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THE THREE-DIMENSIONAL STEADY RADIAL EXPANSION OF A NEWTONIAN GAS FROM A SONIC SOURCE INTO A VACUUM

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Abstract

The three-dimensional steady radial expansion of a viscous, heat-conducting compressible fluid from a spherical sonic source into a vacuum is analysed using the Navier-Stokes equations as a basis. It is assumed that the model fluid is a perfect gas having constant specific heats, a constant Prandtl number of order unity, and viscosity coefficients varying as a power of the absolute temperature. Limiting forms for the flow variable solutions are studied for the case where the Reynolds number based on the sonic source conditions goes to inifinity and the ratio of the constant specific heats goes to one.

Through the use of asymptotic expansions and matching, it is shown that, for the above limit, in what is, to leading approximation, an isothermal process, the velocity goes to zero and the pressure goes to a finite vacuum value, as the radial distance approaches infinity. Three distinct regions span the distance between the sonic source and the vacuum, namely: (1) an inviscid source region, in which there is a convection-pressure gradient balance; (2) a (slightly) viscous downstream

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region, in which there is a convection-pressure gradient-hoop stress balance; and (3) a (fully) viscous far downstream region, in which there is a convection-shear stress-hoop stress balance.

Nomenclature

Variables

$r_1 = r^*r = r^*/t$, radial distance
$u_1 = a^*u = a^*Q^{1/2}$, radial velocity
$\rho_1 = \rho^*\rho$, density
$p_1 = p^*p$, pressure
$T_1 = T^*T$, temperature
$\mu_1 = \mu^*\mu$, first viscosity coefficient
$\lambda_1 = \mu^*\lambda$, second viscosity coefficient
$k_1 = k^*k$, thermal conductivity coefficient

(where f_1 = dimensional variable; f^* = dimensional reference state at the spherical sonic source of inviscid flow limit; and f = non-dimensional variable)

Parameters

$\delta = (\mu^*/\rho^*a^*r^*)$, inverse of sonic source Reynolds number
$\varepsilon = (\gamma-1)/(\gamma+1)$, Newtonian parameter
$\sigma = (\mu_1 c_{p_1}/k_1)$, Prandtl number
$K = 2 + (\lambda_1/\mu_1)$, viscosity coefficient ratio
ω = exponent in viscosity-temperature law

1. Introduction

The radial source flow problem has received considerable theoretical attention in the past decade, with the primary attention being given to the two-dimensional case, since, in this case, the explicit appearance of the radial distance variable can be transformed away and a topological study made of the resulting equations. For the three-dimensional case, however, the existence of a transformation that eliminates the explicit appearance of the radial distance variable has not been shown.

Sakurai (1958), noting the difficulty in expressing the solutions for the three-dimensional case in analytical forms valid over the whole range of the radial variable, attempts to find solutions that span the domain through the finding of approximate solutions for subregions within this domain and connecting these approximate solutions for the various subregions by 'using boundary layer technique'.

This three-dimensional analysis of Sakurai's, which relies heavily on the two-dimensional viscous source flow solutions for expansion into a vacuum as a guide for determining its approximate solutions, unfortunately, does not, itself, describe the expansion of the gas into a vacuum.

This fact is brought out by Ladyshenskii (1962). Although Ladyshenskii presents no (analytical) solutions, he is able to show, for the expansion of a gas into a vacuum, from the integral forms of the three-dimensional equations of motion, that, for $\omega = 0$, and for $0 < \omega < (1+3\varepsilon)/4\varepsilon$ (for ε fixed), the radial velocity goes to zero as the radial distance goes to infinity --a result quite different from the two-dimensional result that the radial velocity tends to a value somewhat less than the corresponding maximum velocity for the inviscid source solution.

