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Turning Magical Hopes into Realistic Expectations: Using and Linking Representations and Manipulatives to Promote Students' Mathematical Understanding

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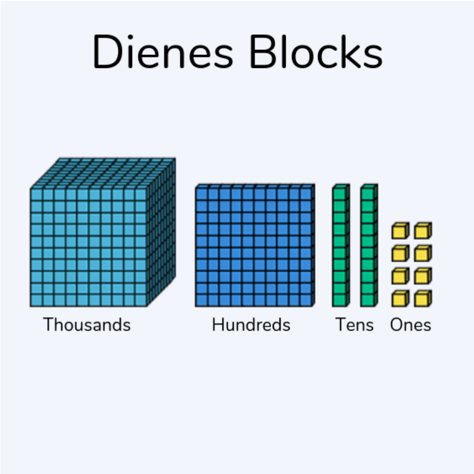
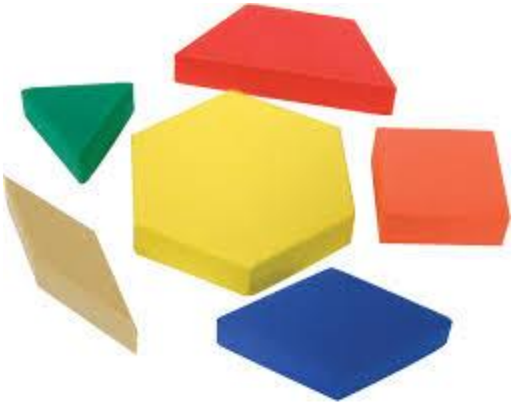
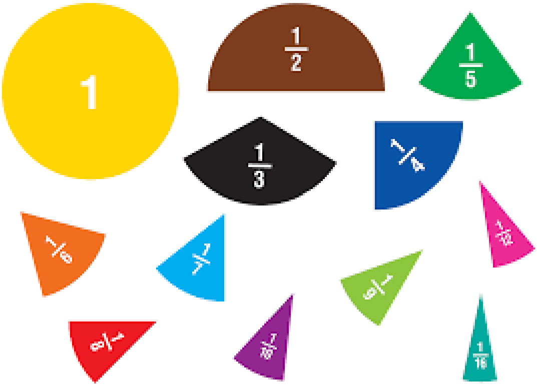
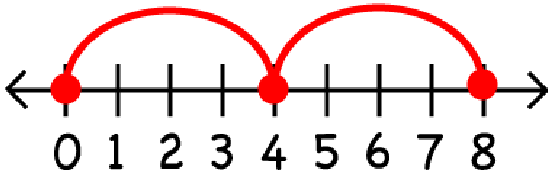
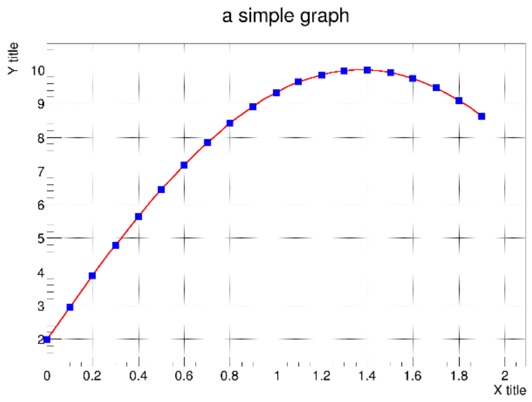
Using Representations and Manipulatives in Our Teaching

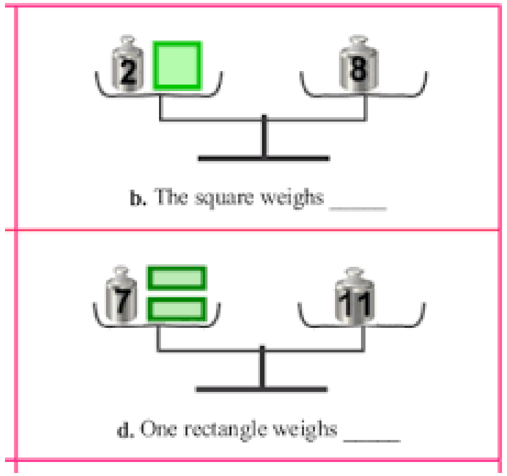












- **A central teaching aid for supporting student learning (NCTM, 2000)**

Using Representations and Manipulatives in Our Teaching

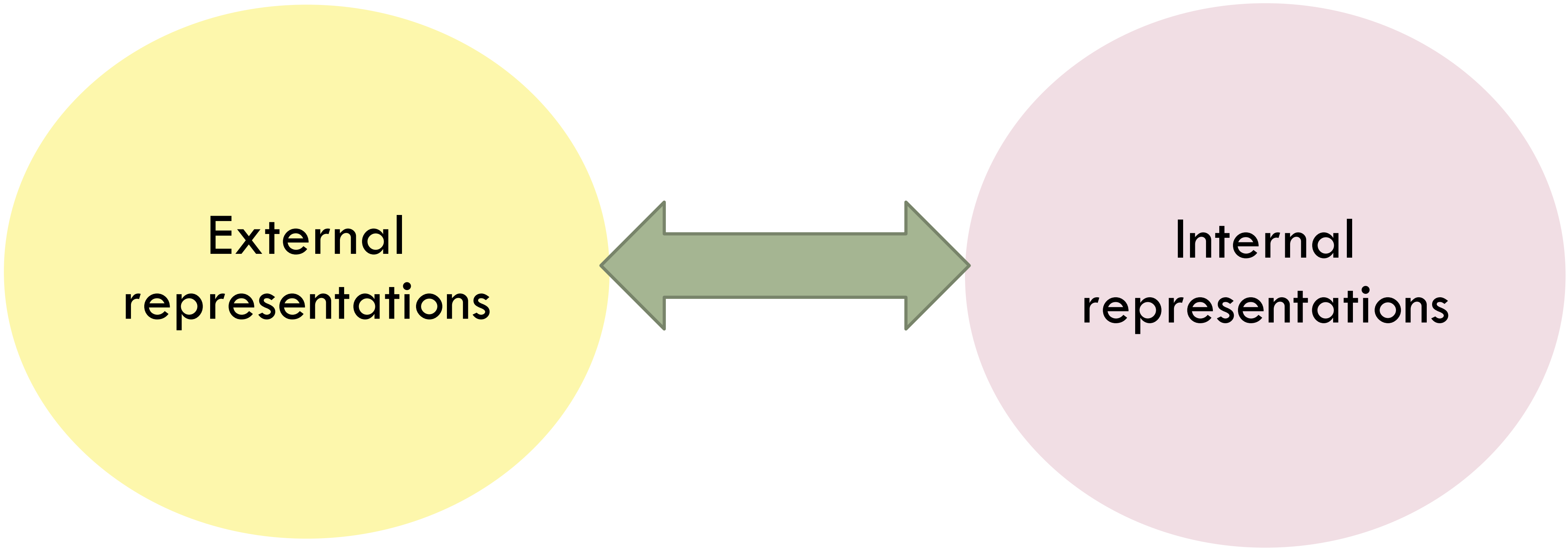
“Manipulatives are often employed as “training wheels” for students’ mathematical thinking. However, most teachers have encountered directly the frustration when the training wheels are removed. Students, rather than riding their mathematical “bicycles” smoothly, fall off, reverting to [using algorithms mindlessly]. […] These training wheels do not work magic. Seeing students work well within the manipulative context can mislead—and later disappoint teachers about what their students know” (p. 18)

