&= \left(|0\rangle_B{}^B\langle 1| + |1\rangle_B{}^B\langle 0|\right)\left(\frac{1}{\sqrt{2}}(|00\rangle_{AB} + i|11\rangle_{AB})\right) \\
&= \frac{1}{\sqrt{2}}\left(|0\rangle_B{}^B\langle 1||00\rangle_{AB} + |1\rangle_B{}^B\langle 0||00\rangle_{AB} + i|0\rangle_B{}^B\langle 1||11\rangle_{AB} + i|1\rangle_B{}^B\langle 0||11\rangle_{AB}\right) \\
&= \frac{1}{\sqrt{2}}\left(|0\rangle_B{}^B\langle 1||0\rangle_A|0\rangle_B + |1\rangle_B{}^B\langle 0||0\rangle_A|0\rangle_B + i|0\rangle_B{}^B\langle 1||1\rangle_A|1\rangle_B + i|1\rangle_B{}^B\langle 0||1\rangle_A|1\rangle_B\right) \\
&= \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B\underbrace{{}^B\langle 1|0\rangle_B}_{0} + |0\rangle_A|1\rangle_B\underbrace{{}^B\langle 0|0\rangle_B}_{1} + i|1\rangle_A|0\rangle_B\underbrace{{}^B\langle 1|1\rangle_B}_{1} + i|1\rangle_A|1\rangle_B\underbrace{{}^B\langle 0|1\rangle_B}_{0}\right) \\
&= \frac{1}{\sqrt{2}}\left(|01\rangle_{AB} + i|10\rangle_{AB}\right)
\end{aligned}$$

Ket-Bra Expressions

- Ket-Bra expressions are operators.
  - Multiplication by $|0\rangle\langle 0| - |1\rangle\langle 1|$ acts as σ_z.
  - Multiplication by $|0\rangle\langle 1| + |1\rangle\langle 0|$ acts as σ_x.
  - To flip qubit B in system AB multiply by $\sigma_x{}^{(B)} = |0\rangle_B{}^B\langle 1| + |1\rangle_B{}^B\langle 0|$.
  - **cnot** from A to B:

$$\begin{pmatrix} |0\rangle\langle 0| & |0\rangle\langle 1| \\ |1\rangle\langle 0| & |1\rangle\langle 1| \end{pmatrix}$$





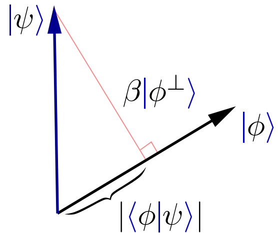
$$\begin{aligned}
\mathbf{cnot}^{(AB)} &= |0\rangle_A{}^A\langle 0| + |1\rangle_A{}^A\langle 1|\sigma_x{}^{(B)} \\
&= |0\rangle_A{}^A\langle 0| + |1\rangle_A{}^A\langle 1|\left(|0\rangle_B{}^B\langle 1| + |1\rangle_B{}^B\langle 0|\right)
\end{aligned}$$

Ex.: $\mathbf{cnot}_{(AB)}(\alpha|0\rangle_A + \beta|1\rangle_A)|0\rangle_B$

$$\begin{aligned}
&= \alpha|0\rangle_A|0\rangle_B \\
&\quad + \qquad \beta|1\rangle_A\,\sigma_{x(B)}|0\rangle_B \\
&= \alpha|0\rangle_A|0\rangle_B + \beta|1\rangle_A|1\rangle_B
\end{aligned}$$

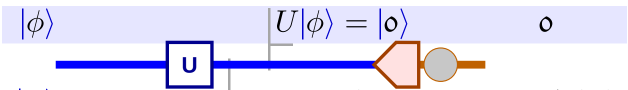
Overlap

- The overlap between $|\psi\rangle$ and $|\phi\rangle$ is $\langle\phi|\psi\rangle$.



$$|\psi\rangle = \langle\phi|\psi\rangle|\phi\rangle + \beta|\phi^\perp\rangle$$

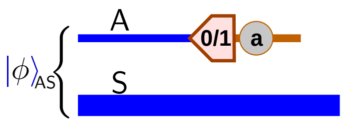
- Consider a measurement to determine if the state is $|\phi\rangle$.



$$U|\psi\rangle = U\left(\langle\phi|\psi\rangle|\phi\rangle + \beta|\phi^\perp\rangle\right) = \langle\phi|\psi\rangle|0\rangle + \beta U|\phi^\perp\rangle \begin{cases} 0: & \text{prob.}=|\langle\phi|\psi\rangle|^2 \\ \neg 0: & \text{prob.}=|\beta|^2=1-|\langle\phi|\psi\rangle|^2 \end{cases}$$

- $|\text{overlap}|^2$ is the prob. of detecting $|\phi\rangle$ if the state is $|\psi\rangle$.