With the above in mind, the purpose of this paper, then, is to present uniformly valid solutions for the three-dimensional case, and, hence, to some extent, to present quantitative details to complete the qualitative picture that Ladyshenskii gives. The method of making this presentation is essentially that which Sakurai proposes. Approximate solutions for regions within the domain are obtained through the use of asymptotic expansion techniques and these solutions are joined through matching techniques for the limiting case of $\delta \to 0$ and $\varepsilon \to 0$. The former limit corresponds to taking the large Reynolds number regime approach to the problem. The latter limit corresponds to taking the gas under consideration to be one whose ratio of specific heats is approaching one--this has been done for reasons of mathematical convenience.

In §2, the basic Navier-Stokes equations of motion, as well as the appropriate boundary conditions, whose validity Ladyshenskii has proven, are given.

In §3, the approximate solutions for the inviscid source flow (sub)region, in which there is a convection-pressure gradient balance, are given.

In §4, the approximate solutions for the geometrical stress (sub)region, in which there is a convection-pressure gradient-hoop stress balance, are given. The matchings of these solutions to those found in §3 are also given.

In §5, the approximate solutions for the shear stress (sub)-region, in which there is a convection-shear stress-hoop stress balance, are given. The matchings of these solutions to those found in §4 are also given.

In §6, a brief discussion of the results is presented.

2. The equations of motion

Consider the three-dimensional steady radial flow of a viscous, compressible gas from a spherical sonic source into a vacuum. Let

$$r_1 = r^*r = r^*/t$$

represent the radial distance, with r^* the radius of the sonic sphere. The radial velocity, pressure, temperature, and density are

$$u_1 = a^*u = a^*Q^{1/2},\ p_1 = p^*p,\ T_1 = T^*T,\ \rho_1 = \rho^*\rho,$$

where a^*, p^*, T^*, and ρ^* are the reference states at the sonic source.

The gas is assumed to be a perfect gas ($p = \rho T$), having (1) constant specific heats, c_{v_1} and c_{p_1} with $\gamma = c_{p_1}/c_{v_1}$ = const.; (2) a constant Prandtl number of order unity (σ = const. = $O(1)$); and (3) its first and second viscosity coefficients proportional to a power, ω, of the absolute temperature ($\mu_1 = \mu^*\mu = \mu^*T^{\omega}$; $\lambda_1 = \mu^*\lambda = -(2-K)\mu^*T^{\omega}$, K = const. = $O(1)$).

The dimensionless Navier-Stokes equations of motion for the radial flow of such a gas can be written (cf., Ladyshenskii (1962)) as two primary equations

$$\begin{aligned}\{Q - (\tfrac{1-\varepsilon}{1+\varepsilon})T\}\frac{dQ}{dt} &+ 2(\tfrac{1-\varepsilon}{1+\varepsilon})Q\{\frac{dT}{dt} + 2\,\frac{T}{t}\} \\ &+ \delta K T^{\omega} Q[\frac{d^2Q}{dt^2} - \frac{1}{2}\,\frac{1}{Q}(\frac{dQ}{dt})^2 - 4\,\frac{Q}{t^2} \\ &\qquad + \frac{\omega}{T}\,\frac{dT}{dt}\{\frac{dQ}{dt} + \frac{4(2-K)}{K}\,\frac{Q}{t}\}] = 0, \qquad (2.01a)\end{aligned}$$

$$T + \left(\frac{\varepsilon}{1-\varepsilon}\right)Q$$

$$+ \delta K T^{\omega}\left[\frac{1}{\sigma K}\frac{dT}{dt} + \left(\frac{\varepsilon}{1-\varepsilon}\right)\left\{\frac{dQ}{dt} + \frac{4(2-K)}{K}\frac{Q}{t}\right\}\right] = \frac{1}{1-\varepsilon}; \qquad (2.01b)^*$$

plus two secondary equations,

$$\rho = \frac{t^2}{Q^{1/2}}, \quad p = \frac{t^2 T}{Q^{1/2}}. \qquad (2.01c,d)$$

In (2.01), δ is the inverse of the sonic source Reynolds number and ε is the Newtonian parameter; both quantities are taken to be small for this analysis, i.e.,

$$\delta = \mu^*/\rho^* a^* r^* << 1, \ \varepsilon = (\gamma-1)/(\gamma+1) << 1. \qquad (2.02a,b)^*$$

Uniformly valid solutions of the system of equations of (2.01), for the limits of (2.02), subject to the boundary conditions at the sonic source and at the vacuum, respectively,

$$Q, T, \rho, p \to 1, \text{ as } t \to 1, \qquad (2.03a)$$

$$Q \to 0, \ p \to P(\delta) << 1, \text{ as } t \to 0, \qquad (2.03b)$$

are determined in the following sections.