Ball, D. L. (1992). Magical hopes: Manipulatives and the reform of math education. *American Educator, 16*(2), 14-18, 46-47.

Questions to Address

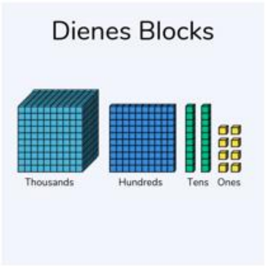
- Why are we using representations and manipulatives?
  - What are representations and manipulatives?
- What representations and manipulatives should we be using in different situations?
  - Does it really matter?
- How should we be using these devices to ensure that when the training wheels are removed from students' mathematical bicycles, students can still ride smoothly and competently?

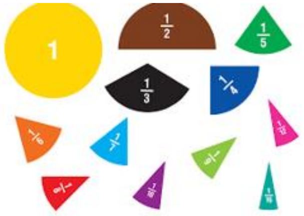
Defining Representations and Manipulatives

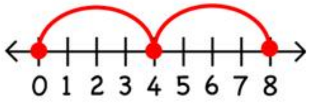


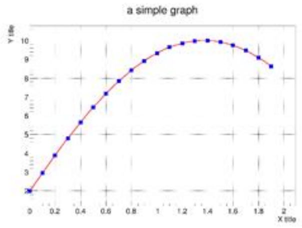
External Representations

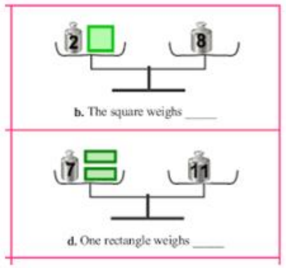
- "A sign or a configuration of signs, characters, or objects [which] can **stand for (symbolize, depict, encode, or represent) something other than itself**" (Goldin & Shteingold, 2001, p. 3)
- Entities that symbolize or **stand for other entities** (Duval, 2006)





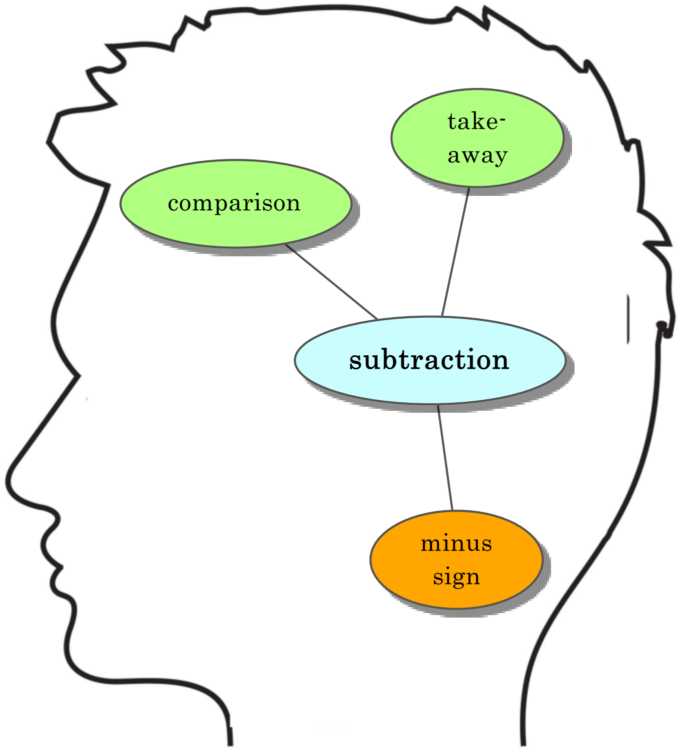






Internal/Mental Representations

- The internal organization of knowledge through cognitive processes (Izsák, 2003)
- "Students' personal symbolization constructs and assignments of meaning to mathematical notions, as wel as their natural language, their visual imagery and spatial representation, their problem-solving strategies and heuristics, and […] their affect in relation to mathematics" (Goldin & Shteingold, 2001, p. 2)



Why Using Representations: Affordances

- **Reinforce students' building of internal representations**
  - Foster connection-making between procedures and concepts or between various strategies (NCTM 2000)
  - Make abstract mathematical concepts more accessible and visible (Flores, 2002)
- **Provide a window into studying and understanding students' internal representations**
  - As teachers "we make *inferences* about students' internal representations on the basis of their interaction with, discourse about, or production of external representations" (Goldin & Shteingold, 2001, pp. 6-7)
- **Provide shared tools and language to communicate important mathematical ideas** (Mathematics Methods Planning Group, 2008, University of Michigan)

Why Using Representations: Challenges (A)

- Students simply **imitate procedures without making connections** to underlying mathematical ideas (Clements & McMillen, 1996; Stein & Bovalino, 2001)
- Representations taught and learned as if they are **ends in themselves**: Learning an external representation results in a proliferation of abstract mathematical rules (Dufour-Janvier et al., 1987; Gregg & Gregg, 2007; NCTM, 2000)
- **Teachers may take for granted that representations will "by default" illuminate underlying mathematical ideas**
  - We're "seeing" concepts that we already understand, but these concepts might be totally obscure to students (Ball, 1992)
  - "Representations do not 'show' the mathematics to students. Rather, the students need to work with each representation extensively in many contexts as well as move between representations in order to understand how they can use a representation to model mathematical ideas and relationships" (NCTM, 2000, p. 209)

Why Using Representations: Challenges (B)

- Can create parallel “worlds”, unconnected to mathematical symbols
- Learning to use the materials becomes the ultimate learning goal
- Can create jargon that obscures communication with parents and others
- Management problems can distract from the mathematics

(Mathematics Methods Planning Group, 2008, University of Michigan)

What is Entailed in Teaching with Representations?