Measuring One Qubit Among Many



prob.$=|\alpha|^2$	prob.$=|\beta|^2$
a=0	a=1
$|\psi_0\rangle_S$	$|\psi_1\rangle_S$

Write $|\phi\rangle_{AS} = |0\rangle_A\left(\alpha|\psi_0\rangle_S\right) + |1\rangle_A\left(\beta|\psi_1\rangle_S\right)$,

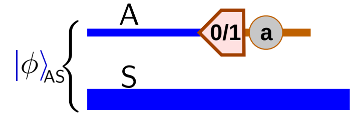
where $|\psi_0\rangle_S$ and $|\psi_1\rangle_S$ are states.

- Computing the probabilities.

$$\begin{aligned} \text{prob}(\mathsf{a}=\mathfrak{b}) &= |{}^{A}\langle\mathfrak{b}||\phi\rangle_{AS}|^2 \\ &= {}^{A}\langle\mathfrak{b}||\phi\rangle_{AS}\,{}^{AS}\langle\phi||\mathfrak{b}\rangle_A \end{aligned}$$

Check: $$\begin{aligned} {}^{A}\langle 0||\phi\rangle_{AS} &= {}^{A}\langle 0|\left(|0\rangle_A\left(\alpha|\psi_0\rangle_S\right) + |1\rangle_A\left(\beta|\psi_1\rangle_S\right)\right) \\ &= \underbrace{{}^{A}\langle 0|0\rangle_A}_{1}\left(\alpha|\psi_0\rangle_S\right) + \underbrace{{}^{A}\langle 0|1\rangle_A}_{0}\left(\beta|\psi_1\rangle_S\right) \\ &= \alpha|\psi_0\rangle_S \end{aligned}$$

Measuring One Qubit Among Many



prob.$=|\alpha|^2$	prob.$=|\beta|^2$
a=0	a=1
$|\psi_0\rangle_S$	$|\psi_1\rangle_S$

$$|\phi\rangle_{AS} = |0\rangle_A\left(\alpha|\psi_0\rangle_S\right) + |1\rangle_A\left(\beta|\psi_1\rangle_S\right)$$

- Bra-ket expression for measurement?

$$\mathbf{meas}(Z{\mapsto}\mathsf{a}) = (0{\mapsto}\mathsf{a})^{A}\langle 0| + (1{\mapsto}\mathsf{a})^{A}\langle 1|$$

  - Effect when applied to $|\phi\rangle_{AS}$:

$$\mathbf{meas}(Z{\mapsto}\mathsf{a})|\phi\rangle_{AS} = (0{\mapsto}\mathsf{a})\alpha|\psi_0\rangle_S + (1{\mapsto}\mathsf{a})\beta|\psi_1\rangle_S.$$

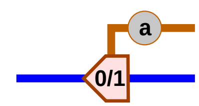
…classically labeled sum of unnormalized states.

- Unnormalized state conventions. Consider $|\omega\rangle$ with $\langle\omega|\omega\rangle \neq 1$.
  1. The intended state is $\frac{1}{\sqrt{\langle\omega|\omega\rangle}}|\omega\rangle$
  2. Amplitude meaningful? Then the state occurs with probability $\langle\omega|\omega\rangle$.

Projective Measurement

- Projective or Von Neumann measurement of σ_z.

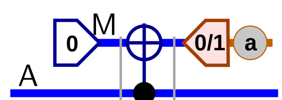
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



prob.$=|\alpha|^2$	prob.$=|\beta|^2$
a=0	a=1
$|0\rangle\propto$	$|1\rangle\propto$
$|0\rangle\langle 0||\psi\rangle$	$|1\rangle\langle 1||\psi\rangle$

  - $P_1 = |0\rangle\langle 0|$, $P_2 = |1\rangle\langle 1|$: Measurement projection operators.
    $P_i = P_i^\dagger, P_i^2 = P_i, P_1 + P_2 = \mathbb{1}$.

- Implementation with destructive measurement.



$\alpha|0\rangle_A + \beta|1\rangle_A$

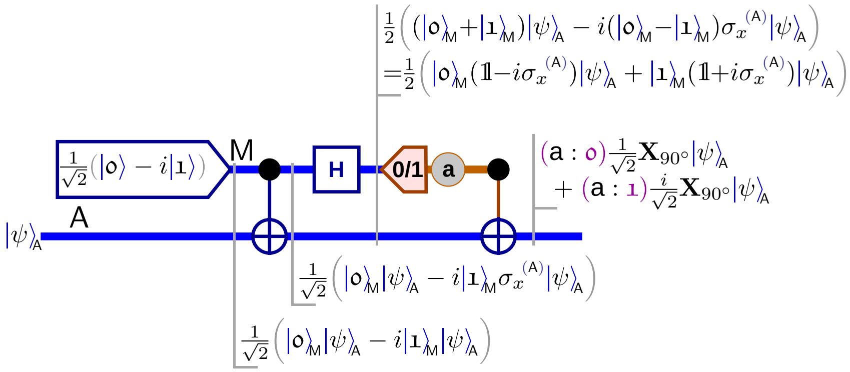
$\alpha|0\rangle_M|0\rangle_A + \beta|1\rangle_M|1\rangle_A$