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* A realistic value for σ is that given by Eucken (1913), namely,

$$\sigma = [1+(5/2)\{\varepsilon/(1+\varepsilon)\}]^{-1}.$$

Thus, for $\varepsilon << 1$, it follows that $(1-\sigma) << 1$.

3. The inviscid source flow solution

Near the sonic source, $t = 1$, the flow is taken to be characterized by the following expansions for the variables:

$$t = t_i ; \tag{3.01a}$$

$$Q = Q_i + \ldots , \quad T = 1 + \varepsilon T_i + \ldots ; \tag{3.01b}$$

$$\rho = \rho_i + \ldots = t_i^2/Q_i^{1/2} + \ldots ,$$
$$p = p_i + \ldots = t_i^2/Q_i^{1/2} + \ldots . \tag{3.01c}$$

With these representations, for $\delta, \varepsilon << 1$, the leading terms in the equations for $Q_i(t_i)$ and $T_i(t_i)$ become those for an inviscid Newtonian source flow, namely,

$$(Q_i - 1) \frac{dQ_i}{dt_i} + 4 \frac{Q_i}{t_i} = 0, \; Q_i(1) = 1; \tag{3.02a}$$

$$T_i = -(Q_i - 1), \; T_i(1) = 0. \tag{3.02b}$$

The solutions of these equations, subject to the boundary conditions at the sonic source, are

$$Q_i - 1 - \log Q_i = 4 \log (1/t_i); \quad T_i = -(Q_i - 1). \tag{3.03}$$

For the solutions' supersonic branch, i.e., for $Q_i \geq 1$, the asymptotic behaviors for Q and T are seen to be

$$Q - 1 = 2[2(1-t_i)]^{1/2} + \dots \to 0,$$
$$(1-T)/\varepsilon = 2[2(1-t_i)]^{1/2} + \dots \to 0, \text{ as } t_i \to 1; \tag{3.04a}$$

$$Q = 4 \log (1/t_i) + \dots \to \infty,$$
$$(1-T)/\varepsilon = 4 \log (1/t_i) + \dots \to \infty, \text{ as } t_i \to 0. \tag{3.04b}$$

In turn,

$$1-\rho = [2(1-t_i)]^{1/2} + \dots \to 0,$$
$$1-p = [2(1-t_i)]^{1/2} + \dots \to 0, \text{ as } t_i \to 1; \tag{3.05a}$$

$$\rho = \frac{1}{2} t_i^2 [\log(1/t_i)]^{-1/2} + \dots \to 0,$$
$$p = \frac{1}{2} t_i^2 [\log(1/t_i)]^{-1/2} + \dots \to 0, \text{ as } t_i \to 0. \tag{3.05b}$$

Further, it is noted that the local Mach number for this region is

$$M = (Q/T)^{1/2} = Q_i^{1/2} + \dots ,$$

so that

$$M-1 = [2(1-t_i)]^{1/2} + \dots \to 0, \text{ as } t_i \to 1;$$
$$M = 2 [\log(1/t_i)]^{1/2} + \dots \to \infty, \text{ as } t_i \to 0.$$

From (3.04b) and (3.05b), it is seen that the solutions of this inviscid source flow region do not satisfy the vacuum boundary conditions of (2.03b), in that: (1) the velocity does not go to zero; (2) the pressure does not approach a small, fixed value. Therefore, it is clear that the flow field for $t \to 0$ must be studied in greater detail in order to determine this flow in the interior (with respect to t) region or regions necessary for the satisfying of (2.03b).