(A.) Representing and solving problems/carrying out mathematical operations

- Recognizing and abiding by the representations' conventions
- Using representations as a means to illuminate certain mathematical ideas involved in a procedure
- Employing appropriate language and notation when using representations
- Decomposing and unpacking mathematical rules and operations through careful use of representations
- Selecting representations that lend themselves to explaining a mathematical procedure

(B.) Creating a context for connecting multiple representations

- Identifying similarities and differences between representations
- Using one representation to help students make sense of another

(C.) Creating a context for generalizing procedures

- Using representations to build generalizations and help students move to a more abstract level
- Selecting and sequencing examples to support student ability to generalize
- Using multiple representations to help students make sense of the underlying meaning of a mathematical procedure

What is Entailed in Teaching with Representations?

(D.) Scaffolding student work on representations and the mathematics

- Using representations to surface student misconceptions and emphasize important mathematical ideas
- Using representations to trigger and remediate student misconceptions
- Flexibly moving between representations to support student understanding
- Providing a balance between explaining the representation conventions and allowing students the space and time to make meaning of the representations and the mathematical ideas they are intended to illuminate
- Examining whether students correctly follow the representations' conventions and ascribe meaning to the representations' manipulations
- Pressing students to articulate the mathematical meaning they are making out of using representations
- Listening to students and unpacking their (promising) productions around using representations
- Differentiating the scaffolding provided to students depending on (a) the anticipated level of transparency of a given representation and (b) students' differential needs and their progress toward abstracting the underlying mathematical ideas the representations are intended to illuminate.

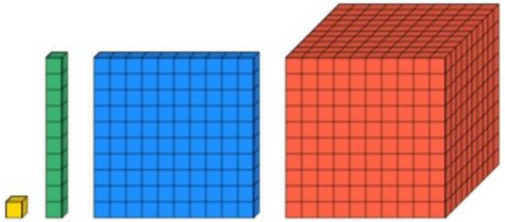
Mitchell, R., Charalambous, C. Y., & Hill, C.H. (2014). Examining the task and knowledge demands needed to teach with representations. *Journal of Mathematics Teacher Education, 17*, 37-60.

What External Representations Should We Be Using? Does it Matter?

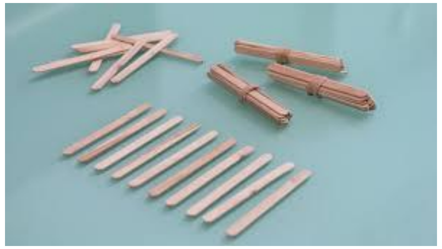
- **Lower elementary grades:** You are teaching subtraction of whole-numbers (e.g., 72-35). Which representation among the following would you use? Why?

(a) Money

(b) Dienes blocks



(c) bundles of sticks



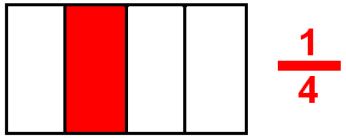
(d) Unifix cubes

What External Representations Should We Be Using? Does it Matter?

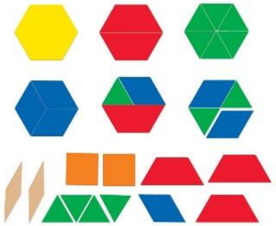
- **Upper elementary grades:** You are teaching addition of fractions with similar denominators. Which representation among the following would you use? Why?

(a) circles/pies

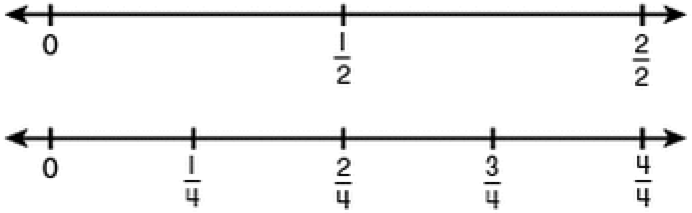
(b) rectangles



(c) pattern blocks



(d) number lines

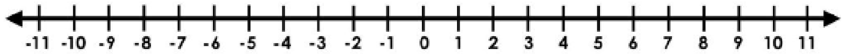


What External Representations Should We Be Using? Does it Matter?

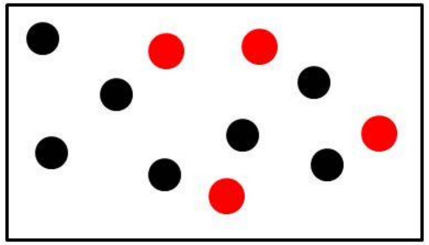
- **Middle grades:** You are teaching integer subtraction. Which representation among the following would you use? Why?

(a) Number lines

5 - (- 4) = ?

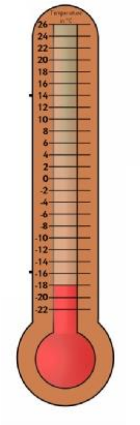


(b) Colored chips



(c) Money (assets and debt)

(d) Thermometer



What External Representations Should We Be Using? Does it Matter?

Time for Reflection

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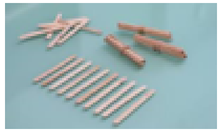
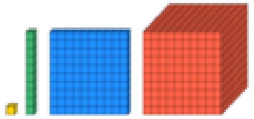
(c) bundles of sticks



(b) Dienes blocks



(d) Unifix cubes

- **Upper elementary grades**: You are teaching addition of fractions with similar denominators. Which representation among the following would you use? Why?