$|0\rangle_M\left(\alpha|0\rangle_A + \beta|1\rangle_A\right)$

prob.$=|\alpha|^2$	prob.$=|\beta|^2$
a=0	a=1
$|0\rangle$	$|1\rangle$

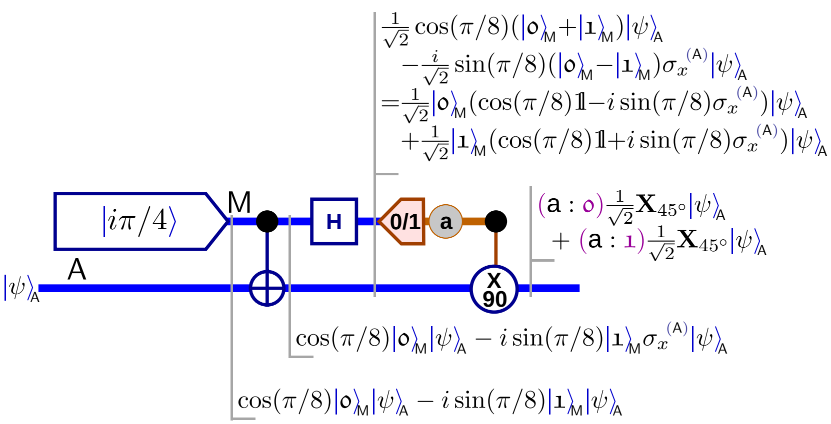
Rotations by State Preparation and Measurement

- $\mathbf{X}_{90^\circ}$ using $|i\pi/4\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$.



Rotations by State Preparation and Measurement

- $\mathbf{X}_{45^\circ}$ using $|i\pi/8\rangle = \cos(\pi/8)|0\rangle - i\sin(\pi/8)|1\rangle$.



Eigenvalues, Eigenvectors

- Let U be a unitary operator.
  - U is diagonalizable, $U = \sum_j e^{-i\lambda_j}|\psi_j\rangle\langle\psi_j|$, with $|\psi_j\rangle$ orthonormal.

$$\underbrace{\begin{pmatrix} \bar{u}_{11} & \bar{u}_{21} & \dots & \bar{u}_{N1} \\ \bar{u}_{12} & \bar{u}_{22} & \dots & \bar{u}_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{u}_{1N} & \bar{u}_{2N} & \dots & \bar{u}_{NN} \end{pmatrix}}_{U} =$$

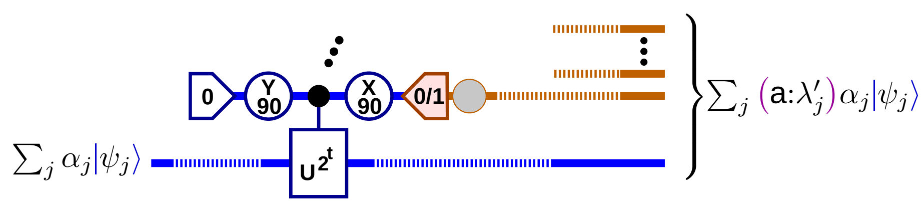
$$\underbrace{\begin{pmatrix} v_{11} & v_{12} & \dots & v_{1N} \\ v_{21} & v_{22} & \dots & v_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ v_{N1} & v_{N2} & \dots & v_{NN} \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} e^{-i\lambda_1} & 0 & \dots & 0 \\ 0 & e^{-i\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{-i\lambda_N} \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} \bar{v}_{11} & \bar{v}_{21} & \dots & \bar{v}_{N1} \\ \bar{v}_{12} & \bar{v}_{22} & \dots & \bar{v}_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{v}_{1N} & \bar{v}_{2N} & \dots & \bar{v}_{NN} \end{pmatrix}}_{V^\dagger}$$

  - There is a complete set of eigenvectors $|\psi_j\rangle = V|j\rangle$.
    $U|\psi_j\rangle = VDVV^\dagger|j\rangle = VD|i\rangle = e^{-i\lambda_j}V|i\rangle = e^{-i\lambda_j}|\psi_j\rangle$
- Projective measurement of the eigenvalues of U?

  For distinct eigenvalues: $|\psi\rangle \rightarrow \sum_j (\lambda_j \mapsto a)|\psi_j\rangle\langle\psi_j|$

Projective Eigenvalue Measurement

- Phase kickback for projective eigenvalue measurement. Assume:
  - U conditionally implementable.
  - $U = \sum_j e^{-i\lambda_j}|\psi_j\rangle\langle\psi_j|$ with $|\psi_j\rangle$ orthonormal.
  - $\min(|\lambda_i - \lambda_j|) > \epsilon$.



  - Requires $O(\log(1/\epsilon)\log\log(1/\epsilon))$ measurements.
  - Can implement $\simeq \sum_j (\lambda'_j \mapsto a)|\psi_j\rangle\langle\psi_j|$ with $\lambda'_j = \lambda_j \pm \epsilon/2$.

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