4. The geometrical stress region

The logarithmic forms of the asymptotic solutions for the inviscid region dependent variables as $t_i \to 0$ suggest that the proper expansions for the variables in a region interior to the inviscid one are:

$$t = D_g t_g,\ D_g << 1; \tag{4.01a}$$

$$Q = \frac{1}{A_g} Q_{g,o}(1 + A_g Q_g + \dots),\ T = 1 + \frac{\varepsilon}{A_g} T_{g,o}(1 + A_g T_g + \dots),$$
$$\varepsilon << A_g << 1,\ Q_{g,o},\ T_{g,o} = \text{consts.}; \tag{4.01b}$$

$$\rho = D_g^2 A_g^{1/2} \rho_g + \dots = D_g^2 A_g^{1/2} \frac{t_g^2}{Q_{g,o}^{1/2}} + \dots ,$$
$$p = D_g^2 A_g^{1/2} p_g + \dots = D_g^2 A_g^{1/2} \rho_g + \dots . \tag{4.01c}$$

With these representations, for $A_g D_g = \delta << 1$, the leading terms in the equations for Q and T are:

$$T_{g,o} + Q_{g,o} = 0,$$
$$Q_{g,o} \frac{dQ_g}{dt_g} + \frac{4}{t_g} - \frac{4KQ_{g,o}}{t_g^2} = 0,$$
$$T_{g,o} T_g + Q_{g,o} Q_g - 1 + \frac{4(2-K)Q_{g,o}}{t_g} = 0. \tag{4.02}$$

It is seen that (4.02) defines a flow (with a local Mach number of $M = (Q_{g,o}/A_g)^{1/2} + \dots >> 1$) in which the inviscid terms are in balance with the geometrical(or hoop) stress terms.

The solutions of Q_g and T_g, found from (4.02), are:

$$Q_g = -\frac{4}{Q_{g,o}} \log t_g + C_g - \frac{4K}{t_g},\ C_g = \text{const.};$$
$$T_g = -\frac{4}{Q_{g,o}} \log t_g - \left(\frac{1-Q_{g,o}C_g}{Q_{g,o}}\right) - \frac{8(K-1)}{t_g} . \tag{4.03}$$

Therefore, as $t_g \to \infty$:

$$Q = \frac{1}{A_g} Q_{g,o}(1 + A_g[- \frac{4}{Q_{g,o}} \log t_g + \ldots] + \ldots),$$

$$T = 1 - \frac{\varepsilon}{A_g} Q_{g,o}(1 + A_g[- \frac{4}{Q_{g,o}} \log t_g + \ldots] + \ldots), \tag{4.04a}$$

Further, as $t_g \to 0$:

$$Q = \frac{1}{A_g} Q_{g,o}(1 + A_g[- \frac{4K}{t_g} + \ldots] + \ldots),$$

$$T = 1 - \frac{\varepsilon}{A_g} Q_{g,o}(1 + A_g[- \frac{8(K-1)}{t_g} + \ldots] + \ldots). \tag{4.04b}$$

To verify that the geometrical stress region formulation is compatible with that of the inviscid source flow region, it must be demonstrated that the solutions of the former region as $t_g \to \infty$ match to the solutions of the latter region as $t_i \to 0$. The matching of the solutions for these two regions is performed through the introduction of the intermediate limit, $\lim_{ig}$, defined by

$$t_{ig} = \frac{t}{D_{ig}} \text{ fixed, } D_g << D_{ig} << 1. \tag{4.05a}$$

In this limit,

$$t_i = D_{ig} t_{ig} \to 0, \; t_g = \frac{D_{ig}}{D_g} t_{ig} \to \infty. \tag{4.05b}$$

The Q-matching requires

$$\lim_{ig}\Big[\{Q_i(D_{ig}t_{ig}) + \ldots\} - \{\frac{1}{A_g} Q_{g,o}(1 + A_g Q_g(D_{ig}t_{ig}/D_g) + \ldots)\}\Big] = 0. \tag{4.06a}$$