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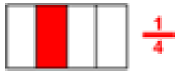
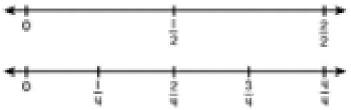
(b) rectangles



(c) pattern blocks

- (d) number lines



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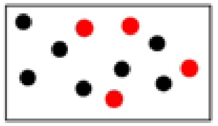

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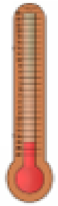

First Scenario: Subtracting Whole-Numbers

Representation	Familiarity	Proportionality in size	Type of model	Logistics
Money	YES	NO	Pre-grouped (trading model)	(+) Less expensive (+) Can facilitate work with large numbers (-) Only some coins are being used
Dienes block	NO	YES	Pre-grouped (trading)	(+) Can facilitate work with large numbers (+) Can illuminate the one-tenness relationship (+) With redefining the unit, can be easily used in decimal operations (-) More expensive
Bundles of sticks	YES	YES	Groupable model (bundling/ unbundling)	(+) Less expensive (+) Can illuminate the one-tenness relationship (-) Rubber bands hard to use
Unifix cubes (interlocking cubes)	NO	YES	Groupable model	(+) Can illuminate the one-tenness relationship (-) More expensive (-) Differences in color might confuse students

Second Scenario: Adding Fractions

Representation	Familiarity	Restrictions in representing fractions	Ease in partitioning	Concept of fraction advanced
Circles/pies	YES	NO	NO	- Part-whole
Rectangles	YES	NO	YES	- Part-whole
Pattern blocks	?	YES	NO	- Part-whole
Number lines	?	NO	YES	- Part-whole - Measurement

Third Scenario: Subtracting Integers

Representation	Familiarity	Interpretation/meaning	Limitations
Number line	?	**Movement:** number sign indicates which way the travelling object should face; operation sign indicates directionality in movement	- The meanings of the "-" sign as both facing backwards and moving backwards might confuse students
Colored chips	NO	**Neutralization representation:** two different colored chips yield a zero ($1 + {}^{+}1 = 0$: additive inverse property)	- Cardinality (how many chips there are) not always the same as value - To subtract, one has to artificially add pairs of chips, a non-intuitive move
Money (assets and debt)	YES	**Neutralization representation:** an asset (+) cancels out a debt (-)	Subtracting a negative cannot be meaningfully/authentically represented (e.g., "*I have $5 in my piggybank, my father decided to strike the $3 I owed him. How much money do I have?*" → $5)
Thermometer	YES (?)	**Temperature and movement:** number sign indicates temperature; operation sign indicates increase/decrease in temperature	Subtracting a negative cannot be meaningfully/authentically represented (e.g., "Yesterday it was 5°C. Today, the temperate has decreased per -3°C" → does not make sense)

How Should We Be Using Representations?

Basic principles

- Not simply using representations but **linking different representations** to avoid creating "parallel" worlds
- Using appropriate **mathematical language** that aligns with the properties of the representation used
- **Highlighting key mathematical ideas** while using and linking representations
- Using representations as a means to foster **multiple solution approaches**

How Should We Be Using Representations?

Examples

1. Subtracting whole-numbers (Dienes blocks)
2. Multiplying multi-digit numbers (Area model)
3. Dividing whole numbers (Unifix cubes)
4. Multiplying fractions (Area model)
5. Dividing fractions (Area model)
6. Subtracting integers (Chips)
7. Linking representations to different solution approaches (the pool boarder problem)

Subtracting Whole Numbers


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  302
- 178
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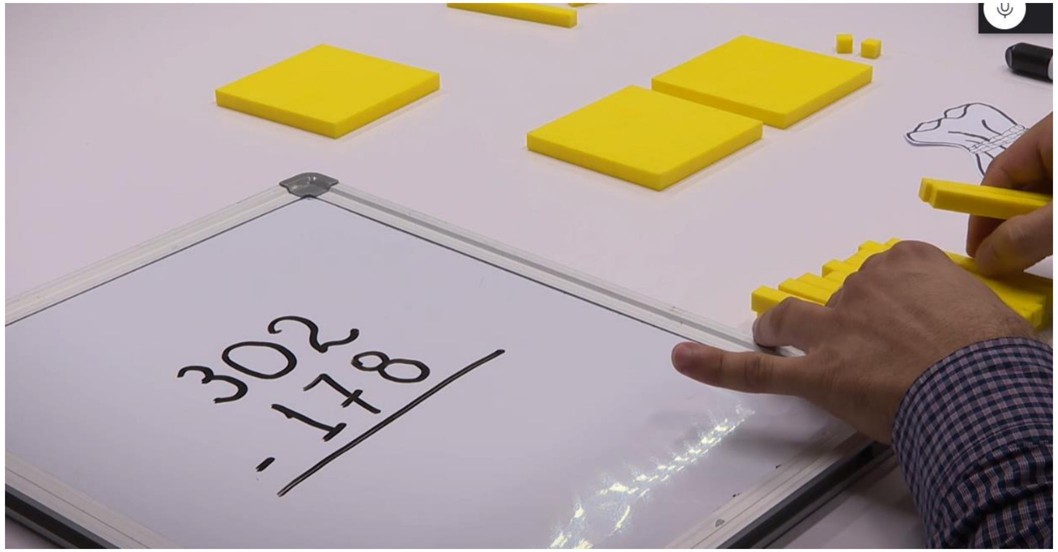
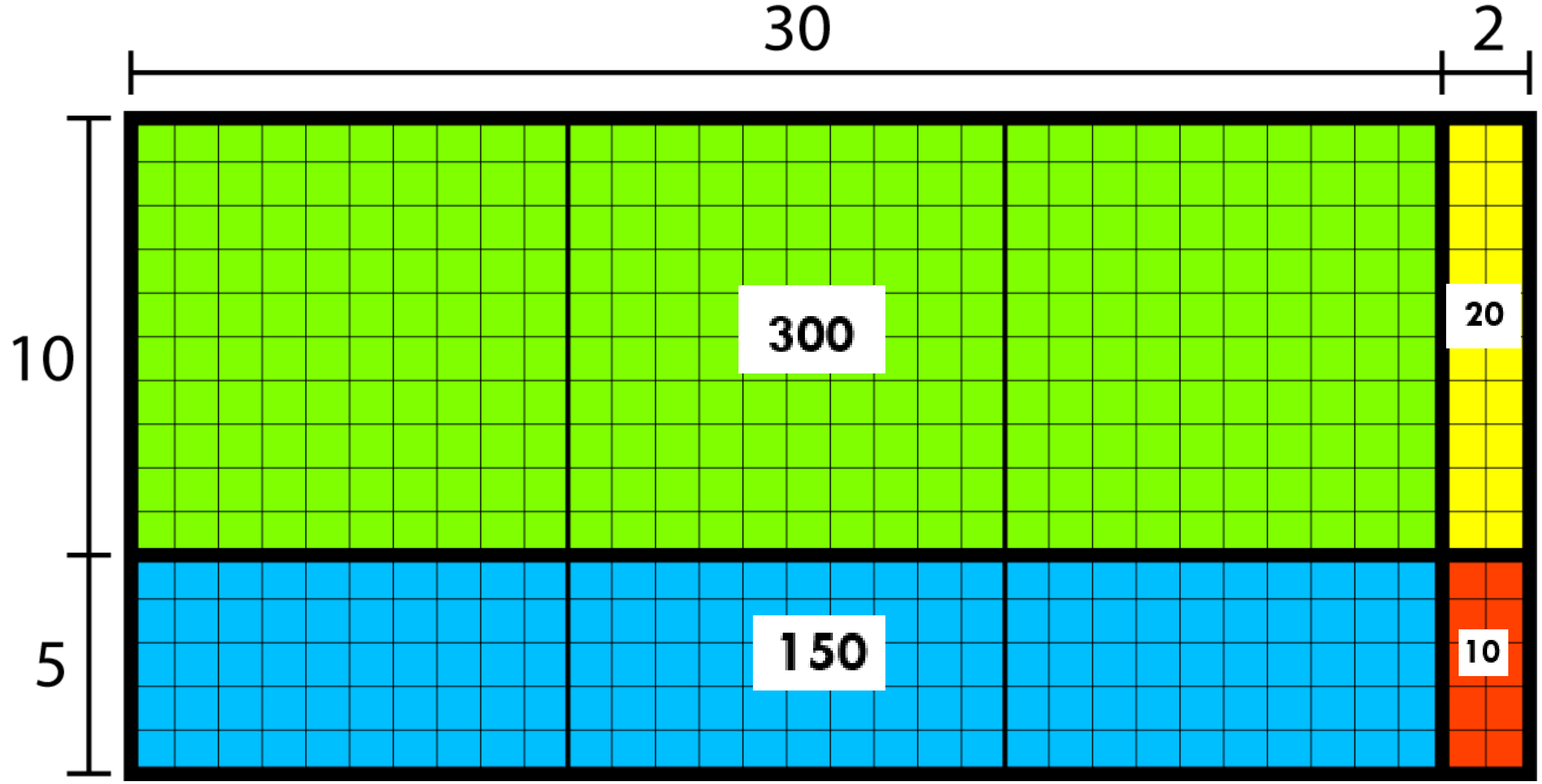