From (3.04b) and (4.04a), it is seen that (4.06a) reduces to

$$\lim_{ig}\Big[\{4\log(\frac{1}{D_{ig}t_{ig}}) + \ldots\} - \{\frac{Q_{g,o}}{A_g}(1 + A_g[-\frac{4}{Q_{g,o}}\log(\frac{1}{D_g}) + \frac{4}{Q_{g,o}}\log(\frac{1}{D_{ig}t_{ig}}) + \ldots])\}\Big] = 0. \tag{4.06b}$$

Hence, there is Q-matching for the inviscid source flow and geometrical stress regions if

$$Q_{g,o} = 4,\ A_g = [\log(\frac{1}{D_g})]^{-1}. \tag{4.07}$$

Without presenting the details, it is noted that the T-matching for these two regions also requires that the conditions of (4.07) be satisfied.

From the conditions that $A_g D_g = \delta$ and $A_g = [\log(\frac{1}{D_g})]^{-1}$, it follows that

$$D_g\ [\log(\frac{1}{D_g})]^{-1} = \delta, \tag{4.08a}$$

or

$$D_g = \delta\log(1/\delta)\ [1 - \frac{\log\{\log(1/\delta)\}}{\log(1/\delta)} + \ldots] \gg \delta,$$

$$A_g = \delta/D_g = [\log(1/D_g)]^{-1}$$
$$= [\log(1/\delta)]^{-1}\ [1 + \frac{\log\{\log(1/\delta)\}}{\log(1/\delta)} + \ldots]. \tag{4.08b}$$

Further, from (4.01b), it has been assumed that $\varepsilon \ll A_g \ll 1$. Thus, the formulation presented here is valid if the parameters ε and δ satisfy the inequalities

$$\varepsilon \ll [\log(1/\delta)]^{-1} \ll 1. \tag{4.09}$$

Again, from the forms that Q and p take, it is seen that this geometrical stress region is not capable of satisfying (2.03b) as $t_g \to 0$. Hence, the examination of the interior region must be extended.

5. The shear stress region

Once again, using the forms of the asymptotic solutions of the geometrical stress region dependent variables as $t_g \to 0$ as guides to the postulation of the expansions for these variables in a region just interior to the geometrical stress one, (4.04b) suggests the following representations:

$$t = D_v t_v, \; D_v << D_g << 1; \tag{5.01a}$$

$$Q = \frac{1}{A_g} Q_v + \ldots , \; T = 1 + \frac{\varepsilon}{A_g} T_v + \ldots ; \tag{5.01b}$$

$$\rho = D_v^2 A_g^{1/2} \rho_v + \ldots = D_v^2 A_g^{1/2} \frac{t_v^2}{Q_v^{1/2}} + \ldots ,$$

$$p = D_v^2 A_g^{1/2} p_v + \ldots = D_v^2 A_g^{1/2} \rho_v + \ldots . \tag{5.01c}$$

With these representations, for $D_v = \delta << D_g = \delta \log(1/\delta) + \ldots .$, the equations for Q and T are:

$$\frac{dQ_v}{dt_v} + K\left[\frac{d^2Q_v}{dt_v^2} - \frac{1}{2}\frac{1}{Q_v}\left(\frac{dQ_v}{dt_v}\right)^2 - 4\frac{Q_v}{t_v^2}\right] = 0; \tag{5.02a}$$

$$T_v + Q_v + K\left[\frac{1}{K}\frac{dT_v}{dt_v} + \frac{dQ_v}{dt_v} + \frac{4(2-K)}{K}\frac{Q_v}{t_v}\right] = 0. \tag{5.02b}$$

Therefore, (5.02) defines a flow (with a local Mach number of $M = (Q_v/A_g)^{1/2} + \ldots)$ in which the inviscid terms are in balance with the shear stress and heat-conduction terms, as well as the geometrical stress terms.