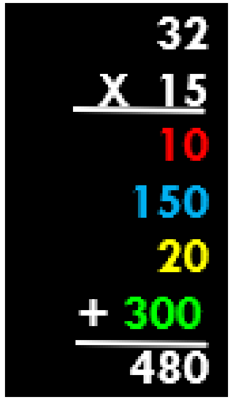
Important to notice

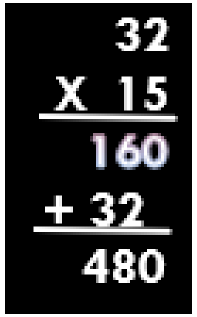
- Language consistent with the representation (trading)
- Emphasis to key mathematical ideas (preservation of value, regardless of the representation of the number)
- Close matching between Dienes and mathematical symbols (notice that even the way the Dienes are placed on the table matches the digits in the algorithm)

Multiplying Multi-Digit Numbers

32 x 15





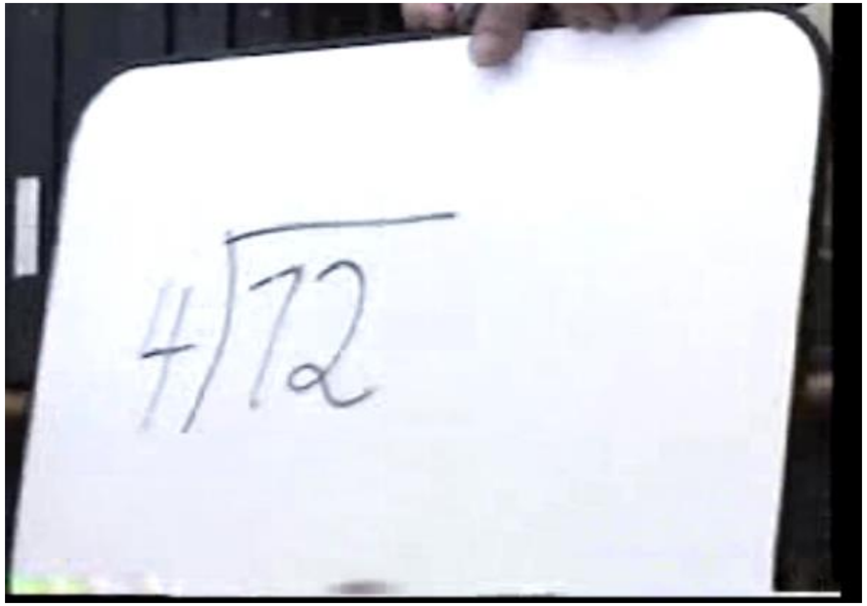


Important to notice

- Drawing connections between the standard algorithm, alternative algorithms, and the representation
- Justifying "mathe-magical" steps: why in the second partial product do we move one place to the left?

Dividing Whole Numbers

72÷4

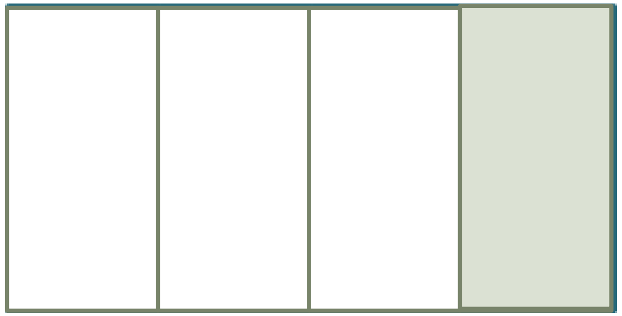


Important to notice

- Addressing the two different meanings of division (**partitive** and measurement) and emphasizing the one corresponding to the long-division algorithm
- Using appropriate language (how many tens, how many not in tens cubes; quotient)
- Recording the steps while using the Unifix cubes
- Clarifying what each number/step in the algorithm corresponds to

Multiplying Fractions

$$\frac{2}{3} \times \frac{1}{4} = \frac{2}{12}$$



Important to notice

- Directly addressing the meaning of fraction multiplication (taking a part of another part)
- Representation is not pre-partitioned into twelfths
  - ¼ is shown first
  - 2/3 of ¼ are illustrated
  - Need to partition the whole into twelfths

See a related online source:
https://www-k6.thinkcentral.com/content/hsp/math/hspmath/na/common/itools_int_9780547584997_/fractions.html