For $Q_v = 4U_v^2$, $t_v = Ks_v$, (5.02a) may be rewritten as the linear equation

$$\frac{d^2U_v}{ds_v^2} + \frac{dU_v}{ds_v} - 2\frac{U_v}{s_v^2} = 0, \tag{5.03}$$

whose complete solution is

$$U_v = c_1 U_v^{(1)} + c_2 U_v^{(2)}, \text{ with } c_1, c_2 = \text{consts.}, \tag{5.04a}$$

where

$$U_v^{(1)} = s_v^2 \exp(-s_v) \underline{F}(2,4,s_v),$$
$$\underline{F} = \text{confluent hypergeometric function}; \tag{5.04b}$$

$$U_v^{(2)} = \left(\frac{2+s_v}{s_v}\right) \exp(-s_v). \tag{5.04c}$$

From (5.04b) and (5.04c), it is seen that the linearly independent solutions, $U_v^{(1)}$ and $U_v^{(2)}$, have the following asymptotic behaviors:

$$U_v^{(1)} \doteq 6\left(1 - \frac{2}{s_v} + \ldots\right) \to \text{const.},$$

$$U_v^{(2)} = \exp(-s_v) \left(1 + \frac{2}{s_v}\right) \to 0, \text{ as } s_v \to \infty; \tag{5.05a}$$

$$U_v^{(1)} = s_v^2 + \ldots \to 0, \; U_v^{(2)} = \frac{2}{s_v} + \ldots \to \infty, \text{ as } s_v \to 0. \tag{5.05b}$$

Thus, for $c_2 = 0$, so that $U_v \to 0$ as $s_v \to 0$, the solution for Q_v is

$$Q_v = [2c_1(t_v/K)^2 \exp\{-(t_v/K)\} \underline{F}(2,4,t_v/K)]^2, \tag{5.06a}$$

where, as $t_v \to 0$,

$$Q_v = [2c_1(t_v/K)^2]^2 + \ldots \to 0. \tag{5.06b}$$

In turn, this means that the solution for p_v is

$$p_v = [(2c_1/K^2) \exp\{-(t_v/K)\} \underline{F}(2,4,t_v/K)]^{-1}, \tag{5.07a}$$

where, as $t_v \to 0$,

$$p_v = (K^2/2c_1) + \ldots \to \text{const.} \tag{5.07b}$$

The behaviors of the velocity and pressure functions, as exhibited in (5.06b) and (5.07b), therefore, satisfy the boundary conditions of (2.03b). For $c_1 = 1/6$ (as will be shown), it follows from these equations, that, as $t \to 0$ $(r \to \infty)$,

$$\begin{aligned} Q &= (1/3K^2)\delta^{-2}[\log(1/\delta)]t^2 + \ldots \\ &= (1/3K^2)\delta^{-2}[\log(1/\delta)]r^{-2} + \ldots; \end{aligned}$$

$$p = 3K^2\delta^2[\log(1/\delta)]^{-1/2} + \ldots = P(\delta) + \ldots \ll 1.$$

That the shear stress region formulation, with the solution for Q_v given in (5.06a), such that, as $t_v \to \infty$,

$$Q = \frac{4}{A_g} (6c_1)^2 \left(1 - \frac{4K}{t_v} + \ldots\right), \tag{5.06c}$$

is compatible with that of the geometrical stress region remains to be demonstrated. These formulations are compatible if the solutions for Q of the shear stress and the geometrical stress regions match as $t_v \to \infty$ and $t_g \to 0$, respectively. The matching of the solutions for Q in these two regions is performed through the introduction of the intermediate limit, $\lim_{gv}$, defined by

$$t_{gv} = \frac{t}{D_{gv}} \text{ fixed, } D_v = \delta \ll D_{gv} \ll D_g = \delta \log(1/\delta) + \ldots . \tag{5.08a}$$

In this limit,

$$t_g = \frac{D_{gv}}{D_g} t_{gv} \to 0, \quad t_v = \frac{D_{gv}}{D_v} t_{gv} \to \infty. \tag{5.08b}$$