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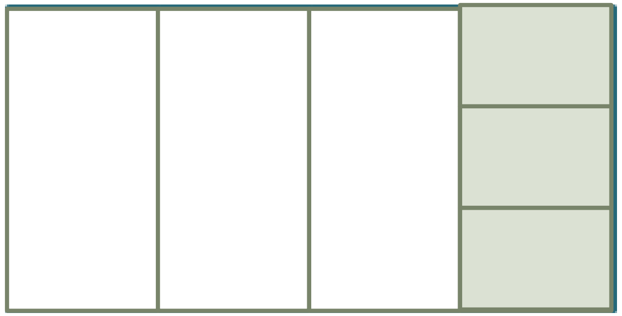

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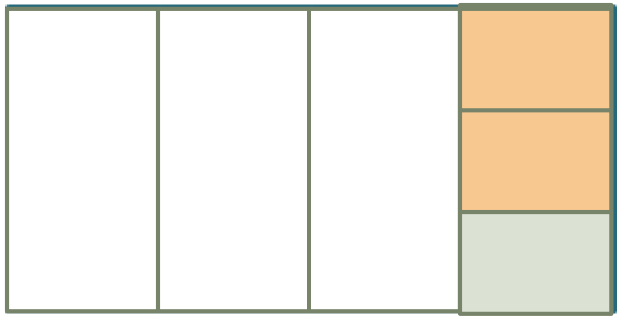

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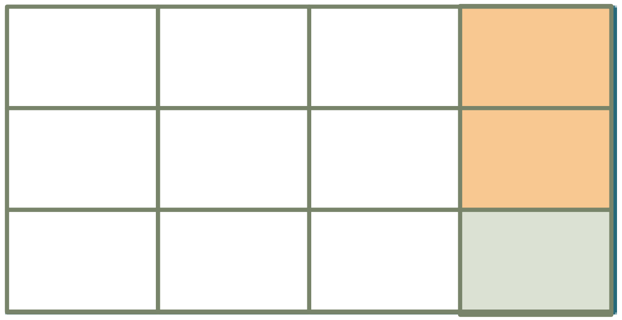

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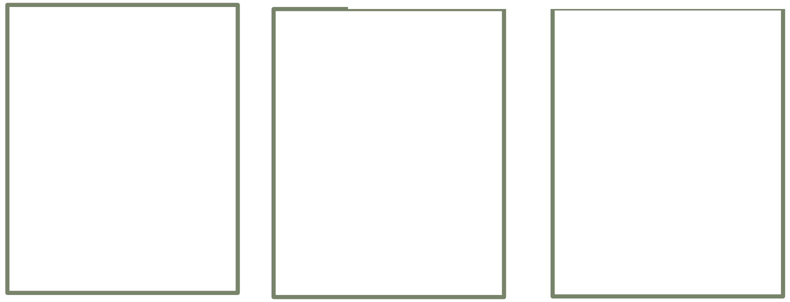
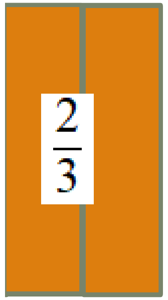
Dividing Fractions

$$3 \div \frac{2}{3} = 3 \text{ X} \frac{3}{2} = \frac{9}{2} = 4\frac{1}{2}$$





Important to notice

- Emphasis given to the measurement meaning of division (how many 2/3 in 3)?
- Distinction between two different units: whole-unit (3) and the reference unit (2/3)
- Meaning given to the different numbers involved in this algorithm

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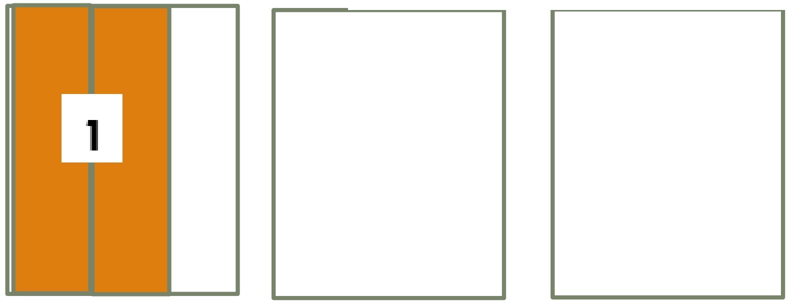

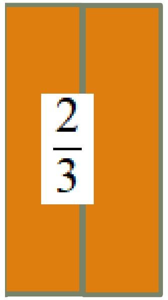

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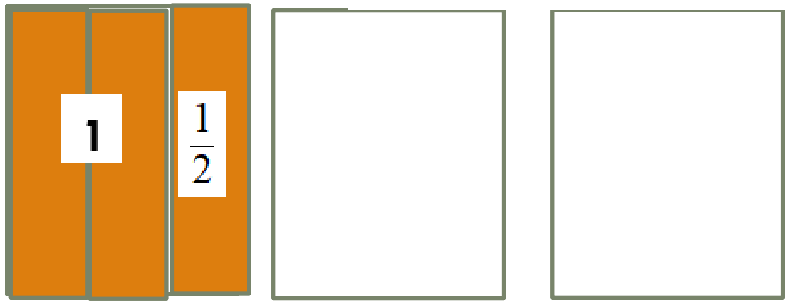
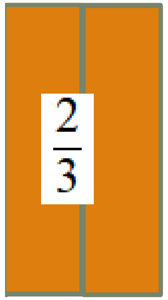
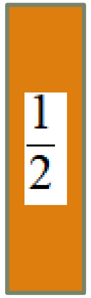




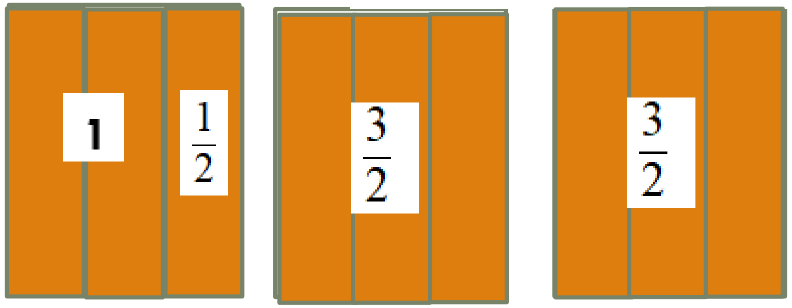


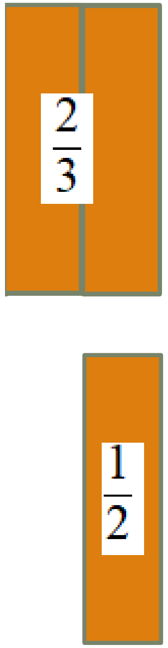
Important to notice

- Emphasis given to the measurement meaning of division (how many 2/3 in 3)?
- Distinction between two different units: whole-unit (3) and the reference unit (2/3)
- Meaning given to the different numbers involved in this algorithm