The Q-matching requires

$$\lim_{gv} \Big[\{ \frac{4}{A_g} (1 + A_g Q_g (D_{gv} t_{gv}/D_g) + \ldots) \} - \{ \frac{1}{A_g} Q_v (D_{gv} t_{gv}/D_v) + \ldots \} \Big] = 0, \tag{5.09a}$$

i.e.,

$$\lim_{gv} \Big[\{ \frac{4}{A_g} (1 + A_g [- 4K \frac{D_g}{D_{gv} t_{gv}} + \ldots] + \ldots) \} - \{ \frac{1}{A_g} [4(6c_1)^2 (1 - 4K \frac{D_v}{D_{gv} t_{gv}} + \ldots) + \ldots \} \Big] = 0. \tag{5.09b}$$

Since it has been taken that $A_g D_g = D_v = \delta$, it follows that the only requirement necessary for matching is that

$$c_1 = 1/6. \tag{5.10}$$

Thus, it has been demonstrated that the forms that Q and p take in this region are capable of satisfying (2.03b) as $t_v \to 0$, and the search for further interior regions need not be continued.

6. Discussion of the results

The results presented in §§3-5 show that the approximate analytical solutions for the proposed regions are compatible with one another and combine to yield uniformly valid solutions over the entire range of the radial variable that satisfy the boundary conditions established by Ladyshenskii for the three-dimensional case.

It is noted that, for the case of $\varepsilon \ll [\log(1/\delta)]^{-1} \ll 1$ considered here, in all three regions, the approximation for the temperature takes the form

$$T = 1 + \ldots ,$$

and, hence, under these conditions, the expansion from a sonic source into a vacuum is, to leading approximation, an isothermal process. Further, to this approximation, the basic momentum equation spanning the domain, a 'composite' equation of the momentum equations considered in the three regions, becomes

$$(Q - 1)\frac{dQ}{dt} + 4\frac{Q}{t} + \delta KQ\left[\frac{d^2Q}{dt^2} - \frac{1}{2}\frac{1}{Q}\left(\frac{dQ}{dt}\right)^2 - 4\frac{Q}{t^2}\right] = 0, \tag{6.01a}$$

or, in terms of the variable u,

$$(u^2 - 1)\frac{du}{dt} + 2\frac{u}{t} + \delta Ku^2\left[\frac{d^2u}{dt^2} - 2\frac{u}{t^2}\right] = 0. \tag{6.01b}$$

To this extent, then, this basic momentum equation is essentially that which Sakurai considers, although Sakurai's formulation neglects to take into account the hoop stress term.

The question arises as to the point at which the analysis presented, based on the Navier-Stokes equations, loses its validity. If the criterion for this validity (c.f., Ladyshenskii) is taken to be that the ratio of the supplementary terms in the Burnett equations to the terms in the Navier-Stokes equations representing the effects of viscosity be small, then, in terms of the variables introduced in this paper, this criterion becomes

$$S = \frac{\mu_1}{p_1}\frac{du_1}{dr_1} = \frac{\delta}{2}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)\frac{1}{T^{1-\omega}}\frac{dQ}{dt} << 1. \qquad (6.02)$$

For the three regions considered, it is found that, to leading approximation,

$$S = \frac{\delta}{2}\frac{dQ_i}{dt_i} + \ldots \sim O(\delta) << 1; \qquad (6.03a)$$

$$S = 2\,[\log(1/\delta)]^{-1}\frac{dQ_g}{dt_g} + \ldots \sim O([\log(1/\delta)]^{-1}) << 1; \qquad (6.03b)$$

$$S = \frac{1}{2}\,[\log(1/\delta)]\frac{dQ_v}{dt_v} + \ldots \sim O([\log(1/\delta)]) >> 1. \qquad (6.03c)$$

Thus, the criterion is satisfied in the inviscid and geometrical stress regions but not in the shear stress region. Nevertheless, as Ladyshenskii points out, these equations, as in the case of the structure of the shock wave, give, in a certain sense, a correct qualitative description of the behavior of the flow.

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