Dividing Fractions

$$3 \div \frac{2}{3} = 3 \text{ X } \frac{3}{2} = \frac{9}{2} = 4\frac{1}{2}$$





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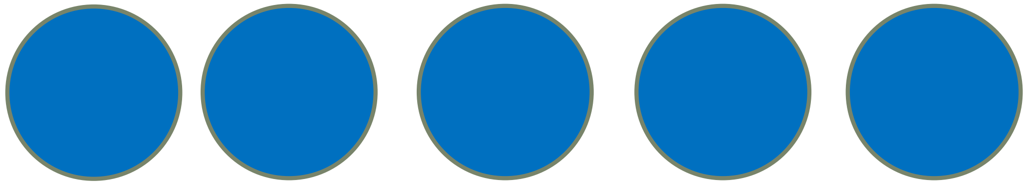
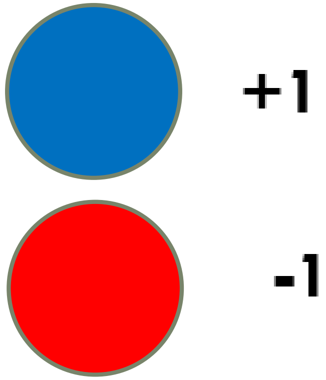
Subtracting Integers

5 – (-3)





<u>Important to notice</u>

- Clarification of what blue and red chips represent
- Reference to the additive inverse property
- Emphasis on the preservation of the value of the minuend

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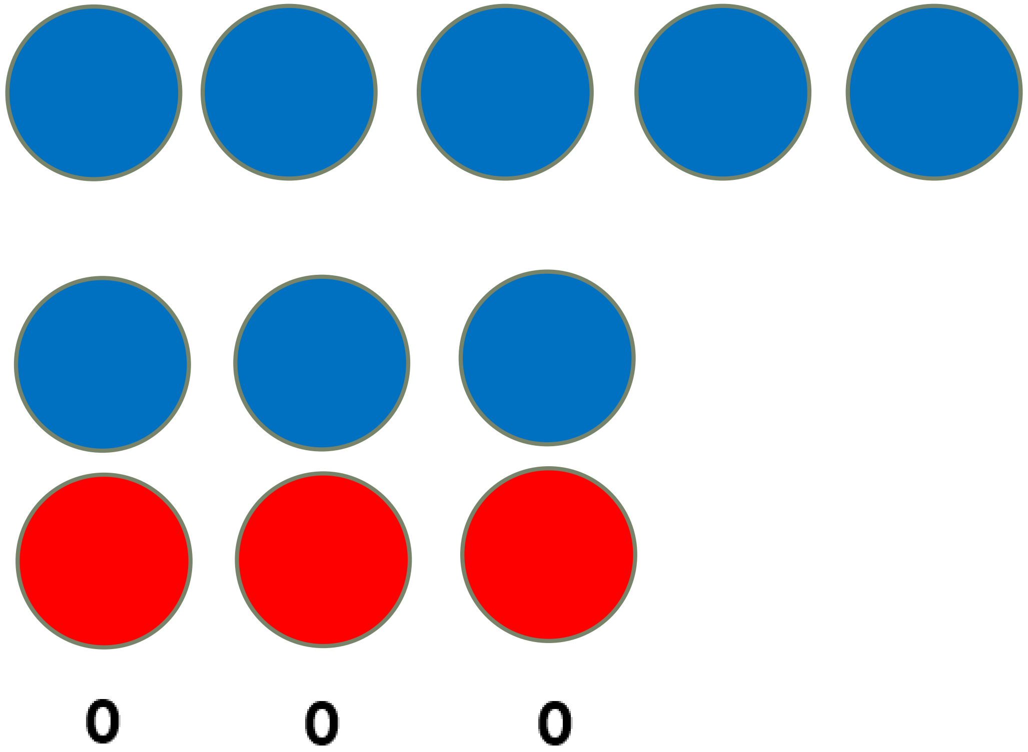

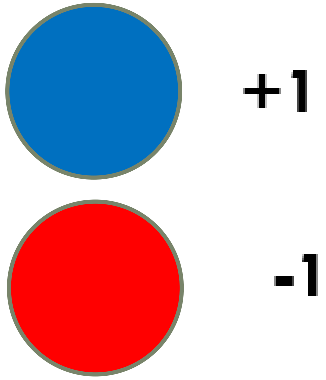

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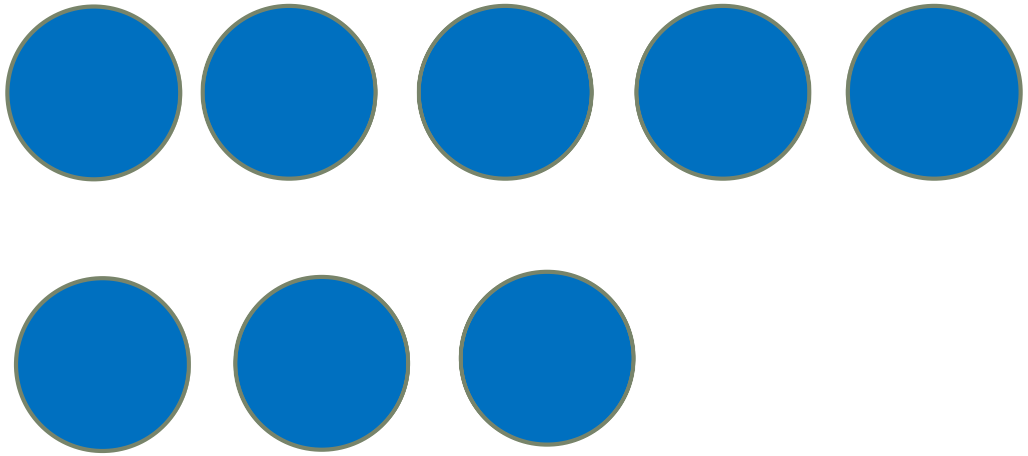

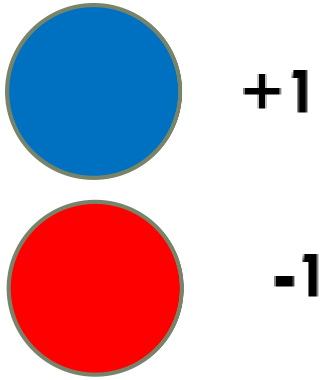

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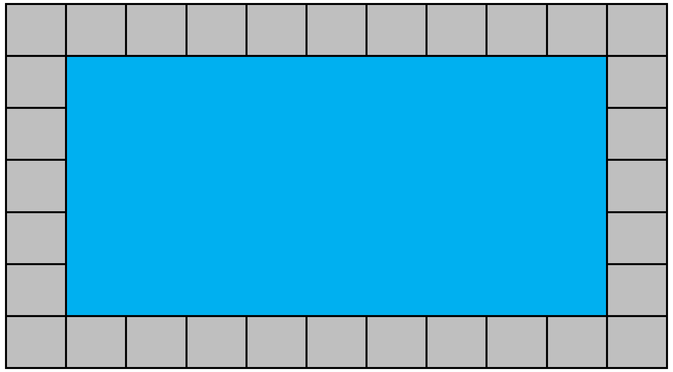
Linking Different Representations to Different Solution Approaches

The pool border problem (NCTM, 2000, pp. 282)

- A rectangular pool is to be surrounded by a ceramic-tile border. The border will be one tile wide all around. How many tiles will be needed for pools of various lengths (L) and widths (W).



Important to notice

- Representations used as a means to foster the identification and the discussion of different solution approaches

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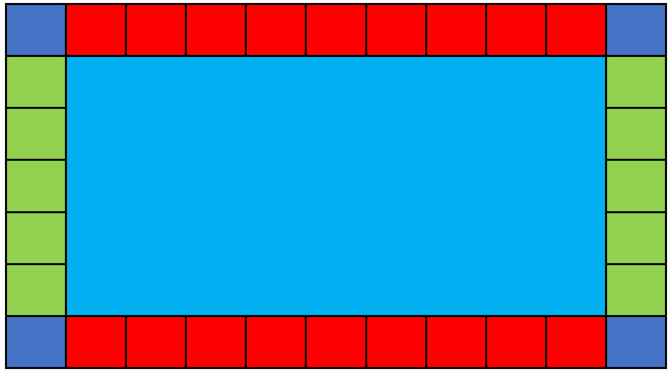

T= 2L + 2W +4

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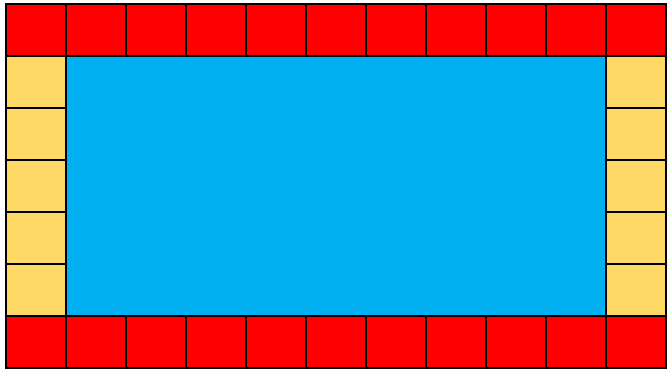

$$T = 2(L+2) + 2W$$

Important to notice

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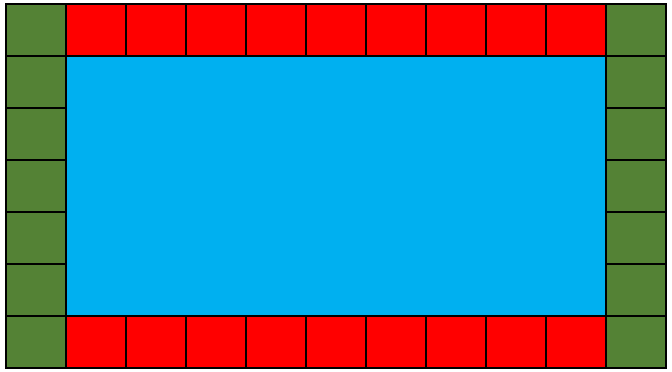

$$T = 2(W+2) + 2L$$

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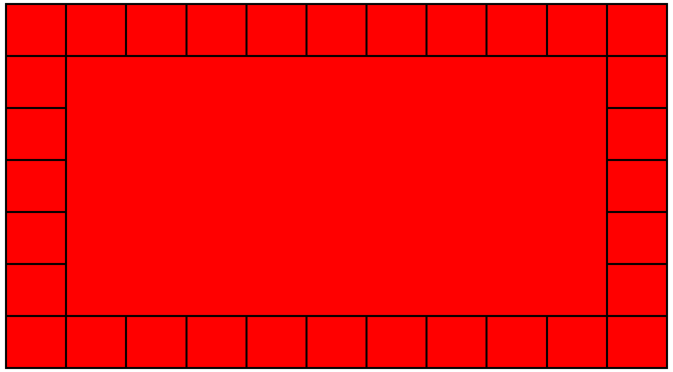

$$T = (L+2)(W+2) - LW$$

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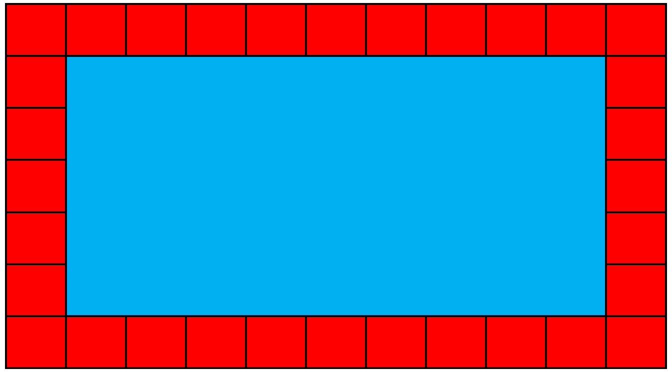

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Important to notice

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Useful Websites for Online Representations and Manipulatives

1. https://www.didax.com/math/virtual-manipulatives.html
2. http://nlvm.usu.edu/en/nav/vlibrary.html (use with Internet Explorer, not Chrome)
3. https://www.visnos.com/demos
4. https://illuminations.nctm.org/ (free for NCTM members)
5. https://www-k6.thinkcentral.com/content/hsp/math/hspmath/na/common/itools_int_9780547584997/main.html
6. https://nrich.maths.org/9084
7. https://mathsframe.co.uk/ (partially free)
8. https://www.mathplayground.com/grade_4_games.html
9. https://www.sheppardsoftware.com/math.htm

Thank you for your attention!

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University of Cyprus

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